ANZIAM J. 51(2009), 261–276
doi:10.1017/S1446181110000088

A TIME-FUEL OPTIMAL CONTROL PROBLEM OF A CRUISE MISSILE BASED ON AN IMPROVED SLIDING MODE VARIABLE STRUCTURE MODEL

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(Received 20 November, 2009; revised 6 February, 2010)

Abstract

The inadequacy of the traditional sliding mode variable structure (SMVS) control method for cruise missiles is addressed. An improved SMVS control method is developed, in which the reaching mode segment of the SMVS control is decomposed into an acceleration accessing segment, a speed keeping segment, and a deceleration buffer segment. A time-fuel optimal control problem is formulated as an optimal control problem involving a switched system with unknown switching times and subject to a continuous state inequality constraint. The new design method is developed based on a control parametrization, a time scaling transform and the constraint transcription method. A sequence of approximate optimal parameter selection problems is obtained with fixed switching time points and a canonical state inequality constraint. Each approximate optimal parameter selection problem can be solved effectively by using existing gradient-based optimization techniques. The convergence of these approximate optimal solutions to the true optimal solution is assured. Simulation results show that the proposed method is highly effective. The response speed of the missile under the control law obtained by the proposed method is improved significantly, while the elevator of the missile is constrained to operate within its permitted range.

2000 *Mathematics subject classification*: primary 49N90; secondary 93B12, 93C95.

Keywords and phrases: cruise missile, optimal control theory, control parametrization, time scaling transform, sliding mode variable structure control.

1. Introduction

A cruise missile is a tactical missile with flight trajectory governed by aerodynamics. Examples are the Tomahawk [5] and the conventional air-launched cruise missiles (CALCMs) [1]. They have been widely used in actual combat [2]. Due to the important

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nature of tactical missiles, many design methods have been investigated and developed in the literature [6, 9, 20, 21].

In flight, the actuator system parameters do not remain constant. Their values will change in response to, for example, the variations in the missile flight altitude, speed and elevator deflection angle. In particular, changes in flight altitude and speed can be huge. There is no known method for predicting such changes. If a traditional control method is used for the design of a controller, each flight state must be taken into account in the design process so that the controller can adjust rapidly enough to avoid performing badly when the values of the parameters are changed. In practice, the desired control law should be robust with respect to uncertainty in the system dynamics. Thus, variable structure control, which is known to be robust against uncertainties in system dynamics, is widely used in missile control design [4, 7, 8, 15, 22]. It is applicable to both linear and nonlinear systems with uncertainties.

With ever increasing complexity in a modern combat environment, the flight trajectory of a missile occurs in a large air space. As the height of the flight trajectory is increased, the air density is decreased. Consequently, a larger elevator deflection is called for. However, the largest allowable deflection angle of an elevator is determined by its mechanical structure, and hence is a hard constraint. Thus, designing a controller for a missile operating in a large air space without violating the allowable limit of the deflection angle is difficult.

In this paper, we propose an improved sliding mode variable structure (SMVS) method for the design of a controller for a cruise missile operating in a large air space with a time-fuel performance index. First, by using a traditional variable structure method, we show that violation of the deflection angle of the actuator from the maximum allowable range cannot be avoided. Then we propose a multiple-stage SMVS method that divides the original system into several subsystems forming a switched system with a continuous inequality constraint. The switching times between these subsystems are to be obtained optimally. Thus, by virtue of the control parametrization technique [16], a time scaling transform method [17] and the constraint transcription method [16], we show that the optimization problem involving this switched system and the continuous inequality constraint can be approximated by a sequence of standard constrained optimal parameter selection problems. Each such problem can be solved by any efficient gradient-based optimization technique. The control parametrization technique detailed in [16] used in conjunction with the time scaling transform introduced in [17] and the constraint transcription method developed in [16] is now known to be an effective approach to developing efficient computational methods for solving various practically significant optimal control problems. See, for example, [3, 10–12, 18, 19]. To deal with the hard constraints on the elevator deflection angle, a linear recursive sliding mode control method is proposed in [14]. However, it does not make use of the time-fuel performance index.

The rest of the paper is organized as follows. Section 2 describes the design of a traditional SMVS controller for a cruise missile and its limitations when the missile is operating in a large air space. Section 3 proposes a design method for improving the design of the SMVS controller. It is formulated as an optimal control problem involving a switched system. In Section 4 a solution procedure is developed for solving this optimal control problem. In Section 5 simulation results are presented, showing the efficiency and effectiveness of the improved variable structure control method proposed.

