Can. J. Math. Vol. **49** (3), 1997 pp. 568–582

A COUNTEREXAMPLE IN L^p APPROXIMATION BY HARMONIC FUNCTIONS

JOAN MATEU

ABSTRACT. For $\frac{n}{n-2} \leq p < \infty$ we show that the conditions $C_{2,q}(G \setminus \mathring{X}) = C_{2,q}(G \setminus X)$ for all open sets G, $C_{2,q}$ denoting Bessel capacity, are not sufficient to characterize the compact sets X with the property that each function harmonic on $\mathring{X}$ and in $L^p(X)$ is the limit in the L^p norm of a sequence of functions which are harmonic on neighbourhoods of X.

1. **Introduction.** Let $X \subset \mathbb{R}^n$ be compact and let $L^p(X)$, $1 < p < \infty$, be the usual Lebesgue space with respect to n-dimensional Lebesgue measure. Set

$$h^p(X) = \{f \in L^p(X) : f \text{ is harmonic on } \mathring{X}\}$$

and denote by $H^p(X)$ the closure in $L^p(X)$ of the set (of restrictions to X) of functions that are harmonic on some neighbourhood of X. Clearly $H^p(X) \subset h^p(X)$ for any X. Many authors have considered the problem of characterizing those X such that

$$H^p(X) = h^p(X), \tag{Ap}$$

which we call the problem of L^p approximation by harmonic functions.

Polking [P, Theorem 2.6, p. 1237] considered and solved the special case of nowhere dense sets X using the Bessel capacity $C_{2,q}$, q being the dual exponent of p. See Section 2 below for the definition of Bessel capacities $C_{\alpha,q}$. In particular Polking [P, Theorem 1.1, p. 1233 and Theorem 2.7, p. 1238] showed that for a general compact set X the condition

$$C_{2,q}(G \setminus \mathring{X}) = C_{2,q}(G \setminus X), \quad \text{for each open set } G, \tag{1}$$

is necessary for (Ap).

In the other direction, Hedberg [H3, Theorem 6.4, p. 76] pointed out the relevant role played by spectral synthesis for Sobolev spaces in the problem of L^p approximation by harmonic functions. Concretely, he showed that once one knows that all closed sets in $\mathbb{R}^n$ admit $(2, q)$ spectral synthesis then condition (1) and

$$C_{1,q}(G \setminus \mathring{X}) = C_{1,q}(G \setminus X), \quad \text{for each open set } G, \tag{2}$$

Received by the editors June 18, 1996.
AMS subject classification: Primary: 41A30; Secondary: 31C15.

are sufficient for (Ap). Some years later it was shown, using deep results on nonlinear potential theory, that all closed sets admit spectral synthesis [H4, Theorem 1.1, p. 237; HW, Theorem 5, p. 166].

It is worth mentioning that Bagby [Ba, Theorem 2.1, p. 764], adapting the constructive methods of Vitushkin [G, Chapter 8] to the L^p case, was able to give necessary and sufficient conditions for (Ap) in terms of a family of polynomials capacities. However, some of these capacities are very difficult to handle and, in particular, they cannot be described in terms of more familiar quantities such as Bessel capacities.

It is not difficult to show, considering an appropriate Swiss cheese [G, p. 25] that (2) does not follow from (Ap).

We can now ask whether (1) alone is sufficient for (Ap). In this respect, given p, $p \geq \frac{n}{n-1}$, Hedberg [H3 Example 6.6, p. 77; AH, Theorem 11.5.5, p. 321] constructed a compact set X in $\mathbb{R}^n$ such that $H^p(X) \neq h^p(X)$. If $p < \frac{n}{n-2}$, then condition (1) holds because $C_{2,q}$ is trivial in that range, and therefore we have an example in which (Ap) does not follow from (1). When $p \geq \frac{n}{n-2}$, is not known whether condition (1) is satisfied for that set. Hence, it is not known if (Ap) follows from (1) for $p \geq \frac{n}{n-2}$.

The main goal of this paper is to present a construction, for $p \geq \frac{n}{n-2}$, different from that of Hedberg, of a compact X satisfying (1) but not (Ap).

THEOREM 1. *Given p, $\frac{n}{n-2} \leq p < \infty$, there exists a compact $X \subset \mathbb{R}^n$ such that $H^p(X) \neq h^p(X)$ and (1) holds.*

Our construction of X will give easily

$$C_{2,q}\big(B(x,r) \setminus \overset{\circ}{X}\big) \leq C\, C_{2,q}\big(B(x,r) \setminus X\big) \tag{3}$$

for all open balls $B(x,r)$ with center $x \in \partial X$ and radius $r \leq r_0$. Here r_0 is a small positive number and C some constant independent of x and r. In the process of showing (1) from (3) we obtain the following result, which seems to be of independent interest.

THEOREM 2. *Let $X \subset \mathbb{R}^n$ be compact and $\frac{n}{n-2} \leq p < \infty$. The following are equivalent.*

(i) $M^p(X) = H^p(X)$.

(ii) $C_{2,q}(B \setminus \overset{\circ}{X}) \leq C\, C_{2,q}(B \setminus X)$ *for all open balls* $B(x,r)$, $x \in \partial X$, $r \leq r_0$.

(iii) $C_{2,q}(G \setminus \overset{\circ}{X}) = C_{2,q}(G \setminus X)$ *for all open sets* G.

Here we have denoted by $M^p(X)$ the closure in $L^p(X)$ of the linear span of the set of functions of the type $f = |x|^{-(n-2)} * \mu$, where μ is a positive measure supported on $\mathbb{R}^n \setminus \overset{\circ}{X}$ and $f \in L^p(R^n)$. For $p = \frac{n}{n-2}$ the definition of $M^p(X)$ must be modified replacing "$f \in L^p(\mathbb{R}^n)$" by "$f \in L^p(\Omega)$" where Ω is any ball such that $\operatorname{dist}(\mathbb{R}^n \setminus \Omega, X) \geq 1$. This is due to the fact that $|x|^{-(n-2)} * \mu$ is not in $L^{n/n-2}$ at ∞ (unless $\mu \equiv 0$).

The statement of the above theorems can be extended to the case of a general homogeneous elliptic operator L with constant coefficients. Let $h^p(X, L) =$

$\{f : Lf = 0 \text{ on } \overset{\circ}{X}\} \cap L^p(X)$ and let $H^p(X, L)$ be the closure in $L^p(X)$ of the set of restrictions to X of functions f satisfying $Lf = 0$ on some neighbourhood of X. Polking [P, Theorem 2.7, p. 1238] has shown that a necessary condition for $h^p = H^p$ is

$$C_{r,q}(G \setminus \overset{\circ}{X}) = C_{r,q}(G \setminus X), \qquad \text{for all open sets } G, \tag{4}$$

where r denotes the order of L. In the other direction, Hedberg and Wolff [H3, Theorem 6.4, p. 76; HW, Theorem 5, p. 166] proved that the conditions

$$C_{j,q}(G \setminus \overset{\circ}{X}) = C_{j,q}(G \setminus X), \qquad \text{for all open sets } G,\ 1 \le j \le r,$$

are sufficient for $H^p(X, L) = h^p(X, L)$. For $\frac{n}{n-1} \le p < \frac{n}{n-r}$ is known that $H^p(X, L) = h^p(X, L)$ does not follow from (4) [H3, Example 6.6, p. 77]. We are able to prove the following extension of Theorem 1.

