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ON GLOBAL ROUGH SOLUTIONS TO A NON-LINEAR SCHRÖDINGER SYSTEM

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Abstract. The non-linear Schrödinger systems arise from many important physical branches. In this paper, employing the 'I-method', we prove the global-in-time well-posedness for a coupled non-linear Schrödinger system in $H^s(\mathbb{R}^n)$ when $n = 2$, $s > 4/7$ and $n = 3$, $s > 5/6$, respectively, which extends the results of J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao (Almost conservation laws and global rough solutions to a nonlinear Schrödinger equation, *Math Res. Lett.* **9**, 2002, 659–682) to the system.

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1. Introduction. In this paper, we consider the following Cauchy problem of a non-linear Schrödinger system:

$$\begin{cases} i\partial_t u_1 + \Delta u_1 = \mu_1 |u_1|^2 u_1 + \beta |u_2|^2 u_1, & x \in \mathbb{R}^n, \quad t > 0, \\ i\partial_t u_2 + \Delta u_2 = \mu_2 |u_2|^2 u_2 + \beta |u_1|^2 u_2, & x \in \mathbb{R}^n, \quad t > 0, \\ u_1(x, 0) = u_{10}(x) \in H^s(\mathbb{R}^n) \quad \text{and} \quad u_2(x, 0) = u_{20}(x) \in H^s(\mathbb{R}^n). \end{cases} \tag{1}$$

Model (1) has applications in many physical problems, especially in non-linear optics. There have been many articles concerning this model in the literature, and here we cite only [**1**, **4**, **5**, **9**, **10**] as references. Recently, the first author and the third author of this paper have obtained some results about the ground state or blow-up solutions of system (1) (see [**11**–**13**]).

From the physical angle, a very interesting question is whether the global-in-time well-posedness holds true in some sense when the initial data enjoys infinite energy. In the pioneering paper [**6**], Colliander, Staffilani, Tao, and the co-authors obtained the global well-posedness for the single Schrödinger equation in $H^s(\mathbb{R}^n)$. The aim of this paper is to extend their results to the coupled non-linear Schrödinger system, exploring the method given in [**6**].

Let us recall some corresponding classical results (see [**7**, **8**, **11**, **14**, **15**]) about the Schrödinger system in $H^1(\mathbb{R}^n)$. The mass conservation law about (1) reads

$$\|u_1(t)\|_{L^2(\mathbb{R}^n)} + \|u_2(t)\|_{L^2(\mathbb{R}^n)} \equiv \|u_{10}\|_{L^2(\mathbb{R}^n)} + \|u_{20}\|_{L^2(\mathbb{R}^n)}, \tag{2}$$

and the energy conservation law is

$$\begin{aligned} E(u_1, u_2)(t) = & \int_{\mathbb{R}^n} \frac{1}{2}(|\nabla u_1|^2 + |\nabla u_2|^2)dx \\ & + \frac{1}{4}\int_{\mathbb{R}^n} (\mu_1|u_1|^4 + \mu_2|u_2|^4 + 2\beta|u_1|^2|u_2|^2)dx \\ \equiv & \, E(u_1, u_2)(0). \end{aligned} \tag{3}$$

Throughout this paper, we make the following assumption on μ_1, μ_2 and β that the matrix:

$$\begin{bmatrix} \mu_1 & \min\{\beta, 0\} \\ \min\{\beta, 0\} & \mu_2 \end{bmatrix} \tag{4}$$

is positive definite. To investigate the global well-posedness in $H^s(\mathbb{R}^n)$ instead of $H^1(\mathbb{R}^n)$, we need to introduce a modification of the energy functional which is 'almost conserved'. Given $s < 1$ and a parameter $N \gg 1$, we define the multiplier operator

$$\widehat{I_N f}(\xi) = M_N(\xi)\hat{f}(\xi), \tag{5}$$

where the multiplier $M_N(\xi)$ is smooth, radially symmetric, non-increasing in $|\xi|$ and

$$M_N(\xi) = \begin{cases} 1, & |\xi| \le N, \\ \left(\frac{N}{|\xi|}\right)^{1-s}, & |\xi| \ge 2N. \end{cases} \tag{6}$$

Note that

$$\begin{aligned} E(I_N u_1, I_N u_2)(t) \le N^{2-2s} & \left(\|u_1(\cdot, t)\|^2_{\dot{H}^s(\mathbb{R}^n)} + \|u_2(\cdot, t)\|^2_{\dot{H}^s(\mathbb{R}^n)}\right) \\ & + \mu_1\|u_1(\cdot, t)\|^4_{L^4(\mathbb{R}^n)} + \mu_2\|u_2(\cdot, t)\|^4_{L^4(\mathbb{R}^n)} + 2|\beta|\|u_1(\cdot, t)u_2(\cdot, t)\|^2_{L^2(\mathbb{R}^n)}, \end{aligned} \tag{7}$$

$$\begin{aligned} & \|u_1(\cdot, t)\|^2_{H^s(\mathbb{R}^n)} + \|u_2(\cdot, t)\|^2_{H^s(\mathbb{R}^n)} \\ & \le CE(I_N u_1, I_N u_2)(t) + C\|u_{10}\|^2_{L^2(\mathbb{R}^n)} + C\|u_{20}\|^2_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{8}$$

To prove the global well-posedness, it is equivalent to obtain a bound on the H^s-norm of the solution which grows at most polynomially in t. To be special, we intend to prove

$$E(I_N u_1, I_N u_2)(t) \preceq (1 + t)^L, \tag{9}$$

for a positive constant N depending on t and a positive constant L depending on $\|u_{10}\|_{H^s(\mathbb{R}^n)} + \|u_{20}\|_{H^s(\mathbb{R}^n)}$. The assumption that matrix (4) is positive definite guarantees that estimate (9) implies the same bound on the H^s-norm.

Our main results in this paper are the following two theorems:

THEOREM 1. *Assume that matrix* (4) *is positive definite. Initial value problem* (1) *is globally well-posed from data* $(u_{10}, u_{20}) \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$ *with* $s > 4/7$.

THEOREM 2. *Assume that matrix* (4) *is positive definite. Initial value problem* (1) *is globally well-posed from data* $(u_{10}, u_{20}) \in H^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3)$ *with* $s > 5/6$.

We remark that our assumption (4) is crucial to the globally well-posed solution to the system. Similar condition has been used in our blow-up result and stability result treated in our previous works [**11**] and [**13**]. Generally speaking, system (1) has more phenomenon than the scalar case to discover.

The rest of this paper is organized as follows: In Section 2, we introduce some notations and preliminaries. In Section 3, we will consider (1) and prove Theorem 1. In Section 4, we will deal with (1) and prove Theorem 2.

