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ON 3-MANIFOLDS WITH SUFFICIENTLY LARGE DECOMPOSITIONS

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In [**6**] it is shown that two (compact) orientable 3-manifolds which are irreducible, boundary irreducible and sufficiently large are homeomorphic if and only if there exists an isomorphism between the fundamental groups which respects the peripheral structure. In this note we extend this theorem to reducible 3-manifolds.

Any compact 3-manifold M has a decomposition into prime manifolds [**1**; **4**].

$$M \approx M_1 \# M_2 \# \dots \# M_n. \tag{1}$$

Here the connected sum of two bounded manifolds N_1, N_2 is defined by removing 3-balls B_1, B_2 in int N_1, int N_2, respectively, and glueing the resulting boundary spheres together. The M_i's which occur in the decomposition (1) are either irreducible or handles (i.e., a fibre bundle over S^1 with fibre S^2). If (1) contains a fake 3-sphere, we assume it to be M_n.

Definitions. (a) M has *sufficiently large decomposition* if in (1) every M_i is sufficiently large (in the sense of [**6**]) or a handle. Thus, in particular, M contains no fake 3-spheres.

(b) We say that M is boundary irreducible if every component of ∂M is an incompressible surface [**6**].

(c) Let M and N be 3-manifolds. An isomorphism $\psi: \pi_1(M) \to \pi_1(N)$ respects the peripheral structure if for each boundary component F of M, there exists a boundary surface G of N such that $\psi(i_*\pi_1(F)) \subset A$, and A is conjugate in $\pi_1(N)$ to $i_*\pi_1(G)$. (i_* denotes inclusion homomorphisms.)

Note that "ψ respects the peripheral structure" does *not* imply "ψ^{-1} respects the peripheral structure".

THEOREM. *Let M, N be compact, orientable, boundary irreducible 3-manifolds. Suppose that ∂M, ∂N contain no 2-spheres. Suppose that M has sufficiently large decomposition. If there exists an isomorphism $\psi: \pi_1(M) \to \pi_1(N)$ which respects the peripheral structure, then $N \approx M \# H$, where H is a homotopy 3-sphere.*

Remarks. (1) The theorem remains true if "orientable" is replaced by "either both of M, N are orientable or non-orientable. In the latter case, assume that M and N do not contain (2-sided) projective planes and no M_i

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is a (twisted) line bundle over a closed surface". This follows by applying [**2**] instead of [**6**].

The requirement that both of M, N are either orientable or not is necessary. For if M is any orientable, irreducible, sufficiently large 3-manifold and N is the non-orientable handle then $M \# S^1 \times S^2$ is not homeomorphic to $M \# N$, but there exists an isomorphism $\psi: \pi_1(M \# S^1 \times S^2) \to \pi_1(M \# N)$ which respects the peripheral structure.

(2) The requirement that M, N be boundary irreducible is necessary. For let M' be any (orientable), (irreducible), (sufficiently large) 3-manifold and let N be the solid torus ($N \approx B^2 \times S^1$); then $M \# (S^2 \times S^1)$ is not homeomorphic to $M \# N$, but there exists an isomorphism $\psi: \pi_1(M \# S^2 \times S^1) \to \pi_1(M \# N)$ which respects the peripheral structure.

(3) The requirement "∂M, ∂N contain no 2-spheres" can be replaced by "∂M, ∂N contain the same number of 2-sphere components".

To prove the theorem we first note the following easily proved lemma.

LEMMA. *Let G be a group, $G = G_1 * G_2$. Let U_i, V_j be non-trivial subgroups of G_i, G_j respectively* ($1 \leqq i, j \leqq 2$). *If $V_i = xU_jx^{-1}$ for some $x \in G$, then $i = j$ and $x \in G_i$.*

Proof of the theorem. (a) Assume that both of M and N do not contain fake 3-balls. Let $M \approx M_1 \# \ldots \# M_n$ be a decomposition of M into prime manifolds. Thus M_i is either irreducible and sufficiently large or a handle $S^2 \times S^1$ ($i = 1, \ldots, n$). By Kneser's conjecture [**5**], which holds for boundary irreducible manifolds [**3**],

$$\pi_1(M) = A_1 * \ldots * A_n,$$

where $A_i \cong \pi_1(M_i)$ is a decomposition of $\pi_1(M)$ into indecomposable factors (i.e., if $A_i \cong A_{i1} * A_{i2}$, then $A_{i1} = 1$ or $A_{i2} = 1$). Let $B_i = \psi(A_i)$ ($i = 1, \ldots, n$). Thus $\pi_1(N) \cong B_1 * \ldots * B_n$. Again by Kneser's conjecture, there exist prime 3-manifolds N_i such that

$$N \approx N_1 \# \ldots \# N_n, \quad \text{and} \quad \pi_1(N_i) \cong B_i$$

(and if $N_i \approx N_i' \# N_i''$ then $N_i' \approx S^3$ or $N_i'' \approx S^3$, since N does not contain fake 3-balls).