THEOREM 3. *Let L be a homogeneous elliptic operator with constant coefficients. Given p, $\frac{n}{n-r} \le p < \infty$, there exists a compact $X \subset \mathbb{R}^n$ such that $H^p(X, L) \neq h^p(X, L)$ and (4) holds.*

The construction of the compact set in the statement of Theorems 1 and 3 is based on a combination of previous examples due to O'Farrell and Hedberg [O, Section 20, p. 203; H3, Example 6.6, p. 77]. Our idea turns out to be useful also in Hölder approximation by solutions of elliptic operators (see [MNOV]).

In Section 2 we collect some background information, definitions and auxiliary results. The proof of Theorem 1 can be found in Section 3. In Section 4 we will show the result of approximation in the space $M^p(X)$, Theorem 2. Finally, Section 5 is devoted to the case of more general elliptic operators and contains the proof of Theorem 3.

2. **Preliminary results.**

2.1. *Capacities.* Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and α be a positive real number. Following Hedberg [H2] we define the (α, p)-Riesz capacity of a subset A of $\mathbb{R}^n$ as

$$\dot{R}_{\alpha,p}(A) = \inf\{\|f\|_p^p : f \in L^p, \quad f \ge 0 \quad \text{and } I_\alpha * f \ge 1 \text{ on } A\},$$

where $I_\alpha(x) = \frac{1}{\gamma(\alpha)} \frac{1}{|x|^{n-\alpha}}$ is the Riesz potential and $\gamma(\alpha) = \pi^{n/2} 2^\alpha \Gamma(\alpha/2) / \Gamma(\frac{n}{2} - \frac{\alpha}{2})$.

For an arbitrary Borel set $A \subset \mathbb{R}^n$ we also define a capacity $R_{\alpha,p}(A)$ by

$$R_{\alpha,p}(A) = \sup_\mu \mu(A)$$

where the supremum is taken over all positive measures supported on A such that $\|I_\alpha * \mu\|_q \le 1$.

As Meyers noted [Me], for a Borel set A one has

$$\big(\dot{R}_{\alpha,p}(A)\big)^{1/p} = R_{\alpha,p}(A).$$

Let G_α be the Bessel kernel of order α. G_α is most conveniently defined in terms of its Fourier transform by $\widehat{G_\alpha}(\xi) = (1+|\xi|^2)^{-\alpha/2}$. If $A \subset \mathbb{R}^n$, the Bessel (α, p) capacity is defined by

$$\dot{C}_{\alpha,p}(A) = \inf\{\|f\|_p^p : f \in L^p, \quad f \geq 0 \quad \text{and } G_\alpha * f \geq 1 \text{ on } A\}$$

and

$$C_{\alpha,p}(A) = \sup\{\mu(A) : \mu \geq 0, \quad \operatorname{spt}\mu \subset A \quad \text{and } \|G_\alpha * \mu\|_q \leq 1\}.$$

As a consequence of the mini-max theorem one obtains that for all Borel sets $A \subset \mathbb{R}^n$

$$C_{\alpha,p}(A) = \left(\dot{C}_{\alpha,p}(A)\right)^{1/p}.$$

The relation between Bessel and Riesz capacity is given by the following:

PROPOSITION A [*e.g.*, AH, PROPOSITION 5.1.4, P. 131]. *Let $\alpha p < n$. Then the following holds.*

(a) For all $A \subset \mathbb{R}^n$

$$R_{\alpha,p}(A) \leq C_{\alpha,p}(A).$$

(b) For each $R > 0$ there is $C(R)$ such that

$$C_{\alpha,p}(A) \leq C R_{\alpha,p}(A)$$

for all $A \subset \mathbb{R}^n$ with diameter at most R.

Some of the set functions introduced above might vanish identically. More precisely, $R_{\alpha,p}(A) = 0$ if $\alpha p \geq n$. To circumvent this undesirable situation we will introduce capacities with respect to a region G.

DEFINITION. *If G is an open bounded set in $\mathbb{R}^n$ and $A \subset G$ a Borel set such that $\bar{A} \subset G$, define*

$$R_{\alpha,p}(A, G) = \sup\{\mu(A) : \mu \geq 0, \quad \operatorname{spt}\mu \subset A \quad \text{and } \|I_\alpha * \mu\|_{q,G} \leq 1\}.$$

We then have

$$K_1 R_{\alpha,p}(A, G) \leq C_{\alpha,p}(A) \leq K_2 R_{\alpha,p}(A, G).$$

A property which holds for all points outside a set A with $C_{\alpha,p}(A) = 0$ is said to hold (α, p)-quasi everywhere or (α, p)-q.e.

2.2. *Quasicontinuity and Kellogg property.* Let the function f be defined $C_{\alpha,p}$-quasi everywhere on $\mathbb{R}^n$ or on some open set. Then f is said to be $C_{\alpha,p}$-quasi continuous, if for every $\varepsilon > 0$ there is an open set G such that $C_{\alpha,p}(G) < \varepsilon$ and $f|_{G^c}$ is continuous in G^c.

It is well known that if $f \in W^{\alpha,p}(\mathbb{R}^n)$, the Sobolev space of functions whose distribution derivatives up to order α are functions in L^p, then f can be represented as $f = G_\alpha * g$,

$g \in L^p$. So, if $g \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, then the potential $G_\alpha * g$, $\alpha > 0$ is (α, p)-quasicontinuous.

In classical potential theory and in non-linear potential theory there are several equivalent definitions of thin sets, see [Br,H1]. We adopt here the following one as suitable for our purposes.

DEFINITION. Let $A \subset \mathbb{R}^n$ and let $1 < p \leq n/\alpha$. Then A is (α, p)-thin at a point $a \in \mathbb{R}^n$ if

$$\int_0^1 \left(\frac{C_{\alpha,p}\big(A \cap B(a,r)\big)}{r^{n-\alpha p}} \right)^{q-1} \frac{dr}{r} < \infty,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

If A is not (α, p)-thin at a it is said to be (α, p)-thick there. The set of points where A is (α, p)-thin is denoted by $e_{\alpha,p}(A)$.

There is a known an useful result which says that given a set $A \subset \mathbb{R}^n$, $C_{\alpha,p}$ quasi-every $x \in A$, x is (α, p)-thick in A. More precisely, one has the following

THEOREM (KELLOGG PROPERTY). *Let $1 < p \leq n/\alpha$ and let $A \subset \mathbb{R}^n$. Then* $C_{\alpha,p}\big(e_{\alpha,p}(A) \cap A\big) = 0$.

In potential theory it is convenient to define the (α, p) fine topology associated to the concept of (α, p)-thinness.

