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SEPARABILITY IN AN ALGEBRA WITH SEMI-LINEAR HOMOMORPHISM

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The purpose of this paper is to outline a simple theory of separability for a non-associative algebra A with semi-linear homomorphism σ. Taking A to be a finite dimensional abelian Lie p-algebra L and σ to be the pth power operation in L, this separability is the separability of [**2**]. Taking A to be an algebraic field extension K over k and σ to be the Frobenius (pth power) homomorphism in K, this separability is the usual separability of K over k. The theory also applies to any unital non-associative algebra A over a field k and any unital homomorphism σ from A to A such that $\sigma(ke) \subset ke$, e being the identity element of A.

Throughout the paper, A is a non-associative algebra over a field k, with or without identity, and σ is a semi-linear homomorphism of non-associative rings from A to A; that is,

$$\sigma(x+y) = \sigma(x) + \sigma(y)$$
$$\sigma(xy) = \sigma(x)\sigma(y)$$
$$\sigma(\alpha x) = \bar{\sigma}(\alpha)\sigma(x)$$

for $x, y \in A$, $\alpha \in k$, $\bar{\sigma}$ being a suitable homomorphism of fields from k to k. In parts of the paper, we assume that A and σ are unital; that is, A has an identity e and $\sigma(e) = e$. Then a *unital subalgebra* of A is a subring B of A such that $e \in B$.

Definition 1. A *σ-subspace* (*σ-subalgebra*) of A is a subspace (subalgebra) V of A such that $\sigma(V) \subset V$.

Definition 2. For $x \in A$, $\langle x \rangle$ is the σ-subspace of A generated by x; that is, the k-span of $x, \sigma(x), \sigma^2(x), \ldots$.

Definition 3. An element x of A is *σ-algebraic* (*σ-separable*; *σ-nilpotent*) if $\langle x \rangle$ is finite dimensional ($\langle x \rangle = \langle \sigma(x) \rangle$; $\sigma^n(x) = 0$ for some positive integer n). If A and σ are unital, an element x of A is *σ-radical* if $\sigma^n(x) \in ke$ for some positive integer n.

We emphasize that all that is said in this paper about σ-radical objects is understood to apply only in the case where A and σ are unital.

Definition 4. The set of σ-algebraic (σ-separable; σ-nilpotent; σ-radical) elements of a σ-subspace V of A is denoted V_{alg} (V_{sep}; V_{nilp}; V_{rad}). If $V = V_{\mathrm{sep}}$ ($V = V_{\mathrm{rad}}$), V is *separable* (*radical*).

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One sees readily that if V is a σ-subspace (σ-subalgebra) of A, then V_{alg}, V_{nilp} and V_{rad} are σ-subspaces (σ-subalgebras) of A. In order to enable us to prove that V_{sep} is also a σ-subspace (σ-subalgebra) of A, we now impose for the remainder of the paper the further condition on A that A be σ-algebraic.

Definition 5. A σ-subspace V is *σ-regular* if $\sigma|V$ is injective and σ maps each basis of V to a basis for V.

PROPOSITION 1. *Let V be a finite dimensional σ-subspace of V. Then the following conditions are equivalent.*

(1) *V is σ-regular;*

(2) *if $x_1, \ldots, x_m$ are linearly independent elements of V, then $\sigma(x_1), \ldots, \sigma(x_m)$ are linearly independent;*

(3) *if $x_1, \ldots, x_m$ span V, then $\sigma(x_1), \ldots, \sigma(x_m)$ span V;*

(4) *$\sigma(V)$ spans V.*

Proof. (1) implies (2) since we can expand $x_1, \ldots, x_m$ to a finite basis $x_1, \ldots, x_n$ $(n \geqq m)$. And (2) implies (3) since we can contract $x_1, \ldots, x_m$ to a minimal spanning set $x_1, \ldots, x_n$ $(n \leqq m)$, a basis, which then is mapped to a basis $\sigma(x_1), \ldots, \sigma(x_n)$ by (2). Clearly, (3) implies (4). And (4) implies (1); for, let $x_1, \ldots, x_n$ be a basis for V. Since $\sigma(V)$ spans V and $x_1, \ldots, x_n$ span V, $\sigma(x_1), \ldots, \sigma(x_n)$ span V. Thus, $\sigma(x_1), \ldots, \sigma(x_n)$ is a basis for V.

COROLLARY. *An element x of A is σ-separable if and only if $\langle x \rangle$ is σ-regular.*

Proof. A spanning set for $\langle x \rangle$ is $x, \sigma(x), \ldots, \sigma^n(x)$ for some n. Then a spanning set for $\langle \sigma(x) \rangle$ is easily seen to be $\sigma(x), \sigma^2(x), \ldots, \sigma^{n+1}(x)$. Now $\langle x \rangle = \langle \sigma(x) \rangle$ if and only if $\langle x \rangle$ is σ-regular by (3) of the above proposition.

