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An orthomodular poset which does not admit a normed orthovaluation

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It is of relevance to studies in the logic of quantum mechanics whether or not every separable completely orthomodular poset admits a normed σ-ortho-valuation. A finite orthomodular poset is constructed which is a counter-example to this proposition.

We review some standard notions: An orthocomplemented poset P is an *orthoposet* if $x \vee y$ exists for any orthogonal $x, y \in P$. An orthocomplemented poset P is a *complete orthoposet* if every orthogonal subset of P has a least upper bound in P. An orthoposet $(P, \leq, \perp)$ is *orthomodular* if for any $x, y \in P$, $x = y^{\perp}$ if $x \leq y^{\perp}$ and $x \vee y = 1$. A complete orthoposet is *completely orthomodular* if it is orthomodular. A poset P is *separable* if every orthogonal subset of P is countable (that is, is finite or countably infinite).

Let $(P, \leq, \perp)$ be an orthoposet, then a real-valued function p on P is a *normed orthovaluation* if:

(i) $p(x) \geq 0$ for all $x \in P$;

(ii) $p(1) = 1$, and

(iii) if $x \neq y$ and $x \leq y^{\perp}$ then $p(x \vee y) = p(x) + p(y)$.

It can be shown that a normed orthovaluation maps P into $[0, 1]$, and in general behaves like a probability function.

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Let $(P, \leq, \perp)$ be a separable complete orthoposet, then a normed orthovaluation p on P is a *normed* σ*-orthovaluation* if (as well as (iii)) $p(\vee X) = \sum_{x \in X} p(x)$ for any orthogonal subset X of P (X must be countable since P is separable). The requirement in (iii) that $x \neq y$ is for practical purposes without significance, but is imposed solely so that

(a) the boolean lattice of all subsets of the empty set (a lattice of one element only) behaves itself (as befits its triviality) by admitting a normed orthovaluation, and

(b) the notion of a normed σ-orthovaluation is (as it is supposed to be) a restriction of the notion of a normed orthovaluation.

In this paper we assume familiarity with Section 1 of Finch [1], which is concerned mainly with the notions of a logical structure and of a logical σ-structure. A logical structure is a set of boolean lattices with a common 0-element and a common 1-element, satisfying a number of conditions, among which is that the partial orderings, orthocomplementations, and $\vee$-functions of any two lattices 'coincide' for the elements in their intersection. 'Combining' the boolean lattices in a logical structure produces an orthomodular poset. For the details the original paper should be consulted. If $L = \{B_\gamma : \gamma \in \Gamma\}$ is a logical structure then the partial ordering, the orthocomplementation, and the $\vee$-function of B_γ will be denoted by $\leq_\gamma$, N_γ and $\vee_\gamma$ respectively. Proofs will be terminated by the sign // .

Finch [2] introduces the notion of a state of a physical system associated with a separable logical σ-structure L, and remarks that any normed σ-orthovaluation on the logic L associated with L (L is always a separable completely orthomodular poset) determines a state of the physical system (although not all of its states arise in this way). In the concluding section of his paper, Finch raised four questions, one of which is: Does every separable completely orthomodular poset admit at least one normed σ-orthovaluation? It is the purpose of this paper to provide a negative answer to this question.