DEFINITION. A function f that is defined on a set F is finely continuous at a point $x \in F$ if the set $\{y \in F;\ |f(y) - f(x)| \geq \varepsilon\}$ is thin at x for all $\varepsilon > 0$.

In the proof of Lemma 2 in Section 4 we will use the following.

PROPOSITION B. *An (α, p)-quasicontinuous function is (α, p)-finely continuous (α, p)-quasi everywhere.*

In [AH, Chapter 6] one can find more information on the continuity of Bessel and Riesz potentials of L^p functions.

2.3. *Hausdorff content.* A measure function is a non-decreasing function $h(t)$, $t > 0$, such that $\lim_{t \to 0} h(t) = 0$. The Hausdorff content Λ^h_∞ related to a measure function h is defined for $A \subset \mathbb{R}^n$ by

$$\Lambda^h_\infty(A) = \inf \sum_i h(\rho_i),$$

where the infimum is taken over all countable coverings of A by open balls $B(x_i, \rho_i)$. When $h(t) = t^\alpha$, $\alpha > 0$, $\Lambda^h_\infty(A) = \Lambda^\alpha_\infty(A)$ is called the α-dimensional Hausdorff content of A.

3. **Proof of Theorem 1.** First we will consider the non integer case. Let $1 < q < \infty$ be a non integer such that $n-2q \geq 0$, $d = [n-q]$ and $s = (n-q)-d$. Then $0 < s < 1$. Let ε be a positive real number satisfying $0 < s+\varepsilon < 1$, and let E_i, $1 \leq i \leq n$, be the linear manifold of dimension i in $\mathbb{R}^n$ given by $E_i = \{(x_1,\dots,x_n) : x_j = 0 \text{ if } j > i\}$. We claim that there exists a compact set Γ satisfying the following conditions:

(a) $\Gamma \subset E_{d+1} \subset E_{n-1}$, with $\Lambda_\infty^{d+s+\varepsilon}(\Gamma) > 0$ and $\Gamma \subset B(0,1/2)$.

(b) There exists a family of balls $B_{jk} = B(a_{jk},\delta_j)$, $a_{jk} \in E_{d+1}$, almost disjoint, such that $S_j = \cup_k B_{jk}$ and $S_j \longmapsto \Gamma$ in the Hausdorff metric.

(c) If $n-2q > 0$, there is a constant C such that for all $x \in \Gamma$ and $\frac{1}{2} > r > 0$ one has $C_{2,q}\big(\cup_j S_j \cap B(x,r)\big) \geq C r^{n-2q}$. (If $n-2q = 0$, the right hand side of the last inequality must be replaced by $C\left(\log(\frac{1}{r})\right)^{1/(1-p)}$).

(d) $\sum\limits_k \delta_j^{d+s} < 2^{-j}$.

(e) Set $\tilde{S}_j = \cup_k B(a_{jk},2\delta_j)$. Then $\tilde{S}_j \cap \tilde{S}_{j'} = \emptyset$ if $j \neq j'$ and $\tilde{S}_j \subset B(0,3/4)$.

Set $X = \overline{B(0,1)} \backslash \cup_j S_j$. First, we will show that condition (1) is fulfilled. By Theorem 2 it is enough to prove that $C_{2,q}(B \setminus \overset{\circ}{X}) \leq C\, C_{2,q}(B \setminus X)$ for each ball B with center $x \in \partial X$ and radius $r \leq 1/2$. This inequality is satisfied if $x \in \Gamma$ as a consequence of (c), and in the other case, namely $x \in \partial X \backslash \Gamma$, because $x \in \partial S_j \cup \partial B(0,1)$ and these are regular sets (finite union of balls) satisfying the cone condition. So, $C_{2,q}(B \setminus \overset{\circ}{X}) \leq C\, C_{2,q}(B \setminus X)$.

Now, we will prove that the equality (Ap) is not satisfied.

Take $\varphi_{jk} \in C_0^\infty\big(B(a_{jk},2\delta_j)\big)$ such that $\sum_k \varphi_{jk} = 1$ in S_j and $|\nabla^l \varphi_{jk}| \leq \frac{c}{\delta_j^l}$, $l = 0,1,2$. Set $\varphi_0 \in C_0^\infty\big(B(0,1)\big)$, satisfying $\varphi_0 \equiv 1$ on $B(0,3/4)$.

Put $\varphi = x_n\varphi_0 - x_n \sum\limits_{j,k} \varphi_{jk}$. Then φ vanishes on $\cup_j S_j$ and $\big(B(0,1)\big)^c$, and so the support of φ is a subset of X. Moreover, $\varphi = x_n$ in $B(0,3/4) \backslash \cup_j \tilde{S}_j$. Consequently the function $\partial\varphi/\partial x_n$ satisfies

$$\partial\varphi/\partial x_n \equiv 1 \text{ on } \Gamma. \tag{5}$$

We will see that the distribution $T = \Delta\varphi$ belongs to $L^q(X)$. Clearly the support of T is a subset of X.

On the other hand, using the definition of φ_{jk}, it is easy to check that $|\Delta(x_n\varphi_{jk})| \leq |\partial_n\varphi_{jk}| + |x_n|\,|\Delta(\varphi_{jk})| \leq \frac{c}{\delta_j}$. Thus, $\|\Delta(x_n\varphi_{jk})\|_q^q \leq \frac{c}{\delta_j^q}\delta_j^n = c\delta_j^{d+s}$. Consequently,

$$\begin{aligned}
\|T\|_q^q &\leq 2^q\Big(\|\Delta(x_n\varphi_0)\|_q^q + \int\Big|\sum_{j,k}\Delta(x_n\varphi_{jk})\Big|^q dx\Big)\\
&\leq C + C\sum_j \int\Big|\sum_k \Delta(x_n\varphi_{jk})\Big|^q dx\\
&\leq C + C\sum_j\sum_k \|\Delta(x_n\varphi_{jk})\|_q^q \leq C + C\sum_j\sum_k \delta_j^{d+s}\\
&\leq C + C\sum_j 2^{-j} \leq C,
\end{aligned}$$

https://doi.org/10.4153/CJM-1997-026-3 Published online by Cambridge University Press

where the second inequality is satisfied because for every j the functions $\sum_k \Delta(x_n\varphi_{jk})$ are supported in $\tilde{S}_j$ and $\tilde{S}_j \cap \tilde{S}_{j'} = \emptyset$ if $j \neq j'$, the third because for a fixed j, the family of balls B_{jk} is almost disjoint, and the next to last because of (d). Hence, T is a function in $L^q(X)$.