PROPOSITION 2. *A σ-subspace V of A is σ-regular if and only if every finite dimensional σ-subspace of V is σ-regular.*

Proof. Let V be σ-regular and let W be a finite dimensional σ-subspace of V. Let $x_1, \ldots, x_n$ be linearly independent elements of W and let S be a basis for V containing $x_1, \ldots, x_n$. Then $\sigma(S)$ is a basis for V and $\sigma(x_1), \ldots, \sigma(x_n)$ are distinct elements of $\sigma(S)$. Thus, $\sigma(x_1), \ldots, \sigma(x_n)$ are linearly independent and W is σ-regular. Suppose conversely that every finite dimensional σ-subspace of V is σ-regular. Let S be a basis for V and let $x_1, \ldots, x_n \in S$. Then $x_1, \ldots, x_n \in W$ where $W = \sum_{i=1}^{n} \langle x_i \rangle$. Since W is a finite dimensional σ-subspace, W is σ-regular and $\sigma(x_1), \ldots, \sigma(x_n)$ are linearly independent. Thus, σ is injective and $\sigma(S)$ is linearly independent. Next, let $x \in V$. Then $x \in W$ where $W = \langle x \rangle$. Thus, x is in the span of $\sigma(W)$, hence in the span of $\sigma(V)$. Since S spans V, it follows that x is in the span of $\sigma(S)$. Thus, $\sigma(S)$ spans V and V is σ-regular.

COROLLARY. *Let V be a σ-regular σ-subspace of V. Then any σ-subspace W of V is σ-regular.*

PROPOSITION 3. *Let V be the sum of a family V_λ of σ-regular σ-subspaces of A. Then V is σ-regular.*

Proof. Let W be a finite dimensional σ-subspace of V. Then $W \subset \sum_{i=1}^n (W_i)$ for suitable finite dimensional σ-subspaces $W_1, \ldots, W_n$ each of which is contained in one of the V_λ. By the above corollary the W_i are σ-regular. Thus, the span of $\sigma(\sum_{i=1}^n W_i) = \sum_{i=1}^n \sigma(W_i)$ is $\sum_{i=1}^n W_i$ and $\sum_{i=1}^n W_i$ is σ-regular. Thus, W is σ-regular. Now V is σ-regular by the above proposition.

In the following discussion, we use the notation V_1V_2 for the span of $\{xy | x \in V_1, y \in V_2\}$ for $V_1, V_2 \subset A$.

PROPOSITION 4. *Let V_1, V_2 be σ-regular σ-subspaces of A. Then V_1V_2 is a σ-regular σ-subspace of A.*

Proof. Obviously, V_1V_2 is a σ-subspace. Let W be a finite dimensional σ-subspace of V_1V_2. Then $W \subset W_1W_2$ for suitable finite dimensional σ-subspaces W_i of V_i $(i = 1, 2)$. (Any finite subset of a σ-subspace V is contained in a finite dimensional σ-subspace of V.) Now the span of $\sigma(W_1W_2)$ contains the span of $\sigma(W_1)\sigma(W_2)$ and the latter is W_1W_2. Since W_1W_2 is finite dimensional it is therefore σ-regular. Thus, W is σ-regular. It follows that V_1V_2 is σ-regular, by Proposition 2.

THEOREM 1. *Let V be a σ-subspace (σ-subalgebra) of A. Then V is σ-separable if and only if V is σ-regular. Moreover, V_{sep} is a σ-subspace (σ-subalgebra) of A.*

Proof. Suppose that V is σ-regular. Then $x \in V$ implies that $\langle x \rangle$ is σ-regular and hence that x is σ-separable. Thus, V is σ-separable. Next, suppose that $x \in V_{\text{sep}}$. Then $\langle x \rangle$ is σ-regular, so that $\langle x \rangle \subset V_{\text{sep}}$ by the above observation. Thus,

$$V_{\text{sep}} = \sum_{x \in V_{\text{sep}}} \langle x \rangle$$

and V_{sep} is a σ-regular σ-subspace of A, by Proposition 3. In particular, if V is σ-separable, then V is σ-regular. Suppose finally that V is a σ-subalgebra of A. Then $V_{\text{sep}}V_{\text{sep}}$ is σ-regular, by Proposition 4. Thus, $V_{\text{sep}}V_{\text{sep}} \subset V_{\text{sep}}$, and V_{sep} is a σ-subalgebra of A.