2. Preliminaries. Let us recall some well-known notations. Denote $\langle A\rangle \equiv (1+A^2)^{\frac{1}{2}}$ and $\langle\nabla^F\rangle$ for the operator with Fourier multiplier $(1+|\xi|^2)^{\frac{1}{2}}$, while the symbol ∇ denotes the spatial gradient. Let $\frac{1}{2}+ \equiv \frac{1}{2}+\varepsilon$, $\frac{1}{2}- \equiv \frac{1}{2}-\varepsilon$ and $\frac{1}{2}-- \equiv \frac{1}{2}-2\varepsilon$ for some universal $0<\varepsilon \ll 1$. Define the weighted Sobolev norms

$$\|\psi\|_{X_{s,b}} \equiv \|\langle\xi\rangle^s\langle\tau-|\xi|^2\rangle^b\widehat{\psi}(\xi,\tau)\|_{L^2(\mathbb{R}^n\times\mathbb{R})}, \tag{10}$$

$$\|\psi\|_{X^\delta_{s,b}} \equiv \inf_{\psi=f \text{ on } [0,\delta]} \|\psi\|_{X^\delta_{s,b}}, \tag{11}$$

$$\|F\|_{L^q_tL^r_x(\mathbb{R}^{n+1})} \equiv \left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}^n}|F(x,t)|^r dx\right)^{\frac{q}{r}} dt\right)^{\frac{1}{q}}. \tag{12}$$

A pair of exponents (q, r) is called *Schrödinger admissible* for R^{n+1} if q and r satisfy

$$\frac{1}{q}+\frac{n}{2r}=\frac{n}{4}, \quad q, r \geq 2, \quad (n, q) \neq (2, 2). \tag{13}$$

For a Schrödinger admissible pair (q, r) and a function $F(x, t)$ on $\mathbb{R}^{n+1}$, we have the $L^q_tL^r_x$ Strichartz estimate (see [**3, 15**])

$$\|F\|_{L^q_tL^r_x(\mathbb{R}^{n+1})} \leq C\|F\|_{X_{0,\frac{1}{2}+}}. \tag{14}$$

We next recall a lemma given by Bourgain [**2**] and Colliander [**6**].

LEMMA 2.1. *Let* $\psi_1, \psi_2 \in X^\delta_{0,\frac{1}{2}+}$ *be supported on spatial frequencies* $|\xi| \sim N_1, N_2$, *respectively. Then for any* $N_1 \leq N_2$

$$\|\psi_1\cdot\psi_2\|_{L^2([0,\delta]\times\mathbb{R}^2)} \leq C\left(\frac{N_1}{N_2}\right)^{\frac{1}{2}}\|\psi_1\|_{X^\delta_{0,\frac{1}{2}+}}\|\psi_2\|_{X^\delta_{0,\frac{1}{2}+}}. \tag{15}$$

holds. Moreover, if we replace the product $\psi_1\cdot\psi_2$ *on the left with either* $\bar{\psi}_1\cdot\psi_2$ *or* $\psi_1\cdot\bar{\psi}_2$, (15) *also holds.*

3. The case of $n = 2$. As argued by Colliander et al. [**6**], to prove Theorem 1, it is sufficient to prove the following proposition:

Proposition 3.1. *Assume that* $4/7 < s$ *and* $u_{10}(x), u_{20}(x) \in C_0^\infty(\mathbb{R}^2)$ *with* $E(Iu_{10}, Iu_{20}) \leq 1$. *Then there exists constant* $\delta = \delta(\|u_{10}\|_{L^2(\mathbb{R}^2)}, \|u_{20}\|_{L^2(\mathbb{R}^2)})$ *such that the solution* (u_1, u_2) *satisfies*

$$u_1(x,t) \in C([0,\delta], H^s(\mathbb{R}^2)), \quad u_2(x,t) \in C([0,\delta], H^s(\mathbb{R}^2))$$

and

$$E(I_N u_1, I_N u_2)(t) = E(I_N u_1, I_N u_2)(0) + O(N^{-\frac{3}{2}+}) \tag{16}$$

for all $t \in [0, \delta]$.

In fact, Proposition 3.1 implies (9) in the following way. By scaling invariance, if (u_1, u_2) is a solution to (1), so does

$$(u_1^{(\lambda)}(x,t), u_2^{(\lambda)}(x,t)) := \left(\frac{1}{\lambda}u_1\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right), \frac{1}{\lambda}u_2\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right)\right). \tag{17}$$

For all $\lambda > 0$, we have the following estimate, which is similar to case of the single Schrödinger equation:

$$\begin{aligned} E(I_N u_{10}^{(\lambda)}, I_N u_{20}^{(\lambda)}) &= \frac{1}{2}\int_{\mathbb{R}^2}(|\nabla I_N u_{10}^{(\lambda)}|^2 + |\nabla I_N u_{20}^{(\lambda)}|^2)dx \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^2}(\mu_1|I_N u_{10}^{(\lambda)}|^4 + \mu_2|I_N u_{20}^{(\lambda)}|^4 + 2\beta|I_N u_{10}^{(\lambda)}|^2|I_N u_{20}^{(\lambda)}|^2)dx \\ &\leq C(\lambda^{-2s}N^{2-2s} + \lambda^{-2})(1 + \|u_{10}\|_{H^s(\mathbb{R}^2)} + \|u_{20}\|_{H^s(\mathbb{R}^2)})^4. \end{aligned} \tag{18}$$

Moreover, we can choose

$$\lambda = \max\left\{\frac{1}{N}, \frac{N^{\frac{1-s}{s}}}{(2C)^{\frac{1}{2s}}}(1 + \|u_{10}\|_{H^s(\mathbb{R}^2)} + \|u_{20}\|_{H^s(\mathbb{R}^2)})^{\frac{2}{s}}\right\} \tag{19}$$

to achieve $E(I_N u_{10}^{(\lambda)}, I_N u_{20}^{(\lambda)}) \leq \frac{1}{2}$. Reapplying Proposition 3.1 at least $C_1 \cdot N^{\frac{3}{2}-}$ times, we arrive at

$$E(I_N u_1^{(\lambda)}, I_N u_2^{(\lambda)})(C_1 \cdot N^{\frac{3}{2}-}\delta) \sim 1.$$

Choosing $N^{\frac{7s-4}{2s}-} \sim T_0$ and re-scaling back to the original energy, we obtain the desired bound

$$E(I_N u_1, I_N u_2)(T) \leq C(1+T)^{\frac{4-4s}{7-4s}+}.$$

For further details, refer to [**6**].

