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DEFORMATIONS OF RINGS

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Let A and A_0 be rings with a surjective homomorphism $A \to A_0$. Given a flat extension B_0 of A_0, a *deformation* of B_0/A_0 over A is a flat extension B of A such that $B \otimes_A A_0$ is isomorphic to B_0. We show that such a deformation will exist if A_0 is an Artin local ring, A is noetherian, and the homological dimension of B_0 over A_0 is $\leqslant 2$. We also show that a deformation will exist if the kernel of A is nilpotent and if A_0 is a finitely generted A_0-algebra whose defining ideal is a local complete intersection.

0. INTRODUCTION

The purpose of this paper is to examine some special cases of the following general question. Given rings A and A_0 with surjection $u: A \to A_0$ and a *flat* A_0-algebra B_0, when does there exist a flat A-algebra B such that $B \otimes_A A_0$ is isomorphic to B_0 as an A_0-algebra? If it exists, such a B is called a *deformation of B_0/A_0 over A*. An important case is when $\ker(u)$ is nilpotent. In this case B is said to be an *infinitesimal deformation.*

One general situation in which a deformation always exists is when u admits a retraction $t: A_0 \to A$ (that is $u \circ t = 1$). In this case $B = B_0 \otimes_{A_0} A$ is a deformation, called the *trivial deformation.* This is the usual setting in algebraic geometry with A_0 a field and the problem is to find the structure of the set of nontrivial deformations.

In this paper, however, we take the ring-theoretic approach and consider the general question of when any deformation exists at all. We prove that a deformation exists in two general cases. One is when A_0 is an Artin local ring and B_0 is of homological dimension $\leqslant 2$ with A noetherian (the noetherian hypothesis can be dropped for an infinitesimal deformation). The other case is the case of an infinitesimal deformation where the defining ideal for B_0 is a local complete intersection. The first case is done in Section 2 while the second is in Section 3.

1. SOME REMARKS ON FLATNESS

We begin with a well-known observation.

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LEMMA 1.1. *Let* A *be a ring and*

$$0 \to K \to L \to M \to 0$$

an exact sequence of A*-modules with* L *flat. Then:*

(1) *given an ideal* $\mathcal{A} \subseteq A$, $\operatorname{Tor}_1^A(M, A/\mathcal{A}) = 0$ *if and only if* $\mathcal{A}L \cap K = \mathcal{A}K$;

(2) M *is flat if and only if* $\mathcal{A}L \cap K = \mathcal{A}K$ *for every ideal* $\mathcal{A}$ *of* A.

PROOF: Tensor the sequence with $A/\mathcal{A}$ and note that $\operatorname{Tor}_1^A(L, A/\mathcal{A}) \cong 0$ so $\operatorname{Tor}_1^A(M, A/\mathcal{A}) = (\mathcal{A}L \cap K)/\mathcal{A}K$. (1) follows, and (2) is a consequence of (1). ∎

DEFINITION: Given a ring A, an ideal $\mathcal{A}$ of A, and an A-module M, we say that M is *idealwise separated for* $\mathcal{A}$ if for every finitely generated ideal I of A, $\cap_{n=1}^{\infty} \mathcal{A}^n(I \otimes_A M) = 0$.

An important special case in which M is ideal-wise separated for $\mathcal{A}$ is when there is a noetherian A-algebra B so that M is a finite B-module and $\mathcal{A}B$ is contained in the Jacobson radial of B.

THEOREM 1.2. *Assume that there is an exact sequence of* A*-modules*

$$0 \to K \to L \to M \to 0$$

where L *is flat. Let* $\mathcal{A}$ *be an ideal in* A, *and suppose that either:*

(i) $\mathcal{A}$ *is nilpotent or*

(ii) A *is noetherian and* M *is idealwise separated for* $\mathcal{A}$.