We define a set L of seven boolean lattices as follows: Let

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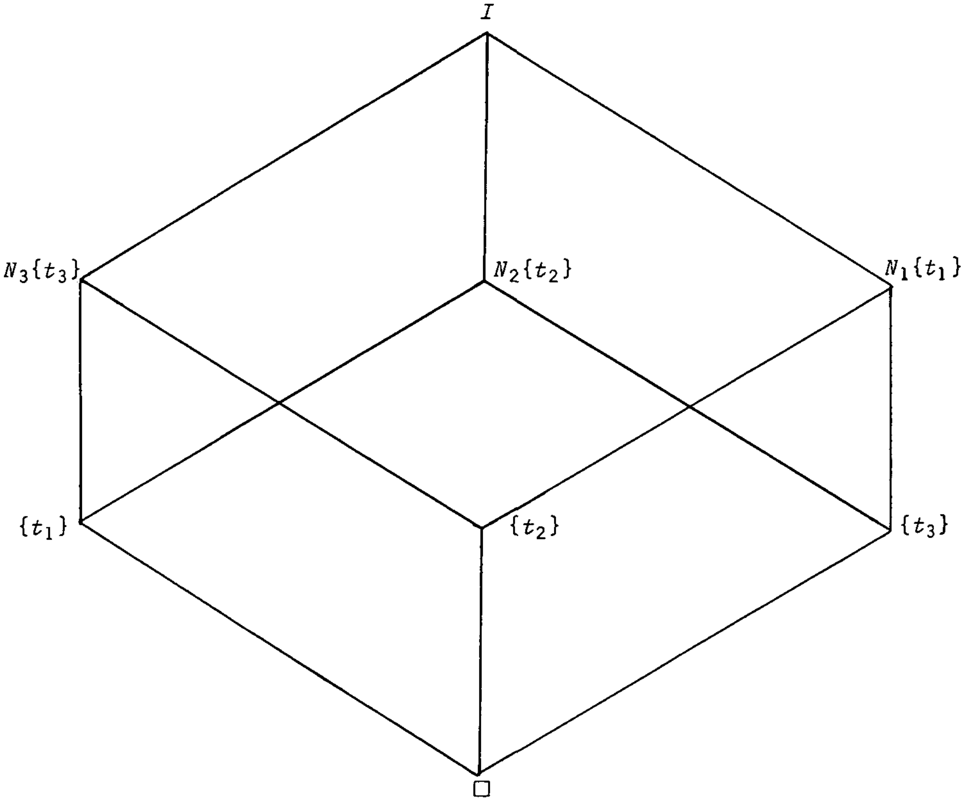
$B_1, B_2, B_3 \in L$ where, for $1 \le i \le 3$, $(B_i, \le_i, N_i)$ is the boolean lattice of all subsets of the set $\{a_i, b_i, c_i, d_i\}$. We identify the 1-elements of each of these lattices, and denote it by I, so that I is the common 1-element of B_1, B_2 and B_3.

Hereinafter i and j will denote arbitrary elements of $\{1, 2, 3\}$, and s and t will denote arbitrary elements of $\{a, b, c, d\}$. The remaining four boolean lattices making up L are defined as follows: Let

$$B_t = \left\{\square, \{t_1\}, \{t_2\}, \{t_3\}, N_1\{t_1\}, N_2\{t_2\}, N_3\{t_3\}, I\right\}$$

(where $\square$ denotes the empty set). The complement of $\{t_i\}$, $N_t\{t_i\}$, is defined to be $N_i\{t_i\}$, and the partial ordering $\le_t$ is defined so as to make $(B_t, \le_t, N_t)$ a boolean lattice with the following structure:



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Thus $L = \{B_1, B_2, B_3, B_a, B_b, B_c, B_d\}$ is a set of boolean lattices. Let $\mathcal{B}_3 = \{B_1, B_2, B_3\}$ and $\mathcal{B}_4 = \{B_a, B_b, B_c, B_d\}$.

LEMMA 1.

(i) If $i \neq j$ *then* $B_i \cap B_j = \{\square, I\}$;

(ii) if $s \neq t$ *then* $B_s \cap B_t = \{\square, I\}$;

(iii) $B_i \cap B_t = \left\{\square, \{t_i\}, N_i\{t_i\}, I\right\}$.

Proof. Apparent. //

LEMMA 2. *L is a logical structure.*

Proof. We must show that L satisfies conditions (i) - (vi) in the definition of a logical structure given on p. 276 of Finch [1]. In this proof, occurrences of small Roman numerals will correspond to the conditions so numbered in the definition. Let $\{B_\gamma : \gamma \in \Gamma\}$ be an enumeration of L .

(i) Each B_γ has the same 0-element, namely, $\square$.

Let $x, y \in B_\alpha \cap B_\beta$. If $\alpha = \beta$ then:

(1) (ii) $x \leq_\alpha y$ if and only if $x \leq_\beta y$;

(2) (iv) $N_\alpha x = N_\beta x$;

(3) (v) $x \vee_\alpha y = x \vee_\beta y$.