2. Traditional variable structure control in flight applications

2.1. Difficulties encountered in a large air space To ensure a smooth flight trajectory for a cruise missile in the air space, the axial thrust and resistance balance condition must be satisfied. The thrust, P, generated by the engine is given by

$$P = \tfrac{1}{2} C_{FD} \rho v^2 S, \tag{2.1}$$

where C_{FD} is the thrust coefficient of the ramjet, ρ is the air density under the current flight condition, v is the flight velocity of the missile, and S is the characteristic area of the missile. The mass loss m_c (mass second-flow) is proportional to the engine thrust, satisfying

$$m_c = \frac{P}{I_{RY}}, \tag{2.2}$$

where I_{RY} is the specific impulse. On the one hand, a higher flight speed is desired for a missile. On the other, a missile is required to carry more payload so that a larger flight radius can be realized. Achieving these two tasks is equivalent to requiring a missile to fly farther and faster with less thrust. From (2.1), we see that it is necessary to increase the flight height so that the air density ρ of the flight environment is reduced. Thus, the design of a controller for a cruise missile operating in a large air space should be considered.

The actuator of a missile is the elevator, installed in the tail of the missile body. By deflecting the elevator, an angle between the elevator and the approaching airflow is formed, producing a rotary moment about the centroid. Due to the existence of the angle between the missile body and the approaching airflow, the missile is subject to the normal aerodynamics. This normal force is usually called the normal overload, denoted by n_y. The composite force of all the external forces (except gravity) is referred to as the control force, denoted by F. The ratio of F to the missile mass is referred to as the overload, which is a vector $\boldsymbol{n}$. The normal overload n_y is the component of $\boldsymbol{n}$ along the longitude of the missile body coordinate system. The transfer function of the elevator deflection angle δ_z to n_y is [13]

$$W_{\delta_z}^{n_y}(s) = \frac{(v/g) K_M}{T_M^2 s^2 + 2 T_M \xi_M s + 1}, \tag{2.3}$$

where T_M (seconds) is the time constant of the missile, ξ_M is called the relative damping coefficient of the missile, K_M (proportional to the air density) is the transfer

https://doi.org/10.1017/S1446181110000088 Published online by Cambridge University Press

coefficient with units s^{-1}, v is the flight velocity of the missile, and g is the acceleration due to gravity. In view of (2.3), it follows that during the smooth flight of the missile, the relationship between δ_z and n_y is $n_y = (v/g)K_M\delta_z$. When the flight height is increased, the air density will be decreased. To maintain the same normal overload, a larger elevator deflection angle is needed. However, the maximum permitted deflection angle of the elevator is limited by its mechanical structure. The maximal permitted elevator deflection angle is denoted by $\hat{\delta}_z$. In this paper, the elevator deflection angle is allowed to range between ± 6 degrees, that is, $-6 \leq \delta_z \leq 6$.

2.2. Traditional SMVS control In the flight course of a missile, the aerodynamic parameters of the missile will change in accordance with change in the external environment. Moreover, the data obtained from a wind tunnel test may not be completely accurate. Thus, the control law that we obtain needs to be robust against unexpected disturbance. For this, the variable structure control method has been used in the design of the missile control law.

Let the normal overload deflection Δn_y be denoted by x_1 and let its derivative $\Delta \dot{n}_y$ be denoted by x_2. Define $\boldsymbol{x} = [x_1\ x_2]^{\mathrm{T}}$, which is referred to as the state. Furthermore, let $\Delta\delta_z$ be the control input, denoting the elevator deflection angle. From the transfer function between $\Delta\delta_z$ and n_y given by (2.3), we obtain

$$\begin{cases} \dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu \\ y = C\boldsymbol{x}, \end{cases} \tag{2.4}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ \dfrac{-1}{T_M^2} & \dfrac{-2\xi_M}{T_M} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \dfrac{vK_M}{gT_M^2} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{2.5}$$

For System (2.4), we choose a hyperplane

$$S(\boldsymbol{x}) = G\boldsymbol{x} = 0$$

where $G = [C_s\ 1]$ and C_s is referred to as the sliding mode parameter. Clearly, for an arbitrary state $\bar{\boldsymbol{x}} = [\bar{x}_1\ \bar{x}_2]^{\mathrm{T}}$ on the hyperplane, if $C_s > 0$, then the state starting from $\bar{x}_1$ and moving on the hyperplane will converge to the origin of the state space. On this basis, we only need to find a control $\bar{u}$ such that the condition $\dot{S}(\boldsymbol{x}) = 0$ is satisfied for the state $\boldsymbol{x}(t)$ starting from any state $\bar{\boldsymbol{x}}$ on the surface $S(\boldsymbol{x}) = 0$ under the control $\bar{u}$. To obtain such a $\bar{u}$, we recall the definition of $S(\boldsymbol{x}) = G\boldsymbol{x} = 0$ and make use of (2.4) to obtain $\dot{S}(\boldsymbol{x}) = GA\boldsymbol{x} + GB\bar{u} = 0$, where $S(\boldsymbol{x}) = 0$. Since

$$GB = \begin{bmatrix} C_s & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \dfrac{vK_M}{gT_M^2} \end{bmatrix} = \frac{vK_M}{gT_M^2} \neq 0, \tag{2.6}$$

GB is invertible. Consequently, we obtain $\bar{u} = -(GB)^{-1}GA\boldsymbol{x}$, where $S(\boldsymbol{x}) = 0$.