We will show that T annihilates $H^p(X)$. Let E be the fundamental solution of the Laplacian and $f \in H^p(X)$. Since $\operatorname{spt}\varphi \cap \operatorname{spt}\Delta f = \emptyset$, $\langle T, f\rangle = \langle T * E, \Delta f\rangle = \langle \varphi, \Delta f\rangle = 0$. The proof will be finished if we can show that T is not orthogonal to $h^p(X)$. By property (a) there exists a compact set Γ satisfying $\Lambda_\infty^{d+s+\varepsilon}(\Gamma) > 0$ and $d+s+\varepsilon > n-q$. Thus, following [AH, Theorem 5.1.13, p. 137] one has that $C_{1,q}(\Gamma) \geq A\Lambda_\infty^{d+s+\varepsilon}(\Gamma) > 0$. It means that there exists a positive measure μ supported on Γ, $\mu(\Gamma) > 0$, satisfying $\|I_1 * \mu\|_p \leq 1$.

Set $g = \partial E/\partial x_n * \mu$. Then $\|g\|_p \leq C\|\mu * I_1\|_p \leq C$ and $\Delta g = \partial\mu/\partial x_n$. Thus $g \in h^p(X)$.

On the other hand, $\langle T, g\rangle = \langle T * E, \Delta g\rangle = \langle \varphi, \partial\mu/\partial x_n\rangle = -\int \partial\varphi/\partial x_n \, d\mu = -\int_\Gamma d\mu = -\mu(\Gamma) \neq 0$, where the forth equality comes from (5). Consequently $g \in h^p(X)\backslash H^p(X)$, and so we get the required result.

When $n-q$ is non integer, the main idea for the construction of Γ comes from [O, Section 20, p. 203]. Following the example of O'Farrell one can build a curve Γ_0 in $\tilde{E}_2 = E_2 \cap \{(x_1,\ldots,x_n) : -\frac{1}{2} \leq x_i \leq \frac{1}{2} \text{ if } i = 1,2\}$, as a limit of polygonals Γ_0^j also supported in $\tilde{E}_2$, satisfying $\Lambda_\infty^{1+s+\varepsilon}(\Gamma_0) > 0$ and $\Lambda_\infty^{1+s}(\Gamma_0^j) = 0$. Take now $\Gamma = \{(x_1,\ldots,x_n) : (x_1,x_2) \in \Gamma_0, -\frac{1}{2} \leq x_j \leq \frac{1}{2} \text{ if } j = 3,\ldots,d+1 \text{ and } x_j = 0 \text{ if } j = d+2,\ldots,n\}$ and $\Gamma^j = \{(x_1,\ldots,x_n) : (x_1,x_2) \in \Gamma_0^j, -\frac{1}{2} \leq x_j \leq \frac{1}{2} \text{ if } j = 3,\ldots,d+1 \text{ and } x_j = 0 \text{ if } j = d+2,\ldots,n\}$.

In fact, Γ and Γ^j are contained in E_{n-1}, because $d+1 \leq n-1$. An easy computation gives that $\Lambda_\infty^{d+s+\varepsilon}(\Gamma) > 0$. Therefore, since Γ^j is a finite union of linear manifolds of dimension d one gets $\Lambda_\infty^{d+s}(\Gamma^j) = 0$.

For each Γ^j one can construct an open set S_j as a finite union of almost disjoint balls $B_{jk} = B(a_{jk},\delta_j)$, $a_{jk} \in \Gamma^j$, such that $\Gamma^j \subset S_j = \cup_k B_{jk}$, $\tilde{S}_j \cap \tilde{S}_{j'} = \emptyset$ if $j \neq j'$, where $\tilde{S}_j = \cup_k B(a_{jk}, 2\delta_j) \subset B(0, 3/4)$, and $\sum_k \delta_j^{d+s} < 2^{-j}$. Clearly, by construction, properties (a), (b), (d) and (e) are satisfied. In order to obtain property (c) we need the following.

LEMMA 1. *Let $1 < q < \infty$ be a real number and $d = [n-q]$. Let $F \subset \mathbb{R}^n$ and B be a ball of center a and radius r. If there exist a positive measure μ_1 in $F \cap B$, and constants c_1 and c_2 such that $\mu_1(F\cap B) \geq c_1 r^d$ and, $\mu_1\big(F \cap \tilde{B}(z,\delta)\big) \leq c_2\delta^d$ for all balls $\tilde{B}(z,\delta)$, then*

(a) If $n-2q > 0$, $C_{2,q}(F\cap B) \geq c_1 r^{n-2q}$.

(b) If $n-2q = 0$, $C_{2,q}(F\cap B) \geq c_1\Big(\log\big(\frac{1}{r}\big)\Big)^{1/(1-p)}$, where $\frac{1}{p}+\frac{1}{q} = 1$.

To obtain (c) applying Lemma 1 we take an open ball $B(x,r)$, $r \leq 1/2$ and consider j_0 (large) such that $\Gamma^{j_0} \cap B(x,\frac{r}{2}) \neq \emptyset$. Since Γ^{j_0} is a finite union of linear manifolds of dimension d, connecting the two components of the boundary of $\{y : \frac{r}{2} < \|y-x\| < r\}$, one has that $\Lambda_\infty^d\big(\Gamma^{j_0} \cap B(x,r)\big) > c_1 r^d$. So, by Frostman Lemma [*e.g.* C, p. 7], there exists a positive measure supported on $\Gamma^{j_0} \cap B(x,r)$ such that $\mu_1\big(\cup S_j \cap B(x,r)\big) \geq c_1 r^d$, and $\mu_1\big(\cup S_j \cap \tilde{B}(z,\delta)\big) \leq c_2\delta^d$ for all balls $\tilde{B}(z,\delta)$. So, by the conclusion of the lemma, property (c) is fulfilled and the compact set Γ satisfies the properties (a)–(e).

PROOF OF LEMMA 1. Let's assume that the hypothesis of the Lemma implies that for every $1 < q' < \frac{n}{n-d}$ $(p' > \frac{n}{d})$ one has

$$C_{(n-d),q'}(F\cap B) \geq R_{(n-d),q'}(F\cap B) \geq C r^{n-(n-d)q'}. \tag{6}$$

Now we will finish the proof of the Lemma using (6). Let q be a real number such that $q \leq \frac{n}{2}$ and consider $\varepsilon > 0$ satisfying $d > n - 2q + \varepsilon$. Set $q' = \frac{2q-\varepsilon}{n-d}$. Then q' satisfies

$$q' = \frac{2q-\varepsilon}{n-d} \leq \frac{n-\varepsilon}{n-d} < \frac{n}{n-d}.$$

On the other hand $q' = \frac{2q-\varepsilon}{n-d} > \frac{n-d}{n-d} = 1$. So, applying (6) and using a known result on comparison of capacities [AH, Theorem 5.5.1, p. 148] one has

$$\text{(a)}\qquad \begin{aligned} C_{2,q}(F\cap B) &\geq C\left(C_{(n-d),q'}(F\cap B)\right)^{n-2q/n-q'(n-d)}\\ &\geq C r^{n-2q} \quad \text{if } n-2q>0.\end{aligned}$$

$$\text{(b)}\qquad \begin{aligned} C_{2,q}(F\cap B) &\geq C\left(\log \frac{A}{C_{(n-d),q'}(F\cap B)}\right)^{1/(1-p')}\\ &\geq C\left(\log\frac{A}{r}\right)^{1/(1-p')} \quad \text{if } n-2q=0.\end{aligned}$$

To complete the proof of the Lemma it is enough to show (6).