Suppose $\alpha \neq \beta$. If $B_\alpha, B_\beta \in \mathcal{B}_3$ or $B_\alpha, B_\beta \in \mathcal{B}_4$ then (by Lemma 1) $x, y \in \{\square, I\}$, so (1) - (3) hold.

If $B_\alpha = B_i$ and $B_\beta = B_t$ then (by Lemma 1) $x, y \in \left\{\square, \{t_i\}, N_i\{t_i\}, I\right\}$, so again (1) - (3) hold.

Similarly if $B_\alpha = B_t$ and $B_\beta = B_i$. Hence for any $x, y \in B_\alpha \cap B_\beta$, (1) - (3) hold.

(iii) Suppose $x \leq_\alpha y$ and $y \leq_\beta z$. We must show that for some

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$\gamma \in \Gamma$, $x \leq_\gamma z$.

If $\alpha = \beta$ then $x \leq_\alpha z$. Suppose $\alpha \neq \beta$. If $B_\alpha, B_\beta \in \mathcal{B}_3$ or $B_\alpha, B_\beta \in \mathcal{B}_4$ then $x, y, z \in \{\Box, I\}$, so $x \leq_\alpha z$. If $B_\alpha = B_i$ and $B_\beta = B_t$ then $y \in B_i \cap B_t = \left\{\Box, \{t_i\}, N_i\{t_i\}, I\right\}$. If $y \in \left\{\Box, \{t_i\}\right\}$ then $x \leq_\beta z$, and if $y \in \left\{N_i\{t_i\}, I\right\}$ then $x \leq_\alpha z$. Similarly if $B_\alpha = B_t$ and $B_\beta = B_i$. Hence for some $\gamma \in \Gamma$, $x \leq_\gamma z$.

(vi) Suppose $y \leq_\alpha N_\alpha x$, $x \leq_\beta z$., and $y \leq_\gamma z$. We must show that for some $\delta \in \Gamma$, $x, y, z \in B_\delta$.

If $\beta = \gamma$ then $x, y, z \in B_\gamma$. Suppose $\beta \neq \gamma$. If $B_\beta, B_\gamma \in \mathcal{B}_3$ or $B_\beta, B_\gamma \in \mathcal{B}_4$ then $z \in \{\Box, I\}$ (by Lemma 1), so $x, y, z \in B_\alpha$.

If $x \in \{\Box, I\}$ then $x, y, z \in B_\gamma$. Suppose $x \notin \{\Box, I\}$. Suppose $B_\beta = B_i$ and $B_\gamma = B_t$ then $z \in \left\{\Box, \{t_i\}, N_i\{t_i\}, I\right\}$ (by Lemma 1). If $z \in \{\Box, I\}$ then $x, y, z \in B_\alpha$; and if $z = \{t_i\}$ then $x \in \left\{\Box, \{t_i\}\right\}$, so $x, y, z \in B_\gamma$.

Suppose $z = N_i\{t_i\}$, then since $B_\gamma = B_t$, either $y \in \left\{\Box, N_i\{t_i\}\right\}$ (in which case $x, y, z \in B_\beta$) or $y \in \left\{\{t_1\}, \{t_2\}, \{t_3\}\right\} \setminus \left\{\{t_i\}\right\}$. Suppose the latter, then $y = \{t_j\}$ for some $j \neq i$, and so $y \in B_j, B_t$ only. Now $x \notin \{\Box, I\}$, $x \in B_\beta = B_i$ and $i \neq j$, so $x \notin B_j$. Since $x, y \in B_\alpha$, $x \in B_j$ or $x \in B_t$. Thus $x \in B_t$, so $x \in \left\{\Box, \{t_i\}, N_i\{t_i\}, I\right\}$ (by Lemma 1), $x \leq_\beta z = N_i\{t_i\}$, and $x \neq \Box$, so $x = N_i\{t_i\} = z$. Thus $x, y, z \in B_\gamma$.

Similarly if $B_\beta = B_t$ and $B_\gamma = B_i$. Hence for some $\delta \in \Gamma$, $x, y, z \in B_\delta$. Since L satisfies the required conditions, L is a logical structure. //

We now define our poset, which will consist of 44 elements. Let $P = B_1 \cup B_2 \cup B_3$, then $P = \bigcup L$. For $x, y \in P$ let $x \leq y$ if and only if for some $\gamma \in \Gamma$, $x \leq_\gamma y$. For $x \in P$ let $x^\perp = N_\gamma x$ for any $\gamma \in \Gamma$ such that $x \in B_\gamma$. In the terminology of Finch [1], $(P, \leq, \perp)$ is the logic associated with the logical structure L.