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Consider the case when the state of the system is outside the sliding surface, that is, $S(\boldsymbol{x}) \neq 0$. In this case, we choose

$$\dot{S}(\boldsymbol{x}) = -C_r \operatorname{sgn}(S(\boldsymbol{x})), \tag{2.7}$$

where the reaching parameter $C_r > 0$, and

$$\operatorname{sgn}(S(\boldsymbol{x})) = \begin{cases} 1 & \text{if } S(\boldsymbol{x}) > 0, \\ 0 & \text{if } S(\boldsymbol{x}) = 0, \\ -1 & \text{if } S(\boldsymbol{x}) < 0. \end{cases}$$

Therefore, given a state $\tilde{\boldsymbol{x}}$ outside the sliding surface, that is, $S(\tilde{\boldsymbol{x}}) \neq 0$, the state starting from $\tilde{\boldsymbol{x}}$ will approach the sliding surface $S(\boldsymbol{x}) = 0$ at a rate C_r.

On the basis of what has been discussed, we know that (2.7) is a sufficient condition for sliding mode convergence. It is necessary to find a control $\tilde{u}$ such that (2.7) is satisfied. Suppose that (2.7) holds. From the fact that $S(\boldsymbol{x}) = G\boldsymbol{x}$,

$$\dot{S}(\boldsymbol{x}) = G(A\boldsymbol{x} + B\tilde{u}) = -C_r \operatorname{sgn}(S(\boldsymbol{x})) \quad \text{where } S(\boldsymbol{x}) \neq 0. \tag{2.8}$$

Thus,

$$GB\tilde{u} = -GA\boldsymbol{x} - C_r \operatorname{sgn}(S(\boldsymbol{x})) \quad \text{where } S(\boldsymbol{x}) \neq 0. \tag{2.9}$$

Since GB is invertible, by (2.9) we obtain

$$\tilde{u} = -(GB)^{-1}GA\boldsymbol{x} - (GB)^{-1}C_r \operatorname{sgn}(S(\boldsymbol{x})) \quad \text{where } S(\boldsymbol{x}) \neq 0. \tag{2.10}$$

This implies that during the reaching phase when the state is moving outside the sliding surface, the control law given by (2.10) will ensure that the sufficient condition (2.7) is satisfied. To conclude, the control law for the whole state space is given by

$$\begin{cases} \bar{u} = -(GB)^{-1}GA\boldsymbol{x}, & S(\boldsymbol{x}) = 0 \\ \tilde{u} = -(GB)^{-1}GA\boldsymbol{x} - (GB)^{-1}C_r \operatorname{sgn}(S(\boldsymbol{x})), & S(\boldsymbol{x}) \neq 0 \end{cases} \tag{2.11}$$

and the control law can be condensed to

$$u = -(GB)^{-1}GA\boldsymbol{x} - (GB)^{-1}C_r \operatorname{sgn}(S(\boldsymbol{x})). \tag{2.12}$$

Since A and B are given, and $G = [C_s \ 1]$, it follows from (2.12) that the control law is specified by C_s and C_r.

During flight under the sliding mode control, the missile movement can be classified into two motion phases: the sliding mode phase when the state is moving on the sliding surface; and the reaching phase when the state is moving outside the sliding surface. The convergence rate analysis for both mode phases is given in [14].

THEOREM 2.1 ([14]). *In the sliding mode phase when the state is moving on the sliding surface, the convergence rate is determined by the sliding parameter C_s.*

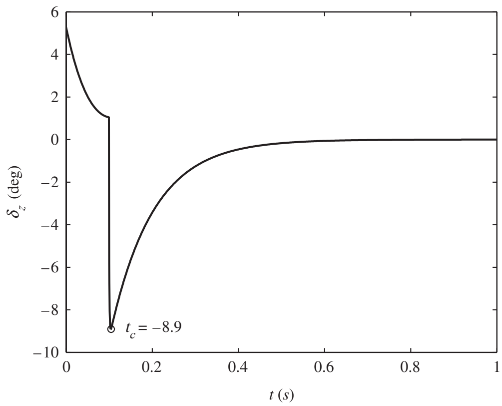

FIGURE 1. The elevator deflection angle obtained by the traditional variable structure method.