To prove Proposition 3.1, we need the following proposition:

Proposition 3.2. *Assume that* $4/7 < s < 1$ *and we are given the data for problem* (1) *with* $E(Iu_{10}, Iu_{20}) \leq 1$. *Then there exist constants* $\delta = \delta(\|u_{10}\|_{L^2(\mathbb{R}^2)}, \|u_{20}\|_{L^2(\mathbb{R}^2)})$ *and* C *such that the solution* (u, v) *satisfies*

$$\|Iu_1\|_{X^\delta_{1,\frac{1}{2}+}} + \|Iu_2\|_{X^\delta_{1,\frac{1}{2}+}} \ll C. \tag{20}$$

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Proof. First we recall some estimates involving the $X^{\delta}_{s,b}$ spaces and functions $f(x)$, $F(x,t)$

$$\|S(t)f\|_{X^{\delta}_{1,\frac{1}{2}+}} \ll \|f\|_{H^1(\mathbb{R}^n)}, \tag{21}$$

$$\left\| \int_0^t S(t-\tau)F(x,\tau)d\tau \right\|_{X_{1,\frac{1}{2}+}} \ll \|F\|_{X^{\delta}_{1,-\frac{1}{2}}}, \tag{22}$$

$$\|F\|_{X^{\delta}_{1,-b}} \ll \delta^P \|F\|_{X^{\delta}_{1,-\alpha}}, \tag{23}$$

where $0 < \alpha < b < \frac{1}{2}$, $P = \frac{1}{2}(1 - \frac{\alpha}{b}) > 0$.

Duhamel's principle gives us

$$\begin{aligned} \|Iu_1\|_{X^{\delta}_{1,\frac{1}{2}+}} &= \|S(t)(Iu_{10}) + \int_0^t S(t-\tau)(\mu_1 I(u_1\bar{u}_1u_1) + \beta I(u_2\bar{u}_2u_1))\|_{X^{\delta}_{1,\frac{1}{2}+}} \\ &\leq \|Iu_{10}\|_{H^1(\mathbb{R}^2)} + \mu_1\|I(u_1\bar{u}_1u_1)\|_{X^{\delta}_{1,-\frac{1}{2}+}} + |\beta|\|I(u_2\bar{u}_2u_1)\|_{X^{\delta}_{1,-\frac{1}{2}+}}. \\ &\leq \|Iu_{10}\|_{H^1(\mathbb{R}^2)} + \delta^{\varepsilon}\mu_1\|I(u_1\bar{u}_1u_1)\|_{X^{\delta}_{1,-\frac{1}{2}++}} + \delta^{\varepsilon}|\beta|\|I(u_2\bar{u}_2u_1)\|_{X^{\delta}_{1,-\frac{1}{2}++}}. \\ \|Iu_2\|_{X^{\delta}_{1,\frac{1}{2}+}} &\leq \|Iu_{20}\|_{H^1(\mathbb{R}^2)} + \delta^{\varepsilon}\mu_2\|I(u_2\bar{u}_2u_2)\|_{X^{\delta}_{1,-\frac{1}{2}++}} + \delta^{\varepsilon}|\beta|\|I(u_1\bar{u}_1u_2)\|_{X^{\delta}_{1,-\frac{1}{2}++}}. \end{aligned} \tag{24}$$

By the definition of the restricted norm, we have

$$\|Iu_1\|_{X^{\delta}_{1,\frac{1}{2}+}} \leq \|Iu_{10}\|_{H^1(\mathbb{R}^2)} + \delta^{\varepsilon}\mu_1\|I(\psi_1\bar{\psi}_1\psi_1)\|_{X^{\delta}_{1,-\frac{1}{2}++}} + \delta^{\varepsilon}|\beta|\|I(\psi_2\bar{\psi}_2\psi_1)\|_{X^{\delta}_{1,-\frac{1}{2}++}}, \tag{25}$$

$$\|Iu_2\|_{X^{\delta}_{1,\frac{1}{2}+}} \leq \|Iu_{20}\|_{H^1(\mathbb{R}^2)} + \delta^{\varepsilon}\mu_2\|I(\psi_2\bar{\psi}_2\psi_2)\|_{X^{\delta}_{1,-\frac{1}{2}++}} + \delta^{\varepsilon}|\beta|\|I(\psi_1\bar{\psi}_1\psi_2)\|_{X^{\delta}_{1,-\frac{1}{2}++}}, \tag{26}$$

where the function ψ_1 agrees with u_1 and ψ_2 agrees with u_2 for $t \in [0,\delta)$ and

$$\|Iu_1\|_{X^{\delta}_{1,\frac{1}{2}+}} \sim \|I\psi_1\|_{X_{1,\frac{1}{2}+}}, \quad \|Iu_2\|_{X^{\delta}_{1,\frac{1}{2}+}} \sim \|I\psi_2\|_{X_{1,\frac{1}{2}+}}. \tag{27}$$

By the results of **[6]**, we know that

$$\|I(\psi_1\bar{\psi}_1\psi_1)\|_{X_{1,-\frac{1}{2}++}} \ll \|I\psi_1\|^3_{X_{1,\frac{1}{2}+}}, \tag{28}$$

$$\|I(\psi_2\bar{\psi}_2\psi_2)\|_{X_{1,-\frac{1}{2}++}} \ll \|I\psi_2\|^3_{X_{1,\frac{1}{2}+}}. \tag{29}$$

Now we only need to show that

$$\|I(\psi_1\bar{\psi}_1\psi_2)\|_{X_{1,-\frac{1}{2}++}} \ll \|I\psi_1\|^2_{X_{1,\frac{1}{2}+}}\|I\psi_2\|_{X_{1,\frac{1}{2}+}}, \tag{30}$$

$$\|I(\psi_2\bar{\psi}_2\psi_1)\|_{X_{1,-\frac{1}{2}++}} \ll \|I\psi_2\|^2_{X_{1,\frac{1}{2}+}}\|I\psi_1\|_{X_{1,\frac{1}{2}+}}. \tag{31}$$

We only prove (30), the proof of (31) is similar. Using the interpolation lemma of **[7]**, we need to prove that

$$\|\psi_1\bar{\psi}_1\psi_2\|_{X_{s,-\frac{1}{2}++}} \ll \|\psi_1\|^2_{X_{s,\frac{1}{2}+}}\|\psi_2\|_{X_{s,\frac{1}{2}+}} \tag{32}$$

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for all $4/7 < s < 1$. However, by duality and 'Leibniz' rule (32) follows from

$$\left| \int_{\mathbb{R}} \int_{\mathbb{R}^2} (\langle \nabla^F \rangle^s \phi_1) \bar{\phi}_2 \phi_3 \phi_4 \, dx \, dt \right| \ll \|\phi_1\|_{X_{s,\frac{1}{2}+}} \|\phi_2\|_{X_{s,\frac{1}{2}+}} \|\phi_3\|_{X_{s,\frac{1}{2}+}} \|\phi_4\|_{X_{0,\frac{1}{2}--}}. \tag{33}$$

Hence (30) is obtained.