PROPOSITION 3. *P is an orthomodular poset.*

Proof. By the previous lemma, L is a logical structure. $P = \bigcup\{B_\gamma : \gamma \in \Gamma\}$ so by the remarks on p. 276 of Finch [1], $(P, \leq, \perp)$ is an orthocomplemented poset. By Theorem 1.1 of the same paper, $(P, \leq, \perp)$ is orthomodular. //

Let S be an orthocomplemented poset, then (following Finch [1, p. 280]) a *frame of* S is a maximal orthogonal subset of $S \setminus \{0\}$.

LEMMA 4. *Let $(S, \leq, \perp)$ be an orthoposet, and let $p : S \to [0, 1]$ be a normed orthovaluation on S. Then for any finite frame F of S,* $\sum_{w \in F} p(w) = 1$.

Proof. By induction on $|F|$. Suppose $|F| = 1$, then $F = \{1\}$, so $\sum_{w \in F} p(w) = p(1) = 1$.

Suppose the lemma holds for all n-element frames of S (with $n \geq 1$). Let F be a frame of S such that $|F| = n + 1$. Let $x, y \in F$ such that $x \neq y$. Now x is orthogonal to y and S is an orthoposet, so $x \vee y$ exists in S. Let $G = (F \setminus \{x, y\}) \cup \{x \vee y\}$, then $|G| = n$. It is easily shown that G is a frame of S.

Since p is a normed orthovaluation on S, $p(x \vee y) = p(x) + p(y)$, so

$$\begin{aligned}\sum_{w \in F} p(w) &= \sum_{w \in G} p(w) - p(x \vee y) + p(x) + p(y) \\ &= \sum_{w \in G} p(w) \\ &= 1\end{aligned}$$

by the inductive hypothesis.

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Thus, by induction, the lemma holds for all finite frames of S . //

PROPOSITION 5. *P is an orthomodular poset which does not admit a normed orthovaluation.*

Proof. P is orthomodular by Proposition 3. The atoms of P are the following 12 unit sets:

$$\begin{array}{cccc} \{a_1\} & \{b_1\} & \{c_1\} & \{d_1\} \\ \{a_2\} & \{b_2\} & \{c_2\} & \{d_2\} \\ \{a_3\} & \{b_3\} & \{c_3\} & \{d_3\} . \end{array}$$

Let $F_i = \Big\{\{a_i\}, \{b_i\}, \{c_i\}, \{d_i\}\Big\}$, and let $F_t = \Big\{\{t_1\}, \{t_2\}, \{t_3\}\Big\}$, then clearly each F_i and each F_t is a frame of P .

Suppose now that P admits a normed orthovaluation p . P is an orthoposet, so by the previous lemma,

$$\sum_{w \in F_i} p(w) = 1 \quad \text{and} \quad \sum_{w \in F_t} p(w) = 1 .$$

Now the F_i are pairwise disjoint, as are the F_t , so

$$\sum\{p(w) : w \in F_1 \cup F_2 \cup F_3\} = \sum_{i=1}^{3} \sum_{w \in F_i} p(w) = 3$$

and

$$\sum\{p(w) : w \in F_a \cup F_b \cup F_c \cup F_d\} = \sum_{t=a}^{d} \sum_{w \in F_t} p(w) = 4 .$$

But $F_1 \cup F_2 \cup F_3 = F_a \cup F_b \cup F_c \cup F_d$, so if P admits a normed orthovaluation then $3 = 4$. Hence P does not admit a normed orthovaluation. //

COROLLARY 6. *P is a completely orthomodular poset which does not admit a normed σ-orthovaluation.*

Proof. P is orthomodular by Proposition 3, so P is an orthoposet. Since P is finite, P is a complete orthoposet, and so P is completely orthomodular. Any normed σ-orthovaluation on a complete

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orthoposet is a normed orthovaluation, so by Proposition 5, P does not admit a normed σ-orthovaluation. //

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