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AN UPPER LIMIT PROPERTY OF THE EULER FUNCTION

BY
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If $\phi(n)$ denotes the Euler function, for $n=p$ a prime we have $\phi(n)/n = (1-1/p)$, which implies that

$$\overline{\lim_{n\to\infty}} \frac{\phi(n)}{n} = 1.$$

In this note we consider a refinement of this result. Namely, we prove that

$$\text{(1)} \qquad \begin{aligned} \overline{\lim_{n\to\infty}} \min\left(\frac{\phi(n+1)}{n+1}, \dots, \frac{\phi(n+k)}{n+k}\right) &= \min\left(\frac{\phi(1)}{1}, \dots, \frac{\phi(k)}{k}\right) \\ &= \frac{\phi(P^*(k))}{P^*(k)} \end{aligned}$$

where $P^*(k)$ is the largest integer of the form $\prod_{i=1}^r p_i \le k$ where $p_1 < p_2 < \cdots < p_r$ are the first r primes in ascending order.

Proof of (1). We first note that for each $1 \le i \le k$, the k integers $n+1, \dots, n+k$ consist of at least i consecutive integers and thus i divides $n+j$ for some j, $1 \le j \le k$, which implies

$$\prod_{p\mid n+j}\left(1-\frac{1}{p}\right) \le \prod_{p\mid i}\left(1-\frac{1}{p}\right)$$

or

$$\text{(2)} \qquad \overline{\lim_{n\to\infty}} \min\left(\frac{\phi(n+1)}{n+1}, \dots, \frac{\phi(n+k)}{n+k}\right) \le \min_{1\le i\le k}\left(\frac{\phi(i)}{i}\right)$$

Thus it suffices to prove that given any $\varepsilon > 0$ there exist arbitrarily large n such that for all $i = 1, \dots, k$

$$\text{(3)} \qquad \frac{\phi(n+i)}{n+i} \ge (1-\varepsilon) \min\left(\frac{\phi(1)}{1}, \dots, \frac{\phi(k)}{k}\right).$$

Let $\varepsilon > 0$ be given and choose $n = k!(\prod_{p\le D} p)t$ where D is a large fixed integer to be chosen later and t is a parameter to be chosen once D is fixed.

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Then

$$n+i=k!\Big(\prod_{p\le D} p\Big)t+i=i\Big(\frac{k!}{i}\Big(\prod_{p\le D} p\Big)t+1\Big).$$

Let $n_i(t)=(k!/i)(\prod_{p\le D} p)t+1$, and note that any prime q which divides $n_i(t)$ is greater than D. Also if $D\ge k$ then for all $i=1,\ldots,k$, $(n_i(t),i)=1$, which in turn gives

$$\frac{\phi(n+i)}{n+i}=\frac{\phi(i)}{i}\frac{\phi\Big(\frac{k!}{i}\Big(\prod_{p\le D} p\Big)t+1\Big)}{\frac{k!}{i}\Big(\prod_{p\le D} p\Big)t+1}=\frac{\phi(i)}{i}\frac{\phi(n_i(t))}{n_i(t)}. \tag{4}$$

Thus (3) will follow if arbitrarily large t can be chosen so that for all $i=1,\ldots,k$

$$\frac{\phi(n_i(t))}{n_i(t)}\ge 1-\varepsilon. \tag{5}$$

This is achieved by producing a t for which (q denotes a prime)

$$\sum_{\substack{q\mid n_i(t)\\ q>D}}\frac{1}{q}<\delta. \tag{6}$$

For then

$$\frac{\phi(n_i(t))}{n_i(t)}=\prod_{q\mid n_i(t)}\Big(1-\frac{1}{q}\Big)=\exp\Big\{\sum_{q\mid n_i(t)}\log\Big(1-\frac{1}{q}\Big)\Big\}$$
$$\ge\exp\Big\{-\sum_{q\mid n_i(t)}\frac{1}{q}\Big\}\ge e^{-2\delta}\ge 1-\varepsilon,$$

for large D and δ small.

To find such a t, fix i and consider

$$\sum_{t\le z}\sum_{\substack{q\mid n_i(t)\\ q>D}}\frac{1}{q}. \tag{7}$$

To obtain an upper bound for (7), interchange the order of summation and note that

$$\sum_{\substack{t\le z\\ n_i(t)\equiv 0(\mathrm{mod}\, q)}} 1\le\begin{Bmatrix}\frac{z}{q} & \text{if} & q\le z\\ 1 & \text{if} & q>z\end{Bmatrix}\le\frac{z}{q}+1.$$

Thus

$$\sum_{t\le z}\sum_{\substack{q\mid n_i(t)\\ q>D}}\frac{1}{q}\le\sum_{D<q<z(n_i(t))}\frac{1}{q}\sum_{\substack{t\le z\\ n_i(t)\equiv 0(\mathrm{mod}\, q)}} 1$$
$$\le\sum_{D<q<z(n_i(t))}\frac{z}{q^2}+\sum_{D<q<z(n_i(t))}\frac{1}{q}.$$

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But by the well known result [1],

$$\sum_{p\le x}\frac{1}{p}=\log\log x+c_1+0(1)$$

it follows that

$$(8)\qquad \begin{aligned}\sum_{t\le z}\sum_{\substack{q\mid n_i(t)\\ q>D}}\frac{1}{q}&\le\frac{z}{D}+c\log\log[z(n_i(t))]\\ &\le z\left[\frac{1}{D}+\frac{c\log\log[z(n_i(t))]}{z}\right].\end{aligned}$$

If $M=$ the number of $t\le z$ such that $\sum_{\substack{q\mid n_i(t)\\ q>D}} 1/q>\delta>0$, δ fixed small, it follows from (8) that

$$M\delta\le\sum_{t\le z}\sum_{\substack{q\mid n_i(t)\\ q>D}}\frac{1}{q}$$

or

$$(9)\qquad M\le z\left[\frac{1}{\delta D}+\frac{c\log\log[z(n_i(t))]}{\delta z}\right].$$

Thus if $D>3k/\delta$ (which is clearly $\ge k$), and z is sufficiently large, then from (9), $M\le z(2/3k)$. Since for a given i, the number of $t\le z$ which are exceptions to (6) is $M\le 2z/3k$, then for all i the number of $t\le z$ which are exceptions to (6) is $Mk\le(\frac{2}{3})z$. Thus there is at least one $t\ge z/6$ such that for all $i=1,\ldots,k$, (6) is satisfied, which completes the proof of (3).

Finally we note that as $\phi(i)/i=\prod_{p\mid i}(1-1/p)$ where each factor $(1-1/p)<1$, the minimum of $\phi(i)$, $i=1,\ldots,k$, is achieved for the value of i which has the largest possible number of prime factors, where the primes are as small as possible, namely $P^*(k)$.

Bibliography

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