In the reaching mode phase when the state is moving outside the sliding surface, with unit overload imposed such that

$$S(\boldsymbol{x}_0) = [C_s\ 1]\begin{bmatrix}1\\0\end{bmatrix} = C_s > 0,$$

the time taken to reach the sliding surface is equal to the ratio C_s/C_r.

From Theorem 2.1, we see that the response speed is directly related to the sliding mode parameter C_s. That is, if the value of C_s is larger, then the system response is quicker. On the other hand, during the reaching mode phase if C_s is increased, the reaching parameter C_r will also need to be increased such that the system state is able to enter into the sliding surface quickly. However, by virtue of (2.12), we note that it is difficult to eliminate chattering as C_r is increased. Thus, C_s should be appropriately chosen.

2.3. Actuator deflection angle We now apply the SMVS law given in Section 2.2 to the cruise missile. The trajectory of the elevator deflection angle obtained with this control law is given in Figure 1, which shows that the maximal deflection angle of the elevator at t_c is -8.91 degrees. This exceeds the maximum permitted elevator deflection angle. Thus, the control law of Section 2.2 cannot be used in practice.

It is known that the maximal deflection angle of the elevator occurs at the cut-in time t_c when the system enters the sliding mode phase from the reaching mode phase, defined by $t_c = \min\{t \mid S(\boldsymbol{x}(t)) = 0\}$. The state at the cut-in time, $\boldsymbol{x}(t_c)$, is called the cut-in point.

Clearly, it is of critical importance that a missile control system has a high system response speed so that the system state can reach the sliding mode phase quickly. Thus, the reaching mode phase should be as short as possible. However, when the system is transferred from the reaching mode phase to the sliding mode phase, the deflection

https://doi.org/10.1017/S1446181110000088 Published online by Cambridge University Press

angle of the elevator at t_c needs to be reduced to -6. However, the transition period of the system state will become much too long. From Section 2.1, we see that to adapt for the change in the flight environment, the flight height of the missile should be raised. However, with the increase in the flight height, the air density is decreased, resulting in saturation of the elevator. In this paper, we propose a multi-stage SMVS control method for the cruise missile which will overcome the issue relating to the saturation of the actuator. The control law obtained is optimal with respect to the time-fuel performance index.

3. An improved SMVS optimal control method

3.1. Reaching mode motion separation Consider the unit overload tracking. From (2.8), the reaching mode movement outside the sliding mode surface, but moving towards the sliding mode surface, should satisfy $\dot{S}(\boldsymbol{x}) = -C_r$. Since $S(\boldsymbol{x}) = G\boldsymbol{x}$ and $G = [C_s\ 1]$, it follows that the state equations become

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -C_s x_2(t) - C_r. \end{cases}$$

To achieve a rapid response speed of the system state when it is required to be transferred from reaching mode to sliding mode phase so that the transition period is short, we propose a multi-stage SMVS method for the design of such a control law. This decomposes the reaching mode phase into an acceleration accessing segment, a speed keeping segment and a deceleration buffer segment.

(i) Acceleration accessing segment $[t_0, t_1)$: the reaching law in this segment is varied with time, and is denoted by $C_r^1(t)$. During this segment, the value of the reaching law parameter C_r is increased so that the system state will reach the sliding surface $S(\boldsymbol{x}) = 0$ with a higher speed. Correspondingly, the sliding mode parameter C_s in this segment is denoted by C_s^1.

(ii) Speed keeping segment $[t_1, t_2)$: the reaching law in this segment is constant, denoted by C_r^2. This is because a continuous increase in the reaching law will result in consumption of more fuel. Correspondingly, the sliding mode parameter C_s in this segment is denoted by C_s^2.

(iii) Deceleration buffer segment $[t_2, t_3)$: as in the acceleration accessing segment, the reaching law of this segment is also varied with time, denoted by $C_r^3(t)$. The purpose of this segment is to ensure the stability of the system, such that at the cut-in time t_c, the absolute value of the elevator deflection angle $|\delta_z|$ is less than or equal to the maximal allowable elevator deflection angle $\hat{\delta}_z$. Correspondingly, the sliding mode parameter C_s in this segment is denoted by C_s^3.

With the separation of the reaching mode motion, the original system becomes a switched system consisting of four subsystems, corresponding to the three intervals above followed by an interval $[t_3, t_4]$ with movement on the sliding surface $S(\boldsymbol{x}) = 0$.

For the first three

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -C_s^i x_2(t) - C_r^i(t), \end{cases} \qquad t \in [t_{i-1}, t_i),\ i = 1, 2, 3, \tag{3.1}$$

while in the last

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -C_s^4 x_2(t), \end{cases} \qquad t \in [t_3, t_f]. \tag{3.2}$$

Here, $t_0 = 0$, t_i, $i = 1, 2, 3$, are to be determined and $t_4 = t_f$.