Then the following are equivalent:

(1) M *is* A*-flat;*

(2) $M/\mathcal{A}M$ *is* $(A/\mathcal{A})$*-flat and* $\mathcal{A}L \cap K = \mathcal{A}K$.

PROOF: Follows from [**6**, (20.C)] and Lemma 1.1. ∎

In general it is not true that condition (2) implies that M is A-flat, even when S is a local ring.

Example. Let k be a field. $k[u,v,w]$ the polynomial ring in 3 variables. Let $\mathcal{M} = (u,v,w)$ and set $A = k[u,v,w]_{\mathcal{M}}$, the localization at $\mathcal{M}$. Take $M = A[x,y]/(ux-1, vy-w)$. Then we have an exact sequence as in Theorem 1.2 with $L = A[x,y]$ and $K = (ux-1, uy-w)$. Note that $M/\mathcal{M}M = (0)$ and $\mathcal{M}L \cap K = \mathcal{M}K$ since $\mathcal{M}$ and K are comaximal. However, $(v,w)L \cap K \neq (v,w)K$ so M is not flat by Lemma 1 (2).

Let A be a ring, $\mathcal{A}$ an ideal of A, $A_0 = A/\mathcal{A}$, and B_0 a finitely generated A_0-algebra which is flat over A_0. In order to deform B_0/A_0 over A we must find a flat A-algebra, B, such that $B/\mathcal{A}B = B_0$. If $B_0 = A_0[x_1, \ldots, x_m]/I_0$ one can easily find an ideal I such that if $B = A[x_1, \ldots, x_m]/I$ then $B/\mathcal{A}B = B_0$. Simply choose preimages for the generators for I_0 under the map $A[x_1, \ldots, x_m] \to A_0[x_1, \ldots, x_m]$. The difficulty is choosing I so that B is A-flat. Later on we will demonstrate that under certain circumstances I can be chosen so that $\mathcal{A}A[x_1, \ldots, x_m] \cap I = \mathcal{A}I$. Hence we are in the context of Theorem 1.2 with $M = B$, $L = A[x_1, \ldots, x_m]$, $K = I$, lacking only hypothesis (i) or (ii). As the example demonstrates, we cannot in general conclude flatness. However, we do have:

THEOREM 1.3. *Let A be a noetherian ring, $B = A[x_1, \ldots, x_m]/I$, $\mathcal{A}$ an ideal of A, $A_0 = A/\mathcal{A}$, and $B_0 = B/\mathcal{A}B$. Suppose that B_0 is A_0-flat and $\mathcal{A}A[x_1, \ldots, x_m] \cap I = \mathcal{A}I$. Then there is an $f \in B$ which is a unit modulo $\mathcal{A}$ and such that B_f is A-flat.*

Hence, there is a finitely generated A-algebra $B' = B_f$ which is a deformation of $B/\mathcal{A}B$ over A.

PROOF: Let P be a prime in B, $Q \subset A[x_1, \ldots, x_m]$ the inverse image of P under the surjection $A[x_1, \ldots, x_m] \to B$, and let $C = A[x_1, \ldots, x_m]_Q$. Then $\mathcal{A}C \cap IC = \mathcal{A}IC$ by the hypothesis.

If $\mathcal{A}B \subseteq P$ then B_P is flat by Theorem 1.2. By [6, Theorem 5.3] there is an open set U in spec (B) containing $V(\mathcal{A}B)$ such that B_P is flat for all $P \in U$. We wish to find an affine open set with the same property. Let spec$(B) - U = V(J)$. Since $V(J) \cap V(\mathcal{A}B) = 0$, we have $J + \mathcal{A}B = B$. Choose $f \in J$ so that $(f) + \mathcal{A}B = B$. Then B_f is A-flat by [6, (3j)]. ∎

2. Deformations in Homological Dimension $\leqslant 2$

Our object is to prove:

THEOREM 2.1. *Given an Artin local ring A_0 and a finitely generated A_0-algebra $A_0[x_1, \ldots, x_m]/I_0$, suppose that B_0 is A_0-flat and that $dh_{A_0[x_1, \ldots, x_m]}(B_0) \leqslant 2$. Let A be a ring and $\mathcal{M} \subseteq A$ an ideal such that $A/\mathcal{M} \cong A_0$. Suppose that either:*

(i) *$\mathcal{M}$ is nilpotent or*

(ii) *A is noetherian.*

Then there is a deformation B of B_0/A_0 over A.