Let $\mathcal{M}^+(F\cap B)$ be the set of all positive measures supported on $F\cap B$. We have

$$R_{n-d,q'}(F\cap B) = \sup_{\mu\in\mathcal{M}^+(E\cap B)} \left(\frac{\mu(F\cap B)}{\|I_{n-d}*\mu\|_{p'}}\right)^{q'},$$

where $\frac{1}{p'} + \frac{1}{q'} = 1$.

Using a standard argument, *e.g.* [AH, Corollary 3.6.3, p. 78], one gets the inequality

$$C^{-1}\|M_{n-d}*\mu\|_{p'} \leq \|I_{n-d}*\mu\|_{p'} \leq C\|M_{n-d}*\mu\|_{p'},$$

where

$$(M_{n-d}*\mu)(x) = \sup_{\rho>0} \frac{1}{\rho^{n-(n-d)}} \int_{B(x,\rho)} d\mu(y).$$

We will give an upper bound for $\|M_{n-d}*\mu_1\|_{p'}$, and so we obtain a lower estimate for the capacity $R_{n-d,q'}(F\cap B)$.

Set $C_j = \{x\in\mathbb{R}^n : jr < |x-a| < (j+1)r\}$. For each $x\in C_j$, $j\geq 1$ we have $|(M_{n-d}*\mu_1)(x)| \leq \frac{\mu_1\left(B(a,r)\right)}{(jr)^d} \leq \frac{C}{j^d}$, and for $x\in C_0$, $|(M_{n-d}*\mu_1)(x)| \leq \sup_{\rho>0}\frac{1}{\rho^d}\int_{B(x,\rho)} d\mu_1(y) \leq C$. So,

$$\begin{aligned}\|M_{n-d}*\mu_1\|_{p'}^{p'} &= \sum_{j=0}^{\infty}\int_{C_j}|M_{n-d}*\mu_1(x)|^{p'}\,dm(x) \leq C\int_{B(a,r)} dm(x)\\ &\quad + \sum_{j=1}^{\infty}\frac{C}{j^{dp'}}j^{n-1}r^n \leq Cr^n + Cr^n\sum_{j=0}^{\infty}\frac{1}{j^{dp'-n+1}} \leq Cr^n,\end{aligned}$$

https://doi.org/10.4153/CJM-1997-026-3 Published online by Cambridge University Press

where the last inequality comes from $p' > n/d$. Thus, $\|M_{n-d} * \mu_1\|_{p'} \leq C r^{n/p'}$, and consequently

$$R_{n-d,q'}(F \cap B) \geq C \left(\frac{r^d}{r^{n/p'}}\right)^{q'} \geq C r^{n-q'(n-d)}.$$

Then, the Lemma follows. ∎

When $d = n - q$ is an integer, it is also possible to construct an example using the same ideas. In this case one must show that there exist a set Γ and sets Γ_j, $\Gamma_j \longmapsto \Gamma$, in the Hausdorff metric such that $\Lambda_\infty^{d+\varepsilon}(\Gamma) > 0$ for a given $\varepsilon > 0$, but $\Lambda_\infty^{d}(\Gamma_j) = 0$. One can do it because $d \geq 2$ (since $n - 2q \geq 0$, $d = n - 2q + q \geq 2$).

4. **Harmonic approximation of potentials of measures.**

4.1. *Proof of* $(ii) \Rightarrow (i)$. Let g be a function such that $\Delta g = 0$ in a neighbourhood of X. Regularizing and multiplying by a function with compact support one can consider that $g \in C_0^\infty$ and is harmonic in a neighbourhood of X. Set $(\Delta g)^+ = \max\{\Delta g, 0\}$ and $(\Delta g)^- = \min\{\Delta g, 0\}$. Then, they are positive measures supported on $\mathbb{R}^n \setminus X \subset \mathbb{R}^n \setminus \mathring{X}$. On the other hand $(\Delta g)^+ * I_2$ and $(\Delta g)^- * I_2$ are in L^p, and so $H^p(X) \subset M^p(X)$. Let's note that in the case $p = \frac{n}{n-2}$ the Riesz potentials only need to be locally in L^p.

In order to get the other inclusion, $M^p(X) \subset H^p(X)$, we will show that the functions orthogonal to $H^p(X)$ annihilate $M^p(X)$. Since $\left(L^p(X)\right)^* = L^q(X)$, we take a function $g \in L^q$ such that $\operatorname{spt} g \subset X$ and g is orthogonal to $H^p(X)$. Set $\varphi = I_2 * g$. Thus $\varphi = 0$ on $\mathbb{R}^n \setminus X$, because $I_2(x) = \frac{1}{|x-y|^{n-2}} \in H^p(X)$ if $y \notin X$.

We will see that g annihilates $M^p(X)$. Let μ be a positive measure such that $f = I_2 * \mu$ is a function in $M^p(X)$. Then

$$\langle g, I_2 * \mu \rangle = \langle \varphi, \mu \rangle = \int_{\partial X} \varphi \, d\mu.$$

To finish the proof we need the following result.

LEMMA 2. *Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, such that $n - 2q \geq 0$ and let $\varphi \in W^{2,q}(\mathbb{R}^n)$ such that $\varphi = 0$ on $\mathbb{R}^n \setminus X$ for some compact set $X \subset \mathbb{R}^n$ satisfying*

$$C_{2,q}(B \setminus \mathring{X}) \leq C\, C_{2,q}(B \setminus X) \text{ for all balls } B(x, r),\ x \in \partial X,\ r \leq 1.$$

Then the following holds.

(a) *$\varphi = 0$ on $\mathbb{R}^n \setminus \mathring{X}$, $C_{2,q}$ almost everywhere.*

(b) *If μ is a positive measure satisfying $I_2 * \mu \in M^p(X)$ and $E = \{x \in \partial X;\ \varphi(x) \neq 0\}$, then $\mu(E) = 0$.*

Clearly, by the definition of φ, one can check that $\varphi \in W^{2,q}(\mathbb{R}^n)$, and so, using the above lemma one obtains

$$\int_{\partial X} \varphi \, d\mu = \int_E \varphi \, d\mu = 0.$$

https://doi.org/10.4153/CJM-1997-026-3 Published online by Cambridge University Press

Thus $\langle f, g\rangle = 0$. On the other hand, if $f_j \in M^p(X)$ and $a_j \in \mathbb{R}$, $j = 1, \dots, N$, then $f = \sum_{j=1}^{N} a_j f_j$ satisfies $\langle f, g\rangle = \sum_{j=1}^{N} a_j \langle f_j, g\rangle = 0$. Now, the proof is complete.

PROOF OF LEMMA 2. (a) Without loss of generality φ can be assumed to be $(2, q)$-quasicontinuous (see 2.2). Thus, by Proposition B, we have that the function φ is $(2, q)$-finely continuous, $(2, q)$-almost everywhere.