Clearly, the trajectory of the missile on the sliding surface is totally determined by the sliding mode parameter C_s^4.

3.2. Formulation as an optimal control problem During the flight course of a missile, an admissible control must satisfy the constraint,

$$|u(t)| \leq 6. \tag{3.3}$$

For the unit overload tracking, the initial state is

$$x_1(0) = 1, \quad x_2(0) = 0. \tag{3.4}$$

Our task may now be formulated as a time-fuel optimal control problem: given System (3.1)–(3.2) and the initial state (3.4), find an admissible control u such that the time-fuel performance index

$$J = t_f + K \int_{t_0}^{t_f} m_c \, dt \tag{3.5}$$

is minimized, where K is the weighting coefficient, $t_0 = 0$ and m_c is proportional to the thrust P. In flight, the missile must satisfy the thrust and resistance balance condition. Thus (3.5) becomes

$$J = t_f + K \int_{t_0}^{t_f} \frac{Q}{I_{RY}} \, dt. \tag{3.6}$$

Here, Q is the resistance experienced by the missile during the flight course of the missile, satisfying

$$Q = qSC_{x0} + qSC_{xi}(\delta_z)\delta_z,$$

where q is the flow dynamic head, S is the characteristic area of the missile, C_{x0} is the zero lift drag coefficient, and C_{xi} is the induced drag coefficient, which is a function of the elevator deflection angle δ_z.

In flight, q, S and C_{x0} are all regarded as known constants. The relationship between C_{xi} and δ_z can be obtained from tables. Usually, C_{xi} is approximately proportional to the square of δ_z. We may set $C_{xi} = \ell\delta_z$, where ℓ is a known constant. Now (3.6) can be written as

$$J = t_f + K \int_{t_0}^{t_f} \left(K_{x0} + K_{xi}\delta_z^2\right) dt, \tag{3.7}$$

where

$$K_{x0} = \frac{qSC_{x0}}{I_{RY}} \quad \text{and} \quad K_{xi} = \frac{qS\ell}{I_{RY}}.$$

As δ_z is denoted by $u(t)$, (3.7) can be rearranged as

$$J = t_f(1 + KK_{x0}) + KK_{xi} \int_{t_0}^{t_f} (u(t))^2 \, dt. \tag{3.8}$$

Since KK_{x0} is greater than zero we may divide through by $1 + KK_{x0}$ to obtain a new performance index J and constant K for which

$$J = t_f + K \int_{t_0}^{t_f} (u(t))^2 \, dt, \tag{3.9}$$

where $t_0 = 0$.

Let $u_i(t)$, $t \in [t_{i-1}, t_i)$, $i = 1, 2, 3, 4$, denote the deflection angle of the elevator for each subsystem. Recall that:

(i) $S(\boldsymbol{x}) > 0$, that is, $\mathrm{sgn}(S(\boldsymbol{x})) = 1$, during the course of the reaching segment;
(ii) $S(\boldsymbol{x}) = 0$, that is, $\mathrm{sgn}(S(\boldsymbol{x})) = 0$, during the course of the sliding segment.

From (2.11) and the fact that $G^i = [C_s^i \ 1]$ for $t \in [t_{i-1}, t_i)$, the constraint (3.3) can be rewritten as

$$-6 \le u_i(t) = \begin{Bmatrix} -\big([C_s^i \ 1]B\big)^{-1}\big([C_s^i \ 1]A\boldsymbol{x} + C_r^i(t)\big), & i = 1, 2, 3 \\ -\big([C_s^i \ 1]B\big)^{-1}[C_s^i \ 1]A\boldsymbol{x}, & i = 4 \end{Bmatrix} \le 6. \tag{3.10}$$

Similarly, the performance index (3.9) can be written as

$$\begin{aligned} J = t_f + K\sum_{i=1}^{3} \int_{t_{i-1}}^{t_i} \Big\{\Big([C_s^i \ 1]B\Big)^{-1}\Big([C_s^i \ 1]A\boldsymbol{x}(t) + C_r^i(t)\Big)\Big\}^2 \, dt \\ + K\int_{t_3}^{t_f} \Big\{\Big([C_s^4 \ 1]B\Big)^{-1}[C_s^4 \ 1]A\boldsymbol{x}(t)\Big\}^2 \, dt. \end{aligned} \tag{3.11}$$

Define

$$\boldsymbol{\rho} = \Big[C_s^1 \ C_s^2 \ C_s^3 \ C_s^4\Big]^{\mathrm{T}}, \qquad \boldsymbol{\mu}(t) = \Big[C_r^1 \ C_r^2 \ C_r^3 \ C_r^4\Big]^{\mathrm{T}},$$
$$\Lambda = \Big\{\boldsymbol{\vartheta} = [t_1 \ t_2 \ t_3 \ t_4]^{\mathrm{T}} \in R^4 : 0 = t_0 \le t_1 \le t_2 \le t_3 \le t_4 = t_f\Big\}.$$

Let Ξ be the set of all vectors $\boldsymbol{\rho}$. Clearly, $\boldsymbol{\mu}(t)$ is vector-valued function from $[0, T]$ into R^4. For the moment, any piecewise continuous function from $[0, T]$ into R^4 can be taken as an admissible choice for $\boldsymbol{\mu}(t)$. Let Ω be the class of all such admissible functions.