PROOF: Let $R_0 = A_0[x_1, \ldots, x_m]$. By hypothesis there is a resolution of the form:

$$0 \to P_2 \to P_1 \to R_0 \to B_0 \to 0$$

where P_1 and P_2 are projective R_0-modules of finite rank. But P_1 and P_2 are free modulo the maximal ideal N of A_0 by [7] or [9]. Since the maximal ideal is nilpotent

P_1 and P_2 must be free. (Apply Nakayama's Lemma to the preimage of a basis mod N). Therefore we have a resolution

$$0 \to R_0^{r-1} \xrightarrow{\phi_2} R_0^r \xrightarrow{\phi_1} R_0 \to B_0 \to 0.$$

We now apply a standard technique using Burch's Theorem. We can find no reference that precisely fits our situation so we give the details (see [1] and [8]). Let $M = [a_{ij}]$ be the $r \times (r-1)$ matrix that represents ϕ_2 and $[f_1, \ldots, f_r]$ the matrix representation for ϕ_1. Let f_i be the minor of M obtained by omitting the ith row of M. By Burch's Theorem ([4, p.148, problem 8]) there is a regular element $f \in R_0$ such that $fd_i = f_i$. We let $R = A[x_1, \ldots, x_m]$ and define $\psi_2 \colon R^{r-1} \to R^r$ by lifting M to a matrix $\tilde{M} = [\tilde{a}_{ij}]$ over R. Let $\tilde{f}$ be a lifting for f and define $g_i = \tilde{f}\tilde{d}_i$ where $\tilde{d}_i$ is the minor of $\tilde{M}$ obtained by omitting the ith row. Then $\psi_1 \circ \psi_2 = 0$ since, for any k

$$g_1 \tilde{a}_{1k} + \ldots + g_r a_{rk} = \tilde{f} \deg \begin{bmatrix} a_{1k} & a_{11} & \ldots & a_{1k} & \ldots & a_{1k-1} \\ \vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\ a_{rk} & a_{r1} & \ldots & a_{rk} & \ldots & a_{rk-1} \end{bmatrix} = 0$$

∎

We now use the following lemma:

LEMMA 2.2. *Let R be a ring, $\mathcal{A} \subseteq R$ an ideal, and $R_0 = R/\mathcal{A}$. Suppose we have a diagram:*

$$\begin{array}{ccccc} G_2 & \xrightarrow{\psi_2} & G_1 & \xrightarrow{\psi_1} & G_0 \\ \downarrow \alpha_2 & & \downarrow \alpha_1 & & \downarrow \alpha_0 \\ F_2 & \xrightarrow{\phi_2} & F_1 & \xrightarrow{\phi_1} & F_0 \end{array}$$

with the bottom row an exact sequence of R_0-modules and the top row a complex of R-modules (that is $\psi_1 \circ \psi_2 = 0$). Suppose further that, for each i, $F_i = F_i \otimes_R R_0$ with α_i the canonical projection.

Then $\mathcal{A}G_0 \cap \operatorname{im}(\psi_1) = \mathcal{A}\operatorname{im}(\psi_1)$.