Setting

$$Q_1(\delta) = \|Iu_1\|_{X^\delta_{1,\frac{1}{2}+}}, \quad Q_2(\delta) = \|Iu_2\|_{X^\delta_{1,\frac{1}{2}+}},$$

we have

$$Q_1(\delta) \le \|Iu_{10}\|_{H^1(\mathbb{R}^2)} + \delta^\varepsilon \mu_1 (Q_1(\delta))^3 + \delta^\varepsilon |\beta| (Q_2(\delta))^2 Q_1(\delta),$$
$$Q_2(\delta) \le \|Iu_{20}\|_{H^1(\mathbb{R}^2)} + \delta^\varepsilon \mu_2 (Q_2(\delta))^3 + \delta^\varepsilon |\beta| (Q_1(\delta))^2 Q_2(\delta).$$

Summing them up, we can obtain that

$$Q_1(\delta) + Q_2(\delta) \le \|Iu_{10}\|_{H^1(\mathbb{R}^2)} + \|Iu_{20}\|_{H^1(\mathbb{R}^2)} + C\delta^\varepsilon \left(Q_1(\delta) + Q_2(\delta)\right)^3. \tag{34}$$

We emphasize that the above inequality is the analogue of the inequality (3.23) in **[6]** related to the single Schrödinger equation. The framework of **[6]** yields the desired result in Proposition 3.2. □

Now we give the proof of Proposition 3.1.

Proof of Proposition 3.1. Applying I to equation (1), we can obtain that

$$\begin{aligned} \partial_t E(Iu_1, Iu_2) = Re \int_{\mathbb{R}^2} \{ \overline{Iu_{1t}} \left(\mu_1 |Iu_1|^2 Iu_1 + \beta |Iu_2|^2 Iu_1 - I(\mu_1 |u_1|^2 u_1 + \beta |u_2|^2 u_1) \right) \\ + \overline{Iu_{2t}} \left(\mu_2 |Iu_2|^2 Iu_2 + \beta |Iu_1|^2 Iu_2 - I(\mu_2 |u_2|^2 u_2 + \beta |u_1|^2 u_2) \right) \} dx. \end{aligned} \tag{35}$$

Integrating (35) from 0 to δ, we have

$$\begin{aligned} &E(Iu_1, Iu_2)(\delta) - E(Iu_1, Iu_2)(0) \\ &= \int_0^\delta \int_{\sum_{j=1}^4 \zeta_j = 0} \left(1 - \frac{M(\zeta_2 + \zeta_3 + \zeta_4)}{M(\zeta_2)\, M(\zeta_3)\, M(\zeta_4)} \right) \mu_1 \widehat{\overline{I\partial_t u_1}}(\zeta_1) \widehat{Iu_1}(\zeta_2) \widehat{\overline{Iu_1}}(\zeta_3) \widehat{Iu_1}(\zeta_4) \\ &+ \int_0^\delta \int_{\sum_{j=1}^4 \zeta_j = 0} \left(1 - \frac{M(\zeta_2 + \zeta_3 + \zeta_4)}{M(\zeta_2)\, M(\zeta_3)\, M(\zeta_4)} \right) \beta \widehat{\overline{I\partial_t u_1}}(\zeta_1) \widehat{Iu_2}(\zeta_2) \widehat{\overline{Iu_2}}(\zeta_3) \widehat{Iu_1}(\zeta_4) \\ &+ \int_0^\delta \int_{\sum_{j=1}^4 \eta_j = 0} \left(1 - \frac{M(\eta_2 + \eta_3 + \eta_4)}{M(\eta_2)\, M(\eta_3)\, M(\eta_4)} \right) \mu_2 \widehat{\overline{I\partial_t u_2}}(\eta_1) \widehat{Iu_2}(\eta_2) \widehat{\overline{Iu_2}}(\eta_3) \widehat{Iu_2}(\eta_4) \\ &+ \int_0^\delta \int_{\sum_{j=1}^4 \eta_j = 0} \left(1 - \frac{M(\eta_2 + \eta_3 + \eta_4)}{M(\eta_2)\, M(\eta_3)\, M(\eta_4)} \right) \beta \widehat{\overline{I\partial_t u_2}}(\eta_1) \widehat{Iu_1}(\eta_2) \widehat{\overline{Iu_1}}(\eta_3) \widehat{Iu_2}(\eta_4). \end{aligned} \tag{36}$$

Denote

$$L_1(\zeta) = \left(1 - \frac{M(\zeta_2 + \zeta_3 + \zeta_4)}{M(\zeta_2)\, M(\zeta_3)\, M(\zeta_4)} \right),$$
$$L_2(\eta) = \left(1 - \frac{M(\eta_2 + \eta_3 + \eta_4)}{M(\eta_2)\, M(\eta_3)\, M(\eta_4)} \right).$$

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Using the equations of (1), we substitute for $\partial_t u_1$ and $\partial_t u_2$ in (36) and we will show that

$$\sum_{j=1}^{8} \mathrm{Term}_j \ll N^{-\frac{3}{2}+}, \tag{37}$$

where

$$\mathrm{Term}_1 = \int_0^\delta \int_{\sum_{j=1}^4 \zeta_j=0} \mu_1 L_1(\zeta)\widehat{\overline{\Delta I u_1}}(\zeta_1)\widehat{Iu_1}(\zeta_2)\widehat{\overline{Iu_1}}(\zeta_3)\widehat{Iu_1}(\zeta_4), \tag{38}$$

$$\mathrm{Term}_2 = \int_0^\delta \int_{\sum_{j=1}^4 \zeta_j=0} \mu_1 L_1(\zeta)\overline{\widehat{I(\mu_1|u_1|^2u_1 + \beta|u_2|^2u_1)}}(\zeta_1)\widehat{Iu_1}(\zeta_2)\widehat{\overline{Iu_1}}(\zeta_3)\widehat{Iu_1}(\zeta_4), \tag{39}$$

$$\mathrm{Term}_3 = \int_0^\delta \int_{\sum_{j=1}^4 \zeta_j=0} \beta L_1(\zeta)\widehat{\overline{\Delta I u_1}}(\zeta_1)\widehat{Iu_2}(\zeta_2)\widehat{\overline{Iu_2}}(\zeta_3)\widehat{Iu_1}(\zeta_4), \tag{40}$$

$$\mathrm{Term}_4 = \int_0^\delta \int_{\sum_{j=1}^4 \zeta_j=0} \beta L_1(\zeta)\overline{\widehat{I(\mu_1|u_1|^2u_1 + \beta|u_2|^2u_1)}}(\zeta_1)\widehat{Iu_2}(\zeta_2)\widehat{\overline{Iu_2}}(\zeta_3)\widehat{Iu_1}(\zeta_4), \tag{41}$$