We may now state the problem as an optimal control problem.

PROBLEM 1. Given System (3.1)–(3.2) and the initial condition (3.4), find a combined element $(\boldsymbol{\rho}, \boldsymbol{\mu}, \boldsymbol{\vartheta}) \in \Xi \times \Omega \times \Lambda$ such that the performance index (3.11) is minimized over $\Xi \times \Omega \times \Lambda$ subject to the constraint (3.10).

4. Optimal solution computation

4.1. Problem transformation Problem 1 is a combined optimal control and optimal parameter selection problem involving a switched system with unknown switching times and subject to a continuous state inequality constraint. The switching instants, sliding mode parameter and parameters associated with the reaching mode phase all need to be chosen such that a time-fuel performance index is minimized. In addition, this optimal control problem is subject to a continuous state inequality constraint. Since the variable switching times are decision variables and the continuous state inequality condition must be satisfied for all $t \in [0, T]$, this optimal problem cannot be solved effectively by classical optimal control methods. In this section, we will present an effective method for solving Problem 1.

We first apply the classical control parametrization scheme [16]. That is, the reaching parameters C_r^i, $i = 1, 2, 3, 4$, are approximated as piecewise constant functions with possible discontinuity points at $\{t_j^i\}_{j=1}^{m_i}$, where

$$t_0 = t_0^1 < \cdots < t_{m_1}^1 = t_1 = t_0^2 < t_{m_2}^2 = t_2 = t_0^3 < \cdots < t_{m_3}^3 = t_3 = t_0^4 < t_{m_4}^4 = t_4 \tag{4.1}$$

and $m_2 = m_4 = 1$, $m_1, m_3 \in R^+$. More specifically, for each $i = 1, 2, 3, 4$, the approximate function is expressed as

$$C_r^i(t) = \sum_{j=1}^{m_i} \sigma_j^i \, \chi_{[t_{j-1}^i, t_j^i)}(t), \tag{4.2}$$

with $\sigma_1^4 = 0$, where $\sigma_j^i \in R^+$, $j = 1, \ldots, m_i$, $i = 1, 2, 3$, are decision parameters and $\chi_I(t)$ denotes the indicator function of I defined by

$$\chi_I(t) = \begin{cases} 1 & t \in I, \\ 0 & \text{elsewhere.} \end{cases}$$

Substituting (4.2) into (3.1), (3.2) yields

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -C_s^i x_2(t) - \displaystyle\sum_{j=1}^{m_i} \sigma_j^i \chi_{[t_{j-1}^i, t_j^i)}(t), \end{cases} \qquad t \in [t_{i-1}, t_i), \ i = 1, 2, 3, 4, \tag{4.3}$$

where the switching times $\{t_j^i\}_{j=1}^{m_i}$ are chosen such that (4.1) is satisfied.

The constraint (3.10) can be written as

$$|\bar{u}_i(t)| \le 6, \qquad \forall t \in [t_{i-1}, t_i), \ i = 1, 2, 3, 4, \tag{4.4}$$

where

$$\bar{u}_i(t) = -\left(\begin{bmatrix} C_s^i & 1 \end{bmatrix} B\right)^{-1} \left[\begin{bmatrix} C_s^i & 1 \end{bmatrix} A\boldsymbol{x}(t) + \sum_{j=1}^{m_i} \sigma_j^i \chi_{[t_{j-1}^i, t_j^i)}(t)\right], \qquad t \in [t_{i-1}, t_i). \tag{4.5}$$

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Similarly, the performance index (3.11) can be transformed into

$$J = t_f + \sum_{i=1}^{4} \int_{t_{i-1}}^{t_i} \bar{u}_i^2(t)\, dt. \tag{4.6}$$

To obtain an approximate optimal control, the parameters σ_j^i, C_s^i and the switching times $\{t_j^i\}_{j=1}^{m_i}$ are to be chosen such that the performance index (4.6) is minimized. However, gradient-based optimization algorithms are not effective for the determination of the optimal switching times. We employ the time scaling transformation introduced in [17] to map these switching times into fixed time points on a new time horizon.