PROOF: Choose $z \in \mathcal{A}G_0 \cap \operatorname{im}(\psi_1)$. Then $z = \psi_1(w)$ for some $w \in G_1$. But $z \in \mathcal{A}G_0 \Rightarrow \alpha_0(z) = 0$. Hence $\phi_1\alpha_1(w) = \alpha_0\psi_1(w) = 0$. By exactness $\alpha_1(w) = \phi_1(u')$ for some $u' \in F_2$. Pick $u \in G_2$ with $\alpha_2(u) = u'$. Then $\alpha_1(w) = \alpha_1(\psi_2(u))$ and $w - \psi_2(u) \in \ker \alpha_1 = \mathcal{A}G_1$ and $\psi_1(w - \psi_2(u)) \in \mathcal{A}\operatorname{im}(\psi_1)$. But $\psi_1(w - \psi_2(u)) = \psi_1(w) = z$ and $z \in \mathcal{A}\operatorname{im}(\psi_1)$.

We have shown that

$$\mathcal{A}G_0 \cap (\psi_1) \subseteq \mathcal{A}\operatorname{im}(\psi_1).$$

The reverse inclusion is obvious. ∎

Now applying the lemma to the sequence

$$\begin{array}{ccccc} R^{r-1} & \xrightarrow{\psi_2} & R^r & \xrightarrow{\psi_1} & R \\ \downarrow & & \downarrow & & \downarrow \\ R_0^{r-1} & \xrightarrow{\phi_2} & R_0^r & \xrightarrow{\phi_1} & R \end{array}$$

and letting $1 = \mathrm{im}(\psi_1)$ we get that $\mathcal{M}R \cap 1 = \mathcal{M}1$. Hence, by Theorem 1.3 in the noetherian case or Theorem 1.2 in the nilpotent case we get the result. ∎

COROLLARY 2.3. *Let A_0 be an Artin local ring and $B_0 = A_0[x_1, x_2]/I_0$ be a flat A_0-algebra. Let A be a ring with an ideal $\mathcal{M}$ such that either:*

(i) *$\mathcal{M}$ is nilpotent, or*

(ii) *A is noetherian.*

Then there is a deformation B of B_0/A_0 over A.

(Note: There are examples of rings A, A_0 as above and $B_0 = A[x_1, x_2, x_3, .x_4, x_5, x_6]/I_0$ for which there is *no* deformation of B_0/A_0 over A. (See [**2**] or [**3**].))

PROOF: We need show only that $dh_{R_0}(B_0) \leqslant 2$ with $R_0 = A_0[x_1, x_2]$. Consider the exact sequence

$$0 \to K \to R_0^n \to R_0 \to B_0 \to 0.$$

Let k be the residue field of A_0.

By the flatness of B_0 we have

$$0 \to K \otimes_{A_0} k \to R_0^n \otimes_{A_0} k \to R_0 \otimes_{A_0} k \to B_0 \otimes_{A_0} k \to 0.$$

$R_0 \otimes_{A_0} k = k[x_1, x_2]$ which has global dimension 2 so $K \otimes_{A_0} k$ is projective (hence free) and that implies that K is free. ∎

3. INFINITESIMAL DEFORMATIONS OF LOCAL COMPLETE INTERSECTIONS

DEFINITION: Let R be a ring and $I \subset R$ an ideal. I is a *local complete intersection* if, for every $P \in \mathrm{spec}(R)$ with $I \subseteq P$, IR_P is generated by an R_P-regular sequence.

Our objective is to prove the following theorem:

THEOREM 3.1. *Given a noetherian ring A_0 and a finitely generated A_0-algebra $B_0 = A_0[x_1, \ldots, x_n]/I_0$, suppose that B_0 is A_0-flat and that I_0 is a local complete intersection in $A_0[x_1, \ldots, x_n]$. Let A be a ring and $\mathcal{A} \subseteq A$ an ideal in A such that $A/\mathcal{A} = A_0$ and $\mathcal{A}$ is nilpotent. Then there is a finitely generated A-algebra B such that B is a deformation of B_0/A_0 over A.*

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Furthermore, if B is any deformation of B_0/A_0 over A and $B = A[y_1, \ldots, y_m]/I$. Then I is a local complete intersection in $A[y_1, \ldots, y_m]$.