Let $x_0 \in \partial X$ be a point where φ is finely continuous. This means that for all $\varepsilon > 0$, the set $F_\varepsilon = \{y : |\varphi(x_0) - \varphi(y)| > \varepsilon\}$ is thin. Consequently, by definition,

$$\int_0^1 \left(\frac{C_{2,q}\big(F_\varepsilon \cap B(x_0, r)\big)}{r^{n-2q}} \right)^{p-1} \frac{dr}{r} < \infty. \tag{7}$$

We will show that φ is not finely continuous at $(2, q)$-almost all $x_0 \in \partial X$ satisfying $\varphi(x_0) \neq 0$. So, φ vanishes on ∂X, $C_{2,q}$ almost everywhere.

Set $x_0 \in \partial X$ such that $\varphi(x_0) \neq 0$ and take $0 < \widetilde{\varepsilon_0} = |\varphi(x_0)|$. Then, for all $0 < \varepsilon < \varepsilon_0 = \min(1, \widetilde{\varepsilon_0})$, one has $F_\varepsilon \supset \{y \in X^c : |\varphi(y) - \varphi(x_0)| > \varepsilon\}$. On the other hand one can see easily, that each open ball $B = B(x_0, r)$ satisfies $F_\varepsilon \cap B \supset B \backslash X$.

Thus

$$\int_0^1 \left(\frac{C_{2,q}(F_\varepsilon \cap B)}{r^{n-2q}} \right)^{p-1} \frac{dr}{r} \geq \int_0^{\varepsilon_0} \left(\frac{C_{2,q}(B \backslash X)}{r^{n-2q}} \right)^{p-1} \frac{dr}{r} + \int_{\varepsilon_0}^1 \left(\frac{C_{2,q}(F_\varepsilon \cap B)}{r^{n-2q}} \right)^{p-1} \frac{dr}{r}.$$

By hypothesis

$$\int_0^{\varepsilon_0} \left(\frac{C_{2,q}(B \backslash X)}{r^{n-2q}} \right)^{p-1} \frac{dr}{r} \geq C \int_0^{\varepsilon_0} \left(\frac{C_{2,q}(B \backslash \overset{\circ}{X})}{r^{n-2q}} \right)^{p-1} \frac{dr}{r} = \infty,$$

$C_{2,q}$ almost all $x_0 \in \partial X$, where the last equality comes from Kellogg property [see 2.2]. So, by (7), this means that φ is not finely continuous at $(2, q)$-almost all $x_0 \in \partial X$ satisfying $\varphi(x_0) \neq 0$. Thus, property (a) follows.

(b) Let $p > \frac{n}{n-2}$ and suppose that $\mu(E) > 0$. Then $0 < \|I_2 * \mu_{|E}\|_p < \infty$. By definition of the Riesz-capacity, $R_{2,q}(E) > 0$, and moreover, by Proposition A, $C_{2,q}(E) > 0$. So, we have a contradiction with (a) and property (b) follows. If $p = \frac{n}{n-2}$ the above argument also holds if we replace $\|\cdot\|_p$ by $\|\cdot\|_{p,\Omega}$ and $R_{2,q}(E)$ by $R_{2,q}(E, \Omega)$. ∎

Let $G^p(X)$ be the closure in $L^p(X)$ of the linear span of the set of functions $f \in L^p(\mathbb{R}^n)$, such that $(I - \Delta)f = 0$ is a positive measure supported on $\mathbb{R}^n \setminus \overset{\circ}{X}$ and $H_2^p(X)$ is the closure in $L^p(X)$ of functions f satisfying the elliptic equation $(I - \Delta)f = 0$ in a neighbourhood of X. It is easy to show that the arguments given in this section can be reproduced to obtain that (ii) implies

$$G^p(X) = H_2^p(X). \tag{iv}$$

https://doi.org/10.4153/CJM-1997-026-3 Published online by Cambridge University Press

4.2. *Proof of* (*i*) $\Rightarrow$ (*iii*). First we will show that (i) implies

$$R_{2,q}(G \setminus \overset{\circ}{X}) = R_{2,q}(G \setminus X) \text{ for each open set } G. \tag{v}$$

When $p = \frac{n}{n-2}$ we replace $R_{2,q}(\cdot)$ by $R_{2,q}(\cdot, \Omega)$.

Let's consider the case $p > \frac{n}{n-2}$. To start we will assume that G is an open bounded set. Let f be a function in $M^p(X)$ such that $f = I_2 * \mu$, where μ is a positive measure, satisfying $\|f\|_{p,\mathbb{R}^n} \leq 1$, $\operatorname{spt} \mu \subset G \setminus \overset{\circ}{X}$ and for a fixed $\varepsilon > 0$, $\langle \mu, 1 \rangle \geq R_{2,q}(G \setminus \overset{\circ}{X}) - \varepsilon$. By hypothesis there exists a family of functions g_n, harmonic in a neighbourhood of X, such that $\|g_n - f\|_{p,X} \longmapsto 0$ if $n \longrightarrow \infty$. Replacing any function g_n by a new function h_n defined as $h_n = g_n$ in a neighbourhood of X, U_n, and $h_n = f$ on U_n^c, one has a family of functions h_n, harmonic in a neighbourhood of X, satisfying $\lim_{n\to\infty} \|h_n - f\|_{p,\mathbb{R}^n} = 0$. Thus, regularizing, we can suppose that h_n is in C_0^∞.

Let $\varphi \in C_0^\infty(G)$, such that $\varphi = 1$ in a neighbourhood of the support of μ. For a function $h \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, the Vitushkin operator is defined as $T_\varphi h = \varphi \Delta h * I_2$. It is known [Ba, Lemma 5.3, p. 773] that this operator has the following property of boundedness

$$\|T_\varphi h\|_{p,\mathbb{R}^n} \leq C \|h\|_{p,G}. \tag{8}$$

We have $T_\varphi f = \varphi \Delta f * I_2 = \varphi \mu * I_2 = f$. Put $f_n = T_\varphi h_n$, then, by (8) we get

(a) $\|f - f_n\|_{p,\mathbb{R}^n} = \|T_\varphi(f - h_n)\|_{p,\mathbb{R}^n} \leq C \|f - h_n\|_{p,G}$ and this quantity tends to zero if n tends to infinity.

(b) $\lim_{n\to\infty} \|f_n\|_{p,\mathbb{R}^n} \leq \lim_{n\to\infty} \|f - f_n\|_{p,\mathbb{R}^n} + \|f\|_{p,\mathbb{R}^n} \leq 1$.

Consequently,

$$\lim_{n\to\infty} |\langle \Delta f_n, 1 \rangle| \leq R_{2,q}(G \setminus X). \tag{9}$$

On the other hand, we claim that

$$\lim_{n\to\infty} \langle \Delta f_n, 1 \rangle = \langle \mu, 1 \rangle. \tag{10}$$

So, from (9) and (10) it is easy to finish the proof in the bounded case because $R_{2,q}(G \setminus \overset{\circ}{X}) - \varepsilon \leq \langle \mu, 1 \rangle = \mu(E) = \lim_{n\to\infty} \langle \Delta f_n, 1 \rangle \leq R_{2,q}(G \setminus X)$. Thus, $R_{2,q}(G \setminus \overset{\circ}{X}) = R_{2,q}(G \setminus X)$ for all open bounded sets G.