We define a monotonic transformation from $t \in [t_0, t_4]$ into $s \in [0, 4]$:

$$\frac{dt^i(s)}{ds} = v^i(s)$$

with initial condition $t(0) = t_0$, and

$$v^i(s) = \sum_{j=1}^{m_i} \delta_j^i \chi_{\left[i-1+\frac{j-1}{m_i},\, i-1+\frac{j}{m_i}\right)}(s) \tag{4.7}$$

where v^i has possible discontinuity points at $s = i - 1 + j/m_i$, $j = 1, \ldots, m_i$, $i = 1, \ldots, 4$, and

$$\delta_j^i \geq 0, \quad j = 1, \ldots, m_i, \ i = 1, \ldots, 4.$$

Clearly, $t_f = \sum_{i=1}^{4} \sum_{j=1}^{m_i} \delta_j^i$. Define

$$\boldsymbol{\sigma} = [\sigma_1^1 \cdots \sigma_{m_1}^1\ \sigma_1^2 \cdots \sigma_1^3 \cdots \sigma_{m_3}^3\ \sigma_1^4]^{\mathrm{T}}$$

and

$$\boldsymbol{\delta} = [\delta_1^1 \cdots \delta_{m_1}^1\ \delta_1^2 \cdots \delta_1^3 \cdots \delta_{m_3}^3\ \delta_1^4]^{\mathrm{T}}.$$

Let Σ be the set of all those vectors $\boldsymbol{\sigma}$, and let Δ be the set of all those vectors $\boldsymbol{\delta}$ such that (4.7) is satisfied.

Define $\tilde{x}_1(s) = x_1(t(s))$, $\tilde{x}_2(s) = x_2(t(s))$ and $\tilde{x}_3(s) = t(s)$. Then, by applying the time scaling transformation to the dynamics (4.3), we obtain

$$\begin{cases} \dot{\tilde{x}}_1(s) = \tilde{x}_2(s) v_i(s), \\ \dot{\tilde{x}}_2(s) = \left(-C_s^i \tilde{x}_2(s) - \displaystyle\sum_{j=1}^{m_i} \sigma_j^i \chi_{\left[i-1+\frac{j-1}{m_i},\, i-1+\frac{j}{m_i}\right)}(s) \right) v_i(s), \\ \dot{\tilde{x}}_3(s) = v_i(s), \end{cases} \tag{4.8}$$

where $s \in [i-1, i)$, $i = 1, 2, 3, 4$, $m_2 = m_4 = 1$, $\sigma_1^4 = 0$. The initial condition (3.4) is changed to

$$\tilde{x}_1(0) = 1, \quad \tilde{x}_2(0) = 0, \quad \tilde{x}_3(0) = 0. \tag{4.9}$$

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The constraint (4.4) is equivalent to

$$\phi(s, \tilde{\boldsymbol{x}}(s)) = V_i^2(s) - 36 \leq 0, \quad s \in [i-1, i),\ i = 1, 2, 3, 4, \tag{4.10}$$

where $\tilde{\boldsymbol{x}}(s) = [\tilde{x}_1(s), \tilde{x}_2(s)]^{\mathrm{T}}$ and, for $i = 1, 2, 3, 4$,

$$V_i(s) = -\Big([C_s^i\ 1]B\Big)^{-1}\Bigg\{[C_s^i\ 1]A\tilde{\boldsymbol{x}}(s) + \sum_{j=1}^{m_i} \sigma_j^i\, \chi_{\left[i-1+\frac{j-1}{m_i}, i-1+\frac{j}{m_i}\right)}(s)\Bigg\}.$$

Similarly, the performance index (4.6) is transformed into

$$J = \sum_{i=1}^{4}\sum_{j=1}^{m_i} \delta_j^i + \sum_{i=1}^{4}\int_{i-1}^{i} V_i^2(\tau) v_i(\tau)\, d\tau. \tag{4.11}$$

The continuous state inequality constraint (4.10) must be satisfied for all s. By virtue of [16, Chapter 8.3], the continuous state inequality constraint (4.10) is approximated by

$$G_\varepsilon(\sigma, \delta) = -\gamma + \int_0^N L_\varepsilon(s, \tilde{x}(s))\, ds \leq 0, \tag{4.12}$$

where $\gamma > 0$, $\varepsilon > 0$, $N = 4$ and

$$L_\varepsilon(s, \tilde{\boldsymbol{x}}(s)) = \begin{cases} 0 & \text{if } \phi(s, \tilde{\boldsymbol{x}}(s)) < -\varepsilon, \\ \dfrac{(\phi(s, \tilde{\boldsymbol{x}}(s)) + \varepsilon)^2}{4\varepsilon} & \text{if } -\varepsilon \leq \phi(s, \tilde{\boldsymbol{x}}(s)) \leq \varepsilon, \\ \phi(s, \tilde{\boldsymbol{x}}(s)) & \text{if } \phi(s, \tilde{\boldsymbol{x}}(s)) > \varepsilon. \end{cases}$$

REMARK 1. Under appropriate conditions, it is known (see [16, Lemma 8.3.3]) that for any $\varepsilon > 0$ there exists a $\gamma(\varepsilon) > 0$ such that if $(\boldsymbol{\sigma}, \boldsymbol{\delta}) \in \Sigma \times \Delta$ satisfies (4.11) with γ such that $0 < \gamma < \gamma(\varepsilon)$, then it satisfies the continuous state inequality constraint (4.10).