$$\mathrm{Term}_5 = \int_0^\delta \int_{\sum_{j=1}^4 \zeta_j=0} \mu_2 L_2(\eta)\widehat{\overline{\Delta I u_2}}(\eta_1)\widehat{Iu_2}(\eta_2)\widehat{\overline{Iu_2}}(\eta_3)\widehat{Iu_2}(\eta_4), \tag{42}$$

$$\mathrm{Term}_6 = \int_0^\delta \int_{\sum_{j=1}^4 \zeta_j=0} \mu_2 L_2(\eta)\overline{\widehat{I(\mu_2|u_2|^2u_2 + \beta|u_1|^2u_2)}}(\eta_1)\widehat{Iu_2}(\eta_2)\widehat{\overline{Iu_2}}(\eta_3)\widehat{Iu_2}(\eta_4), \tag{43}$$

$$\mathrm{Term}_7 = \int_0^\delta \int_{\sum_{j=1}^4 \zeta_j=0} \beta L_2(\eta)\widehat{\overline{\Delta I u_2}}(\eta_1)\widehat{Iu_1}(\eta_2)\widehat{\overline{Iu_1}}(\eta_3)\widehat{Iu_2}(\eta_4), \tag{44}$$

$$\mathrm{Term}_8 = \int_0^\delta \int_{\sum_{j=1}^4 \zeta_j=0} \beta L_2(\eta)\overline{\widehat{I(\mu_2|u_2|^2u_2 + \beta|u_1|^2u_2)}}(\eta_1)\widehat{Iu_1}(\eta_2)\widehat{\overline{Iu_1}}(\eta_3)\widehat{Iu_2}(\eta_4). \tag{45}$$

By the results of **[6]**, we know that

$$\begin{aligned}&\left|\int_0^\delta \int_{\sum_{j=1}^4 \zeta_j=0} \left(1 - \frac{M(\zeta_2+\zeta_3+\zeta_4)}{M(\zeta_2)\,M(\zeta_3)\,M(\zeta_4)}\right)\widehat{\overline{\phi_1}}(\zeta_1)\widehat{\phi_2}(\zeta_2)\widehat{\overline{\phi_3}}(\zeta_3)\widehat{\phi_4}(\zeta_4)\right| \\ &\leq CN^{-\frac{3}{2}}(N_1N_2N_3N_4)^{0-}\|\phi_1\|_{X_{-1,\frac{1}{2}+}}\|\phi_2\|_{X_{1,\frac{1}{2}+}}\|\phi_3\|_{X_{1,\frac{1}{2}+}}\|\phi_4\|_{X_{1,\frac{1}{2}+}}\end{aligned} \tag{46}$$

for any functions ϕ_j $(j = 1, \ldots, 4)$ with positive spatial Fourier transforms supported on

$$\langle \xi \rangle \sim 2^{k_j} \equiv N_j \tag{47}$$

for some $k_j \in \{0, 1, \ldots\}$.

Letting $\phi_1 = \Delta I u_k$ $(k = 1, 2)$, $\phi_j = Iu_k$ $(j = 2, 3, 4, k = 1, 2)$ in (46), we obtain that

$$\mathrm{Term}_1 + \mathrm{Term}_3 + \mathrm{Term}_5 + \mathrm{Term}_7 \leq CN^{-\frac{3}{2}}. \tag{48}$$

Now we need to estimate Term_2, Term_4, Term_6, Term_8.

First we consider Term_2.

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It is easy to see that $|\mathrm{Term}_2| \le |\mathrm{Term}_{21}| + |\mathrm{Term}_{22}|$ with

$$\mathrm{Term}_{21} = \int_0^\delta \int_{\sum_{j=1}^4 \zeta_j=0} \mu_1 L_1(\zeta) \overline{\widehat{I(\mu_1|u_1|^2u_1)}}(\zeta_1)\widehat{Iu_1}(\zeta_2)\overline{\widehat{Iu_1}}(\zeta_3)\widehat{Iu_1}(\zeta_4),$$

$$\mathrm{Term}_{22} = \int_0^\delta \int_{\sum_{j=1}^4 \zeta_j=0} \mu_1 L_1(\zeta) \overline{\widehat{I(\beta|u_2|^2u_1)}}(\zeta_1)\widehat{Iu_1}(\zeta_2)\overline{\widehat{Iu_1}}(\zeta_3)\widehat{Iu_1}(\zeta_4).$$

By the results of [**6**], we know that

$$\left|\int_0^\delta \int_{\sum_{j=1}^6 \zeta_j=0} L_1(\zeta) P_{N_{123}} \overline{\widehat{I(\phi_1\phi_2\phi_3)}}(\zeta_1+\zeta_2+\zeta_3)\widehat{I\phi_4}(\zeta_4)\overline{\widehat{I(\phi_5)}}(\zeta_5)\widehat{I\phi_6}(\zeta_6)\right|$$
$$\le CN^{-\frac{3}{2}} N_4^{0-} \prod_{j=1}^6 \|I\phi_j\|_{X_{1,\frac{1}{2}+}}, \tag{49}$$

where $0 \le \hat{\phi}_j$ is supported for $|\xi_j| \sim N_j = 2^{k_j}$ and

$$N_4 \ge N_5 \ge N_6 \quad \text{and } N_4 \ge CN, \tag{50}$$

while let $P_{N_{123}}$ be the projection onto functions supported in the N_{123} dyadic spatial frequency shell.

Letting $\phi_j = u_1 (j = 1, 2, \ldots, 6)$ in Term_{21} and $\phi_1 = \phi_2 = u_2$, $\phi_j = u_1$ (j = 3, 4, 5, 6) in Term_{22}, using (49), and proceeding exactly as in [**6**], we finally obtain that

$$|\mathrm{Term}_2| \le CN^{-\frac{3}{2}}. \tag{51}$$

Similarly, we can get the bounds for $\mathrm{Term}_4, \mathrm{Term}_6, \mathrm{Term}_8$. Proposition 3.1 is proved. □

4. The results in $\mathbb{R}^3$. Similar to the proof of Theorem 1, we only need to prove the following proposition:

PROPOSITION 4.1. *Assume that $s > 5/6$, $N \gg 1$, $(u_{10}, u_{20}) \in C_0^\infty(\mathbb{R}^3) \times C_0^\infty(\mathbb{R}^3)$ with $E(I_N u_{10}, I_N u_{20}) \le 1$. Then there exists a universal constant δ such that*

$$(u, v) \in C([0, \delta], H^s(\mathbb{R}^3)) \times C([0, \delta], H^s(\mathbb{R}^3))$$

satisfies

$$E(I_N u_1, I_N u_2)(t) = E(I_N u_1, I_N u_2)(0) + O(N^{-1+}) \tag{52}$$

for all $t \in [0, \delta]$.