PROPOSITION 5. *For $x \in A$, $\sigma^n(x)$ is separable for some n.*

Proof. Since $\langle x \rangle$ is finite dimensional, there exists a positive integer n such that

$$\langle x \rangle \supset \langle \sigma(x) \rangle \supset \ldots \supset \langle \sigma^n(x) \rangle = \langle \sigma^{n+1}(x) \rangle = \ldots .$$

For such an n, $\sigma^n(x)$ is σ-separable.

Definition 6. Let V, W be unital σ-subalgebras of A. Then V is *σ-separable* if V is the W-span of $\sigma(V)$; that is,

$$V = \{\textstyle\sum_{i=1}^m \sigma(v_i)w_i | m \geqq 1, v_1, \ldots, v_m \in V, w_1, \ldots, W_m \in W\}.$$

We now give necessary and sufficient conditions for a finite dimensional unital σ-algebra A to decompose as $A = A_{\text{sep}} \otimes_k A_{\text{rad}}$ (internal tensor product). The counterpart for fields is [**1**, p. 50].

THEOREM 2. *Let A and σ be unital and suppose that A is finite dimensional and σ injective. Then $A = A_{\text{sep}} \otimes_k A_{\text{rad}}$ (internal tensor product) if and only if A/A_{rad} is σ-separable.*

Proof. Suppose first that $A = A_{\text{sep}} \otimes_k A_{\text{rad}}$. Then since the k-span of $\sigma(A)$ contains A_{sep}, the A_{rad}-span of $\sigma(A)$ contains $A_{\text{sep}} \otimes_k A_{\text{rad}} = A$. (Note here that $A_{\text{rad}} \supset ke$.) Thus, A/A_{rad} is σ-separable. Suppose, conversely, that A/A_{rad} is σ-separable. Let $b_1, \ldots, b_m$ span A over A_{rad}. Take n such that $\sigma^n(b_1), \ldots, \sigma^n(b_m)$ are σ-separable. Now $b_1, \ldots, b_m$ span A over A_{rad}, so that $\sigma^n(b_1), \ldots, \sigma^n(b_m)$ span A over A_{rad} by the σ-separability of A/A_{rad}. Thus, $A \subset A_{\text{sep}}A_{\text{rad}}$. It remains to show that A_{sep} and A_{rad} are linearly disjoint over k. For this, let $a_1, \ldots, a_m$ be linearly independent elements of A_{sep} and suppose that $\sum_{i=1}^n a_i c_i = 0$ where the c_i are in A_{rad}. Choose n such that $\sigma^n(c_i) \in ke$ for all i. Then $\sum_{i=1}^n \sigma^n(c_i)\sigma^n(a_i) = 0$. By the linear independence over k of the $\sigma^n(a_i)$, $\sigma^n(c_i) = 0$ for all i. But σ is injective, so that the c_i are all 0. Thus, A_{sep} and A_{rad} are linearly disjoint over k and $A = A_{\text{sep}} \otimes_k A_{\text{rad}}$.

We conclude with a decomposition theorem which is a form of Fitting's lemma. It's counterpart for Lie p-algebras yields the decomposition of a linear transformation into its semi-simple and nilpotent parts (cf. [**2**, p. 120]).

THEOREM 3. *Let A be finite dimensional and $\bar{\sigma}$ surjective. Then $A = A_{\text{sep}} \oplus A_{\text{nilp}}$ (internal direct sum).*

Proof. For any n, $\sigma^n(A)$ and

$$\text{Kern } \sigma^n = \{x | \sigma^n(x) = 0\}$$

are k-subspaces of A, since $k = \bar{\sigma}^n(k)$. Thus,

$$\sigma(A) \supset \sigma^2(a) \supset \ldots$$

and

$$\text{Kern } \sigma \subset \text{Kern } \sigma^2 \ldots$$

are chains of subspaces of A. Since A is finite dimensional, $\sigma^n(A) = \sigma^{n+1}(A)$ and Kern σ^n = Kern σ^{n+1} for some n. Now $\sigma^n A = A_{\text{sep}}$, by Theorem 1, and Kern $\sigma^n = A_{\text{nilp}}$. Let $x \in A$ and choose $y \in A_{\text{sep}}$ such that $\sigma^n(x) = \sigma^n(y)$. This is possible since

$$A_{\text{sep}} = \sigma^n(A) = \sigma^{2n}(A) = \sigma^n(A_{\text{sep}}).$$

Now $x = y + (x - y)$ with $y \in A_{\text{sep}}$. And $x - y \in A_{\text{nilp}}$ since

$$\sigma^n(x - y) = \sigma^n(x) - \sigma^n(y) = 0.$$

Since $A_{\text{sep}} \cap A_{\text{nilp}} = \{0\}$, it follows that $A = A_{\text{sep}} \oplus A_{\text{nilp}}$.

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