Let F be a component of ∂M. F is an incompressible component of ∂M_i, for some i ($1 \leqq i \leqq n$). Let $i: M_i \to M$ be inclusion. Then $i_*\pi_1(M_i) = wA_iw^{-1}$, for some $w \in \pi_1(M)$. Since ψ respects the peripheral structure there exists a component G of ∂N such that $\psi(i_*\pi_1(F)) \subset A$ and $A = zi_*\pi_1(G)z^{-1}$, for some $z \in \pi_1(N)$. But $G \subset N_j$, for some $j (1 \leqq j \leqq n)$ and $i_*\pi_1(N_j) = vB_jv^{-1}$, for some $v \in \pi_1(N)$. Consider the decompositions

$$\pi_1(M) \approx A_1' * \ldots * A_n', \text{ where } A_k' = wA_kw^{-1} \ (k = 1, \ldots, n),$$

$$\pi_1(N) \approx B_1' * \ldots * B_n', \text{ where } B_k' = vB_kv^{-1} \ (k = 1, \ldots, n).$$

Let T_x be the inner automorphism $y \to x^{-1}yx$ and consider

$$\phi_{ij} = T_w \circ \psi \circ T_{v^{-1}}: A_1' * \ldots * A_n' \to B_1' * \ldots * B_n'.$$

Then (since $i_*\pi_1(F) \subset A_i'$) we have $\phi_{ij}(i_*\pi_1(F)) \subset B_i'$. On the other hand, $\phi_{ij}(i_*\pi_1(F)) =$

$$v\psi(w^{-1})\psi i_*\pi_1(F)\psi(w)v^{-1} \subset v\psi(w^{-1})zi_*\pi_1(G)z^{-1}\psi(w)v^{-1} \subset uB_j'u^{-1},$$

where $u = v\psi(w^{-1})z$. By the lemma we have $i = j$ and $u \in B_i'$. Thus $\phi_{ij}|A_i'$ is an isomorphism of $i_*\pi_1(M_i) \to i_*\pi_1(N_i)$ such that for every component F of ∂M_i there exists a component G of ∂N_i and $\phi_{ij}(i_*\pi_1(F)) \subset B$ and B is conjugate in $i_*\pi_1(N_i)$ to $i_*\pi_1(G)$.

Therefore, for $i = 1, \ldots, n$, there are isomorphisms $\phi_i: \pi_1(M_i) \to \pi_1(N_i)$ which respect the peripheral structure.

If $M_i \approx S^2 \times S^1$ then $N_i \approx S^2 \times S^1$, since otherwise N_i being prime would be irreducible. But the only irreducible 3-manifold having infinite cyclic fundamental group is the solid torus, which is not boundary irreducible.

If M_i is irreducible then (by the same argument) N_i is irreducible.

If N_i is bounded then it is sufficiently large and by Waldhausen's theorem [**6**, Corollary 6.5], $M_i \approx N_i$.

If N_i is closed then $\phi_i^{-1}: \pi_1(N_i) \to \pi_1(M_i)$ respects the peripheral structure and since M_i is sufficiently large it follows again that $M_i \approx N_i$.

(b) By definition M does not contain fake 3-balls. If N contains fake 3-balls then $N \approx N' \# H$, where N' does not contain fake 3-balls and H is a (fake) homotopy 3-sphere. Since any isomorphism $\pi_1(M) \to \pi_1(N)$ which respects the peripheral structure yields an isomorphism $\pi_1(M) \to \pi_1(N')$ which respects the peripheral structure, the proof of the theorem is complete.

References

1. W. Haken, *Ein Verfahren zur Aufspaltung einer 3-Mannigfaltigkeit in irreduzible 3-Mannigfaltigkeiten*, Math. Z. *76* (1961), 427–467.
2. W. H. Heil, *On P^2-irreducible 3-manifolds*, Bull. Amer. Math. Soc. *75* (1969), 772–775.
3. ——— *On Kneser's conjecture for bounded 3-manifolds* (to appear).
4. J. Milnor, *A unique decomposition theorem for 3-manifolds*, Amer. J. Math. *84* (1962), 1–7.
5. J. Stallings, *Grushko's theorem* II. *Kneser's conjecture*, Notices Amer. Math. Soc. *6* (1959), Abstract 559–165, 531–532.
6. F. Waldhausen, *On irreducible 3-manifolds which are sufficiently large*, Ann. Math. *87* (1968), 56–88.

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