Recall too $(u_1^{(\lambda)}(x, t), u_2^{(\lambda)}(x, t))$ as the scaled solution defined in (17). For $n = 3$ and all $\lambda > 0$, we have

$$E(I_N u_{10}^{(\lambda)}, I_N u_{20}^{(\lambda)}) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla I_N u_{10}^{(\lambda)}|^2 + |\nabla I_N u_{20}^{(\lambda)}|^2) dx$$
$$\le C_0 \lambda^{1-2s} N^{2-2s} (1 + \|u_{10}\|_{H^s(\mathbb{R}^3)} + \|u_{20}\|_{H^s(\mathbb{R}^3)})^4. \tag{53}$$

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Moreover, we can choose

$$\lambda = \frac{N^{\frac{2s-2}{1-2s}}}{(2C_0)^{\frac{1}{1-2s}}}(1 + \|u_{10}\|_{H^s(\mathbb{R}^3)} + \|u_{20}\|_{H^s(\mathbb{R}^3)})^{-\frac{4}{1-2s}}\} \tag{54}$$

such that $E(I_N u_{10}^{(\lambda)}, I_N u_{20}^{(\lambda)}) \leq 1/2$. Reapplying Proposition 4.1 at least $C_1 \times N^{1-}$ times, we get

$$E(I_N u_1^{(\lambda)}, I_N u_2^{(\lambda)})(C_1 N^{1-} \times \delta) \sim 1. \tag{55}$$

Choosing $N^{(\frac{5-6s}{1-2s})-} \sim T_0$ and re-scaling back to the original energy, and noticing that in three dimensions

$$E(I_N u_1^{(\lambda)}, I_N u_2^{(\lambda)})(\lambda^2 t) = \frac{1}{\lambda} E(I_N u_1^{(\lambda)}, I_N u_2^{(\lambda)})(t),$$

we can obtain that

$$E(I_N u_1^{(\lambda)}, I_N u_2^{(\lambda)})(T_0) \leq \lambda E(I_N u_1^{(\lambda)}, I_N u_2^{(\lambda)})(\lambda^2 T_0) \leq C T_0^{\frac{1-s+}{3(s-5/6)}}.$$

Hence we can achieve that

$$\|u_1\|_{H^s(\mathbb{R}^3)}(T) + \|u_2\|_{H^s(\mathbb{R}^3)}(T) \leq C(1+T)^{\frac{1-s+}{6(s-5/6)}}. \tag{56}$$

To prove Proposition 4.1, we need the following proposition:

PROPOSITION 4.2. *Assume that* $5/6 < s < 1$ *and* (u_{10}, u_{20}) *satisfies* $E(u_{10}, u_{20}) \leq 1$. *Then there exist a universal constant* $\delta > 0$ *such that*

$$\|\nabla u_1\|_{X^\delta_{0,\frac{1}{2}+}} + \|\nabla u_2\|_{X^\delta_{0,\frac{1}{2}+}} \leq C. \tag{57}$$

Proof. Using Duhamel's principle, we get that

$$\|\nabla I u_1\|_{X^\delta_{1,\frac{1}{2}+}} \leq \|\nabla I u_{10}\|_{H^1(\mathbb{R}^2)} + \delta^\varepsilon \mu_1 \|\nabla I(u_1 \bar{u}_1 u_1)\|_{X^\delta_{1,-\frac{1}{2}++}} + \delta^\varepsilon |\beta| \|\nabla I(u_2 \bar{u}_2 u_1)\|_{X^\delta_{1,-\frac{1}{2}++}}, \tag{58}$$

$$\|\nabla I u_2\|_{X^\delta_{1,\frac{1}{2}+}} \leq \|\nabla I u_{20}\|_{H^1(\mathbb{R}^2)} + \delta^\varepsilon \mu_2 \|\nabla I(u_2 \bar{u}_2 u_2)\|_{X^\delta_{1,-\frac{1}{2}++}} + \delta^\varepsilon |\beta| \|\nabla I(u_1 \bar{u}_1 u_2)\|_{X^\delta_{1,-\frac{1}{2}++}}. \tag{59}$$

By the definition of the restricted norm, we have

$$\|\nabla I u_1\|_{X^\delta_{1,\frac{1}{2}+}} \leq \|\nabla I u_{10}\|_{H^1(\mathbb{R}^2)} + \delta^\varepsilon \mu_1 \|\nabla I(\psi_1 \bar{\psi}_1 \psi_1)\|_{X^\delta_{1,-\frac{1}{2}++}} + \delta^\varepsilon |\beta| \|\nabla I(\psi_2 \bar{\psi}_2 \psi_1)\|_{X^\delta_{1,-\frac{1}{2}++}}, \tag{60}$$

$$\|\nabla I u_2\|_{X^\delta_{1,\frac{1}{2}+}} \leq \|\nabla I u_{20}\|_{H^1(\mathbb{R}^2)} + \delta^\varepsilon \mu_2 \|\nabla I(\psi_2 \bar{\psi}_2 \psi_2)\|_{X^\delta_{1,-\frac{1}{2}++}} + \delta^\varepsilon |\beta| \|\nabla I(\psi_1 \bar{\psi}_1 \psi_2)\|_{X^\delta_{1,-\frac{1}{2}++}}, \tag{61}$$

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where the function ψ_1 agrees with u_1 and ψ_2 agrees with u_2 for $t \in [0, \delta)$ and

$$\|\nabla I u_1\|_{X^\delta_{1,\frac{1}{2}+}} \sim \|\nabla I \psi_1\|_{X_{1,\frac{1}{2}+}}, \quad \|\nabla I u_2\|_{X^\delta_{1,\frac{1}{2}+}} \sim \|\nabla I \psi_2\|_{X_{1,\frac{1}{2}+}}. \tag{62}$$

By the results of [**6**], we know that

$$\|\nabla I(u_1\bar{u}_1u_1)\|_{X^\delta_{0,-\frac{1}{2}++}} \ll \|\nabla I u_1\|^3_{X_{0,\frac{1}{2}+}}, \tag{63}$$

$$\|\nabla I(u_2\bar{u}_2u_2)\|_{X^\delta_{0,-\frac{1}{2}++}} \ll \|\nabla I u_2\|^3_{X_{0,\frac{1}{2}+}}. \tag{64}$$

Now we only need to prove that

$$\|\nabla I(u_1\bar{u}_1u_2)\|_{X^\delta_{0,-\frac{1}{2}++}} \ll \|\nabla I u_1\|^2_{X_{0,\frac{1}{2}+}} \|\nabla I u_2\|_{X_{0,\frac{1}{2}+}}, \tag{65}$$

$$\|\nabla I(u_2\bar{u}_2u_1)\|_{X^\delta_{0,-\frac{1}{2}++}} \ll \|\nabla I u_2\|^2_{X_{0,\frac{1}{2}+}} \|\nabla I u_1\|_{X_{0,\frac{1}{2}+}}. \tag{66}$$

By the results of [**6**], and applying a Leibnitz rule for the operator ∇I and duality, one can show that

$$\|(\nabla I)(\phi_1)\cdot\bar{\phi}_2\cdot\phi_3\cdot\psi\|_{L^1(\mathbb{R}^{3+1})} \le C\|\psi\|_{X_{0,\frac{1}{2}--}} \prod_{j=1}^{3} \|\nabla\phi_j\|_{X_{0,\frac{1}{2}+}}. \tag{67}$$

Letting $\phi_j = u_k$ $(j = 1, 2, 3, k = 1, 2)$ in (67) and proceeding as in [**6**], we can obtain (65) and (66).