If G is an open and not bounded set, then there exists a family, $G_N = G \cap B(0, N)$, of open bounded sets such that $G_N \subset G_{N+1}$ and $\bigcup_{N\in\mathbb{N}} G_N = G$. Using a well known property of the capacity, *e.g.* [AH, Theorem 2.3.10 (d), p. 28] and the above case one has

$$R_{2,q}(G \setminus \overset{\circ}{X}) = \lim_{N\to\infty} R_{2,q}(G_N \setminus \overset{\circ}{X}) = \lim_{N\to\infty} R_{2,q}(G_N \setminus X) \leq R_{2,q}(G \setminus X).$$

Consequently, $R_{2,q}(G \setminus \overset{\circ}{X}) = R_{2,q}(G \setminus X)$ for all open sets G.

To finish the proof we must show (10).

Take $\psi \in C_0^\infty(G)$ such that $\psi \equiv 1$ in a neighbourhood of the set $\operatorname{spt} \Delta f_n \cup \operatorname{spt} \Delta f$. Then, if $\frac{1}{p} + \frac{1}{q} = 1$, we can write

$$\begin{aligned} |\lim_{n\to\infty} \langle \Delta f_n - \mu, 1\rangle| &= |\lim_{n\to\infty} \langle \Delta(f_n - f), 1\rangle| \\ &= |\lim_{n\to\infty} \langle \Delta(f_n - f), \psi\rangle| \\ &= |\lim_{n\to\infty} \int_G (f_n - f)\Delta\psi| \\ &\leq \lim_{n\to\infty} \|f_n - f\|_{p,G} \, \|\Delta\psi\|_{q,G} = 0, \end{aligned}$$

where the second equality comes from the definition of ψ and the last from (a). So, claim (10) is proved.

When $p = \frac{n}{n-2}$ the above arguments follow replacing $\| \cdot \|_{p,\mathbb{R}^n}$ by $\| \cdot \|_{p,\Omega}$.

Now, we have shown that (i) implies (v). Clearly, by Proposition A, one has that (v) implies (ii), and, as we have noted at the end of Section 4.1, (ii) implies (iv). To finish we only have to repeat the arguments given in this section to obtain that (iv) implies (iii).

5. **L^p-approximation by solutions of elliptic operators.** The purpose of this section will be to derive the results obtained in Sections 3 and 4 for the case of a homogeneous elliptic operator. The ideas to improve these results are simple variants of the above, and for this reason we will not give all the details of the proofs and we only sketch some of them. Let L be a homogeneous elliptic operator of order r, $r < n$ and $1 < p$, $q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. We define $M^p(X, L)$ as the closure in $L^p(X)$ of the linear span of the set of functions $f \in L^p(\mathbb{R}^n)$ such that Lf is a positive measure supported on $\mathbb{R}^n \setminus \overset{\circ}{X}$. For $H^p(X, L)$, we mean the closure in $L^p(X)$ of functions f such that $Lf = 0$ in some neighbourhood of X. We come now to the approximation theorem for the space $M^p(X, L)$. For this purpose we need to introduce a new capacity. Let E be the fundamental solution of the operator L. For an arbitrary Borel set A

$$C_{L,p}(A) = \sup_{\mu} (A)$$

where the supremum is taken over all positive measures supported on A such that $\|E * \mu\|_q \leq 1$. For $p = \frac{n}{n-r}$ we consider the corresponding version of $M^p(X, L)$ and $C_{L,p}(\cdot)$.

The generalization of Theorem 2 for an elliptic operator is given in the following result.

THEOREM 4. *Let $1 < p$, $q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, L a constant coefficient elliptic homogeneous operator of order r, such that $n - rq \geq 0$, and $X \subset \mathbb{R}^n$ a compact set. The following are equivalent.*

(i) $M^p(X, L) = H^p(X, L)$.

(ii) $C_{r,q}(B \setminus \overset{\circ}{X}) \leq C\, C_{r,q}(B \setminus X)$ for each ball $B(x, r)$, $x \in \partial X$, $r \leq 1/2$.

https://doi.org/10.4153/CJM-1997-026-3 Published online by Cambridge University Press

(iii) $C_{L,q}(G \setminus \overset{\circ}{X}) = C_{L,q}(G \setminus X)$ *for each open set* G.

PROOF. We will start with (iii) $\Rightarrow$ (ii). Since on compact sets Riesz and Bessel capacities are equivalent, in order to obtain this implication it is enough to get the following: there is a constant C such that for a Borel set A

$$C^{-1}R_{r,q}(A) \leq C_{L,q}(A) \leq C R_{r,q}(A). \tag{11}$$

These inequalities can be obtained easily, by introducing a decomposition of E in terms of the Riesz potential of order r and a Calderón-Zygmund operator.

The Fourier transform of the fundamental solution of L has the form

$$\hat{E}(\xi) = \frac{c}{L(\xi)}$$

where $L(\xi)$ is the homogeneous polynomial associated to the operator.

Set

$$m(\xi) = \frac{|\xi|^r}{L(\xi)}$$

and write

$$\hat{E}(\xi) = c\frac{m(\xi)}{|\xi|^r}.$$

Then, m is a homogeneous multiplier of degree 0 and it is infinitely differentiable on the sphere, and so by [S, Theorem 6, p. 75] the operator T defined by $\widehat{(Tf)} = m\hat{f}$ is the sum of a constant times f and the action of a Calderón-Zygmund operator T_1 on f. Consequently, for a positive measure μ such that $E * \mu \in L^p$ we have the equality

$$E * \mu = C(I_r * \mu) + T_1(I_r * \mu). \tag{12}$$

Since $\frac{1}{m}$ is also a homogeneous multiplier of degree 0 and infinitely differentiable on the sphere, we also have that for a positive measure μ, such that $I_r * \mu \in L^p$, the following equality holds

$$(I_r * \mu) = C(E * \mu) + T_2(E * \mu), \tag{13}$$

where T_2 is a Calderón-Zygmund operator.

Now, we can proceed to prove (11). Let μ be a positive measure such that spt $\mu \subset A$ and $\|E * \mu\|_q \leq 1$. By (13) and the invariance of L^p-spaces under Calderón-Zygmund operators we have that $\|I_r * \mu\|_q \leq C$. So,

$$R_{r,q}(A) \geq \frac{1}{C} C_{L,q}(A).$$

If we repeat the same argument with (12) we will show the other inequality:

$$C_{L,q}(A) \geq \frac{1}{A} R_{r,q}(A).$$

Thus, (10) has been obtained and so (iii) $\Rightarrow$ (ii) has been proved.

(i) $\Rightarrow$ (iii) follows using the ideas of (i) $\Rightarrow$ (iii) in Theorem 2. In this case we must consider the Vitushkin operator for a general elliptic equation. For the proof of the boundedness of this operator on L^p spaces the reader can see [Ba, 5.3, p. 733].