In order to prove Theorem 3.1 we need to use the upper cotangent functor $T^i(B/A, M)$ associated with a ring homomorphism $A \to B$ and a B-module M. (see [**5**]). We state two results from [**5**].

THEOREM 3.2. *([**5**, 3.2.2]) Let A be a noetherian ring and B a finitely generated A-algebra. The following conditions are equivalent:*

(a) $B = A[x_1, \ldots, x_n]/I$ *where I is a local complete intersection in* $A[x_1, \ldots, x_n]$*;*

(b) *for any surjection $A[y_1, \ldots, y_m] \to B$ the kernel is a local complete intersection of $A[y_1, \ldots, y_m]$;*

(c) $T^2(B/A, M) = 0$ *for every B-module M.*

Before stating the next theorem we need a little explanation. Consider a sequence of ring homomorphisms $A \to A_0 \to B_0$ where $A_0 = A/\mathcal{A}$ with $\mathcal{A}^2 = 0$. This gives a long exact change of rings sequence of cotangent functors. In particular, we have a map

$$\delta\colon T^1(A_0/A, \mathcal{A} \otimes_{A_0} B_0) \to T^2(B_0/A_0, \mathcal{A} \otimes_{A_0} B_0).$$

(Note that $\mathcal{A} = \mathcal{A}/\mathcal{A}^2$ is an A_0-module). It is known that $T^1(A_0/A, \mathcal{A} \otimes_{A_0} B_0) \cong \operatorname{End}_{B_0}(\mathcal{A}_0, \mathcal{A} \otimes_{A_0} B)$. Hence there is a canonical element $\delta(1) \in T^2(B_0/A_0, \mathcal{A} \otimes_{A_0} B_0)$.

THEOREM 3.3. *([**5**, 4.3.3]) Let A be a ring and let $A_0 - A/\mathcal{A}$ with $\mathcal{A}^2 = 0$. Suppose that B_0 is a flat A_0-algebra. Then there is a deformation of B_0/A_0 over A if and only if $\delta(1) = 0$ in $T^2(B_0/A_0, \mathcal{A} \otimes_{A_0} B_0)$.*

PROOF OF THEOREM 3.1: We first prove the last statement. So assume that B is a deformation of B_0/A_0 over A. By Theorem 3.2 it is sufficient to prove that $T^2(B/A, M) = 0$ for every B-module M. Since $\mathcal{A}$ is nilpotent there is an integer k such that $\mathcal{A}^k M = 0$. Our proof is by induction on k.

For $k = 1$, M is a $B/\ AB \cong B_0$-module. Since B is flat over A we can conclude by [**5**, 2.3.2] that $T^2(B/A, M) \cong T^2(B_0/A_0, M)$. Therefore, by Theorem 3.2, I is a local complete intersection.

Now suppose $k > 1$ and consider the exact sequence of B-modules

$$0 \to \mathcal{A}M \to M \to M/\mathcal{A}M \to 0.$$

By the induction hypothesis $T^2(B/A, \mathcal{A}M) = T^2(B/A, M/\mathcal{A}M) = 0$ and by the long exact sequence of T^i ([**5**, 2.3.6]) $T^2(B/A, M) = 0$ and I is a local complete intersection.

Now assume that B_0, A_0, and A are as in the theorem and we wish to show that a deformation exists. Let k be such that $\mathcal{A}^k = 0$. Again, we proceed by induction on k. If $k \leqslant 2$ we are done by Theorem 3.2 and Theorem 3.3 so assume that $k > 2$ and let B_{k-1} be a deformation of B_0/A_0 over $A_{k-1} = A/\mathcal{A}^{k-1}$ of the form $B_{k-1} = A_{k-1}[y_1, \ldots, y_n]/I_{k-1}$. By the first part of the proof I_{k-1} is a local complete intersection and we can find a finitely generated deformation of B_{k-1}/A_{k-1} over A. ∎

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