Setting

$$Q_1(\delta) = \|\nabla I u_1\|_{X^\delta_{1,\frac{1}{2}+}}, \quad Q_2(\delta) = \|\nabla I u_2\|_{X^\delta_{1,\frac{1}{2}+}},$$

we have

$$Q_1(\delta) \le \|\nabla I u_{10}\|_{H^1(\mathbb{R}^2)} + \delta^\varepsilon\mu_1(Q_1(\delta))^3 + \delta^\varepsilon|\beta|(Q_2(\delta))^2 Q_1(\delta),$$
$$Q_2(\delta) \le \|\nabla I u_{20}\|_{H^1(\mathbb{R}^2)} + \delta^\varepsilon\mu_2(Q_2(\delta))^3 + \delta^\varepsilon|\beta|(Q_1(\delta))^2 Q_2(\delta).$$

Summing them up, we obtain

$$Q_1(\delta) + Q_2(\delta) \le \|\nabla I u_{10}\|_{H^1(\mathbb{R}^2)} + \|\nabla I u_{20}\|_{H^1(\mathbb{R}^2)} + C\delta^\varepsilon\left(Q_1(\delta) + Q_2(\delta)\right)^3. \tag{68}$$

And by continuity we obtain

$$\|\nabla I u_1\|_{X^\delta_{1,\frac{1}{2}+}} + \|\nabla I u_2\|_{X^\delta_{1,\frac{1}{2}+}} \le C. \qquad \square$$

Proof of Proposition 4.1. Applying I to equation (1), we can obtain

$$\partial_t E(Iu_1, Iu_2) = Re\int_{\mathbb{R}^3} \Big\{\overline{Iu_{1t}}\left(\mu_1|Iu_1|^2 Iu_1 + \beta|Iu_2|^2 Iu_1 - I(\mu_1|u_1|^2u_1 + \beta|u_2|^2u_1)\right)$$
$$+ \overline{Iu_{2t}}\left(\mu_2|Iu_2|^2 Iu_2 + \beta|Iu_1|^2 Iu_2 - I(\mu_2|u_2|^2u_2 + \beta|u_1|^2u_2)\right)\Big\}\,dx. \tag{69}$$

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Integrating (69) from 0 to δ, we have

$$\begin{aligned}
&E(Iu_1, Iu_2)(\delta) - E(Iu_1, Iu_2)(0)\\
&\quad = \int_0^\delta \int_{\sum_{j=1}^4 \zeta_j=0} \left(1 - \frac{M(\zeta_2+\zeta_3+\zeta_4)}{M(\zeta_2)\,M(\zeta_3)\,M(\zeta_4)}\right) \mu_1 \widehat{\overline{I\partial_t u_1}}(\zeta_1)\widehat{Iu_1}(\zeta_2)\widehat{\overline{Iu_1}}(\zeta_3)\widehat{Iu_1}(\zeta_4)\\
&\qquad + \int_0^\delta \int_{\sum_{j=1}^4 \zeta_j=0} \left(1 - \frac{M(\zeta_2+\zeta_3+\zeta_4)}{M(\zeta_2)\,M(\zeta_3)\,M(\zeta_4)}\right) \beta \widehat{\overline{I\partial_t u_1}}(\zeta_1)\widehat{Iu_2}(\zeta_2)\widehat{\overline{Iu_2}}(\zeta_3)\widehat{Iu_1}(\zeta_4)\\
&\qquad + \int_0^\delta \int_{\sum_{j=1}^4 \eta_j=0} \left(1 - \frac{M(\eta_2+\eta_3+\eta_4)}{M(\eta_2)\,M(\eta_3)\,M(\eta_4)}\right) \widehat{\overline{I\partial_t u_2}}(\eta_1)\widehat{Iu_2}(\eta_2)\widehat{\overline{Iu_2}}(\eta_3)\widehat{Iu_2}(\eta_4)\\
&\qquad + \int_0^\delta \int_{\sum_{j=1}^4 \eta_j=0} \left(1 - \frac{M(\eta_2+\eta_3+\eta_4)}{M(\eta_2)\,M(\eta_3)\,M(\eta_4)}\right) \beta \widehat{\overline{I\partial_t u_2}}(\eta_1)\widehat{Iu_1}(\eta_2)\widehat{\overline{Iu_1}}(\eta_3)\widehat{Iu_2}(\eta_4). \quad (70)
\end{aligned}$$

Denote

$$\begin{aligned}
L_1(\zeta) &= \left(1 - \frac{M(\zeta_2+\zeta_3+\zeta_4)}{M(\zeta_2)\,M(\zeta_3)\,M(\zeta_4)}\right),\\
L_2(\eta) &= \left(1 - \frac{M(\eta_2+\eta_3+\eta_4)}{M(\eta_2)\,M(\eta_3)\,M(\eta_4)}\right).
\end{aligned}$$

Using the equations given in (1), we substitute for $\partial_t u_1$ and $\partial_t u_2$ in (70) and we will show that

$$\sum_{j=1}^{8} \mathrm{Term}_j \ll N^{-1++}, \tag{71}$$

where

$$\mathrm{Term}_1 = \int_0^\delta \int_{\sum_{j=1}^4 \zeta_j=0} \mu_1 L_1(\zeta)\widehat{\overline{\Delta I u_1}}(\zeta_1)\widehat{Iu_1}(\zeta_2)\widehat{\overline{Iu_1}}(\zeta_3)\widehat{Iu_1}(\zeta_4), \tag{72}$$

$$\mathrm{Term}_2 = \int_0^\delta \int_{\sum_{j=1}^4 \zeta_j=0} \mu_1 L_1(\zeta)\overline{\widehat{I(\mu_1|u_1|^2u_1 + \beta|u_2|^2u_1)}}(\zeta_1)\widehat{Iu_1}(\zeta_2)\widehat{\overline{Iu_1}}(\zeta_3)\widehat{Iu_1}(\zeta_4), \tag{73}$$