To obtain (ii) $\Rightarrow$ (i) one can use the arguments of (ii) $\Rightarrow$ (i) in Theorem 2. The main difference will be in the proof that for a function $g \in L^p$, satisfying that $\operatorname{spt} g \subset X$ and such that g is in the orthogonal of $H^p(X, L)$, one has $\varphi = E * g \in W^{r,p}(\mathbb{R}^n)$. This comes from (12), since for every multi index α, $|\alpha| \leq r$, $\partial^\alpha \varphi = \partial^\alpha I_r * g + T(\partial^\alpha I_r * g)$. So, since g has compact support, it is not difficult to check that $\|\partial^\alpha I_r * g\|_p \leq C\|g\|_p$. ∎

The second part of this section will be devoted to sketch the ideas of the proof of Theorem 3. Actually, we will prove something more than it is stated. We will show that for every integer α_0, $0 < \alpha_0 < r$, there exists a compact set X such that $h^p(X, L) \neq H^p(X, L)$, $C_{L,q}(G \setminus \overset{\circ}{X}) = C_{L,q}(G \setminus X)$ for all open sets G, and for $0 < \alpha < r$, $\alpha \neq \alpha_0$

$$C_{\alpha,q}(B \setminus \overset{\circ}{X}) \leq C\, C_{\alpha,q}(B \setminus X) \text{ for all balls } B(x, r),\ x \in \partial X,\ r \leq 1/2. \tag{14}$$

PROOF OF THEOREM 3. Let α_0 be an integer such that $0 < \alpha_0 < r$ and let $1 < q < \infty$ be a noninteger. Set $d = [n - \alpha_0 q]$. One can build a compact set $X = \overline{B(0,1)} \setminus \cup_j S_j$, where S_j is a union of almost disjoint balls, with the same radius for a fixed j, centered on a union of linear manifolds of dimension d. The inner boundary of X is a compact set Γ, of Hausdorff dimension $n - \alpha_0 q + \varepsilon$, $0 < \varepsilon < 1$, such that $n - \alpha_0 q + \varepsilon < d + 1$. For more details on the construction of Γ and the properties of X see Section 3.

Now, a slight variant of Lemma 1 can be easily proved.

LEMMA 3. *Let $1 < q < \infty$ be a real number, α_0 an integer, and $d = [n - \alpha_0 q]$. Let $F \subset \mathbb{R}^n$ and let B be a ball of center a and radius r. If there exist a positive measure μ_1 in $F \cap B$, and constants c_1 and c_2 such that $\mu_1(A \cap B) \geq c_1 r^d$ and, $\mu_1\big(F \cap \tilde{B}(z, \delta)\big) \leq c_2 \delta^d$ for each ball $\tilde{B}(z, \delta)$, then for any α such that $\alpha > \alpha_0$ the following holds.*

(a) If $n - \alpha q > 0$, then $C_{\alpha,q}(F \cap B) \geq c_1 r^{n - \alpha q}$.

(b) If $n - \alpha q = 0$, then $C_{\alpha,q}(F \cap B) \geq c_1 \big(log(\frac{1}{r})\big)^{\frac{1}{(1-p)}}$*, where* $\frac{1}{p} + \frac{1}{q} = 1$.

Therefore, by the above lemma, one has for every α; $\alpha > \alpha_0$, $C_{\alpha,q}(B \setminus X) \geq r^{n-\alpha q}$ for each ball $B(x, r)$, $x \in \Gamma$, $r \leq 1/2$. Moreover, since S_j has the cone property, (14) is satisfied for $\alpha > \alpha_0$. Take now $\alpha < \alpha_0$, then $C_{\alpha,q}(\Gamma) = 0$, since the Hausdorff dimension of Γ is smaller than $d + 1$. So, (14) is satisfied, because

$$\begin{aligned} C_{\alpha,q}(B \setminus \overset{\circ}{X}) &= C_{\alpha,q}\big((B \setminus X) \cup (B \cap \partial S_i) \cup (B \cap \Gamma)\big) \\ &\leq C_{\alpha,q}\big((B \setminus X) \cup (B \cap \partial S_i)\big) \\ &\leq C\, C_{\alpha,q}(B \setminus X). \end{aligned}$$

To show that $h^p(X, L) \neq H^p(X, L)$ one only needs to slightly modify the arguments in Section 3. ∎

ACKNOWLEDGEMENT. I would like to express my gratitude to J. Orobitg and J. Verdera for their stimulating interest in the problems dealt with in this paper. I am

https://doi.org/10.4153/CJM-1997-026-3 Published online by Cambridge University Press

also grateful for the support given by grant PB96-1183 from DGICYT and by grant 1996SRG00026 from CIRIT.

REFERENCES

[AH] D. R. Adams, L. I. Hedberg, *Function spaces and potential theory*, Springer, Berlin and Heidelberg, 1996.

[Ba] T. Bagby, *Approximation in the mean by solutions of elliptic equations*, Trans. Amer. Math. Soc. **281** (1984), 761–784.

[Br] M. Brelot, *Sur les ensembles effilés*, Bull. Sci. Math. **68**(1944), 12–36.

[C] L. Carleson, *Selected problems on exceptional sets*, Van Nostrand, Princeton, New Jersey, 1967.

[G] T. W. Gamelin, *Uniform Algebras*, Prentice Hall, Englewood Cliff, New Jersey, 1969.

[H1] L. I. Hedberg, *Non-linear potentials and approximation in the mean by analytic functions*, Math. Z. **129**(1972), 299–319.

[H2] ______, *Approximation in the mean by solutions of elliptic equations*, Duke Math. J. **40**(1973), 9–16.

[H3] ______, *Two approximation problems in function spaces*, Ark. Mat. **16**(1978), 51–81.

[H4] ______, *Spectral synthesis in Sobolev spaces, and uniqueness of solutions of the Dirichlet problem*, Acta Math. **147**(1981), 237–264.

[HW] L. I. Hedberg and T. Wolff, *Thin sets in nonlinear potential theory*, Ann. Inst. Fourier (Grenoble) (4)**33**(1983), 161–187.

[MNOV] J. Mateu, Y. Netrusov, J. Orobitg and J. Verdera, *BMO and Lipschitz approximation by solutions of elliptic equations*, Ann. Inst. Fourier (Grenoble) (4)**46**(1996), 1057–1081.

[Me] N. G. Meyers, *A theory of capacities for potentials of functions in Lebesgue classes*, Math. Scand. **26**(1970), 225–292.

[O] A. G. O'Farrell, *Hausdorff content and rational approximation in fractional Lipschitz norms*, Trans. Amer. Math. Soc. **228**(1977), 187–206.

[P] J. C. Polking, *Approximation in L^p by solutions of elliptic partial differential equations*, Amer. J. Math. **94**(1972), 1231–1244.

[S] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, New Jersey, 1970.

Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra (Barcelona)
Spain
e-mail: mateu@manwe.mat.uab.es