$$\mathrm{Term}_3 = \int_0^\delta \int_{\sum_{j=1}^4 \zeta_j=0} \beta L_1(\zeta)\widehat{\overline{\Delta I u_1}}(\zeta_1)\widehat{Iu_2}(\zeta_2)\widehat{\overline{Iu_2}}(\zeta_3)\widehat{Iu_1}(\zeta_4), \tag{74}$$

$$\mathrm{Term}_4 = \int_0^\delta \int_{\sum_{j=1}^4 \zeta_j=0} \beta L_1(\zeta)\overline{\widehat{I(\mu_1|u_1|^2u_1 + \beta|u_2|^2u_1)}}(\zeta_1)\widehat{Iu_2}(\zeta_2)\widehat{\overline{Iu_2}}(\zeta_3)\widehat{Iu_1}(\zeta_4), \tag{75}$$

$$\mathrm{Term}_5 = \int_0^\delta \int_{\sum_{j=1}^4 \zeta_j=0} \mu_2 L_2(\eta)\widehat{\overline{\Delta I u_2}}(\eta_1)\widehat{Iu_2}(\eta_2)\widehat{\overline{Iu_2}}(\eta_3)\widehat{Iu_2}(\eta_4), \tag{76}$$

$$\mathrm{Term}_6 = \int_0^\delta \int_{\sum_{j=1}^4 \zeta_j=0} \mu_2 L_2(\eta)\overline{\widehat{I(\mu_2|u_2|^2u_2 + \beta|u_1|^2u_2)}}(\eta_1)\widehat{Iu_2}(\eta_2)\widehat{\overline{Iu_2}}(\eta_3)\widehat{Iu_2}(\eta_4), \tag{77}$$

$$\mathrm{Term}_7 = \int_0^\delta \int_{\sum_{j=1}^4 \zeta_j=0} \beta L_2(\eta)\widehat{\overline{\Delta I u_2}}(\eta_1)\widehat{Iu_1}(\eta_2)\widehat{\overline{Iu_1}}(\eta_3)\widehat{Iu_2}(\eta_4), \tag{78}$$

$$\mathrm{Term}_8 = \int_0^\delta \int_{\sum_{j=1}^4 \zeta_j=0} \beta L_2(\eta)\overline{\widehat{I(\mu_2|u_2|^2u_2 + \beta|u_1|^2u_2)}}(\eta_1)\widehat{Iu_1}(\eta_2)\widehat{\overline{Iu_1}}(\eta_3)\widehat{Iu_2}(\eta_4). \tag{79}$$

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By the results of [**6**], we know that

$$\left| \int_0^\delta \int_{\sum_{j=1}^4 \zeta_j = 0} \left(1 - \frac{M(\zeta_2+\zeta_3+\zeta_4)}{M(\zeta_2)\,M(\zeta_3)\,M(\zeta_4)}\right) \widehat{\overline{\phi_1}}(\zeta_1)\widehat{\phi_2}(\zeta_2)\widehat{\overline{\phi_3}}(\zeta_3)\widehat{\phi_4}(\zeta_4) \right|$$

$$\leq CN^{-1}C(N_1,N_2,N_3,N_4)\|\phi_1\|_{X_{-1,\frac{1}{2}+}} \prod_{j=2}^4 \|\nabla\phi_j\|_{X_{0,\frac{1}{2}+}} \tag{80}$$

for sufficiently small $C(N_1,N_2,N_3,N_4)$ and any functions $\phi_j\,(j=1,\ldots,4)$ with positive spatial Fourier transforms supported on

$$\langle\xi\rangle \sim 2^{k_j} \equiv N_j \tag{81}$$

for some $k_j \in \{0,1,\ldots\}$.

Letting $\phi_1 = \Delta Iu_k\,(k=1,2)$, $\phi_j = Iu_k\,(j=2,3,4,k=1,2)$ in (80), we obtain

$$\mathrm{Term}_1 + \mathrm{Term}_3 + \mathrm{Term}_5 + \mathrm{Term}_7 \leq CN^{-1}. \tag{82}$$

Now we need to estimate $\mathrm{Term}_2,\ \mathrm{Term}_4,\ \mathrm{Term}_6,\ \mathrm{Term}_8$.

First we consider Term_2. It is easy to see that $|\mathrm{Term}_2| \leq |\mathrm{Term}_{21}| + |\mathrm{Term}_{22}|$ with

$$\mathrm{Term}_{21} = \int_0^\delta \int_{\sum_{j=1}^4 \zeta_j=0} \mu_1 L_1(\zeta)\widehat{\overline{I(\mu_1|u_1|^2u_1)}}(\zeta_1)\widehat{Iu_1}(\zeta_2)\widehat{\overline{Iu_1}}(\zeta_3)\widehat{Iu_1}(\zeta_4),$$

$$\mathrm{Term}_{22} = \int_0^\delta \int_{\sum_{j=1}^4 \zeta_j=0} \mu_1 L_1(\zeta)\widehat{\overline{I(\beta|u_2|^2u_1)}}(\zeta_1)\widehat{Iu_1}(\zeta_2)\widehat{\overline{Iu_1}}(\zeta_3)\widehat{Iu_1}(\zeta_4).$$

By the results of [**6**], we know that

$$\left| \int_0^\delta \int_{\sum_{j=1}^6 \zeta_j=0} L_1(\zeta)P_{N_{123}}\widehat{\overline{I(\phi_1\phi_2\phi_3)}}(\zeta_1+\zeta_2+\zeta_3)\widehat{I\phi_4}(\zeta_4)\widehat{\overline{I(\phi_5)}}(\zeta_5)\widehat{I\phi_6}(\zeta_6) \right|$$

$$\leq CN^{-\frac{3}{2}}N_4^{0-}\prod_{j=1}^6 \|I\phi_j\|_{X_{1,\frac{1}{2}+}}, \tag{83}$$

where $0 \leq \hat{\phi}_j$ is supported for $|\xi_j| \sim N_j = 2^{k_j}$ and

$$N_4 \geq N_5 \geq N_6 \quad \text{and } N_4 \geq CN, \tag{84}$$

while let $P_{N_{123}}$ be the projection onto functions supported in the N_{123} dyadic spatial frequency shell.

Letting $\phi_j = u_1\,(j=1,2,\ldots,6)$ in Term_{21} and $\phi_1=\phi_2=u_2$, $\phi_j=u_1$ (j = 3,4,5,6) in Term_{22}, using (83) and proceeding as in [**6**], we can obtain

$$|\mathrm{Term}_2| \leq CN^{-1}. \tag{85}$$

Similarly, we can get the bounds for $\mathrm{Term}_4, \mathrm{Term}_6, \mathrm{Term}_8$. Proposition 2 is proved. □

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