Problem 1 is now approximated by Problem 2 below.

PROBLEM 2. For $\varepsilon > 0$ and $\gamma > 0$, given System (4.8) with initial condition (4.9), find a combined control parameter vector and switching vector $(\boldsymbol{\rho}, \boldsymbol{\sigma}, \boldsymbol{\delta}) \in \Xi \times \Sigma \times \Delta$ to minimize the performance index (4.11) subject to the inequality constraint (4.12).

To solve Problem 1, we will solve a sequence of approximate problems in the form of Problem 2 as follows.

We first choose an $\varepsilon > 0$ and a $\gamma > 0$. Then, we solve Problem 2 with such ε and γ. Let $(\boldsymbol{\sigma}^{\varepsilon,\gamma,*}, \boldsymbol{\delta}^{\varepsilon,\gamma,*})$ be the solution obtained. Then we check whether the continuous state inequality constraint (4.10) is satisfied or not. If it is not satisfied, we will reduce the value of γ to $\gamma/2$ and return to solve Problem 2 with γ taken as $\gamma/2$. By virtue of Remark 1, we see that this reduction step will only be required a finite number of times

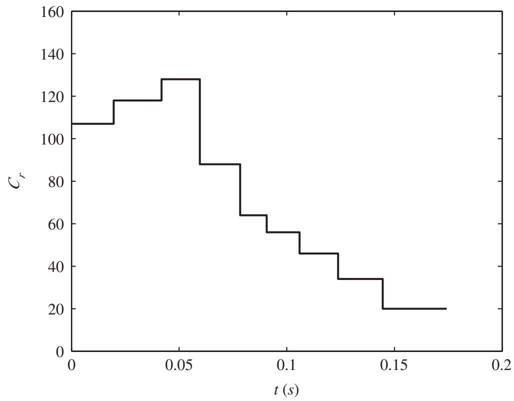

FIGURE 2. The trajectory of the reaching parameter C_r.

before the continuous state inequality constraint (4.10) is satisfied. We then reduce the value of ε and repeat the process until a satisfactory approximate optimal solution is obtained.

It is known that the approximate optimal cost will converge to the true optimal cost as $\varepsilon \to 0$.

5. Numerical simulation

We now return to Problem 1. The approximate Problem 2 is a standard optimal parameter selection problem, which is solved by using a program written in Matlab.

Figure 2 shows the trajectory of the reaching parameter C_r. For clarity, we only draw the trajectory of the reaching parameter during the reaching segment. After the system state enters into the sliding surface $S(\boldsymbol{x}) = 0$, the reaching parameter is required to be larger than zero. Thus, we choose the terminal value of the reaching segment as the reaching parameter in our simulation.

Figure 3 shows the state trajectory of System (3.1)–(3.2), where the dots are used to label the optimal switching instants of the system state. The optimal switching instants obtained are $t_1 = 0.06$ s, $t_2 = 0.078$ s, and $t_3 = 0.174$ s.

The first graph of Figure 4 depicts the trajectory of the elevator deflection angle δ_z. We can see that δ_z is less than the maximal permitted value $\hat{\delta}_z$. Thus, the problem of violation of the maximal permitted range of the actuator is overcome effectively.

The second graph of Figure 4 shows trajectories of the real overload n_y simulated with the recursive and multi-stage SMVS methods. The time taken by the missile to adjust for the unit overload is 0.36 s for the recursive SMVS method and 0.27 s for the multi-stage SMVS method proposed in this paper. This shows that the system response speed is improved significantly for the control law obtained by using the proposed method.

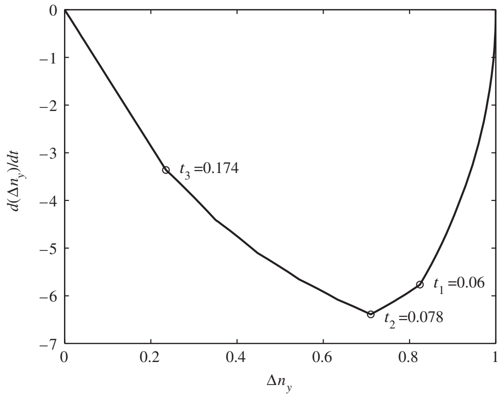

FIGURE 3. The state trajectory of System (3.1), (3.2).

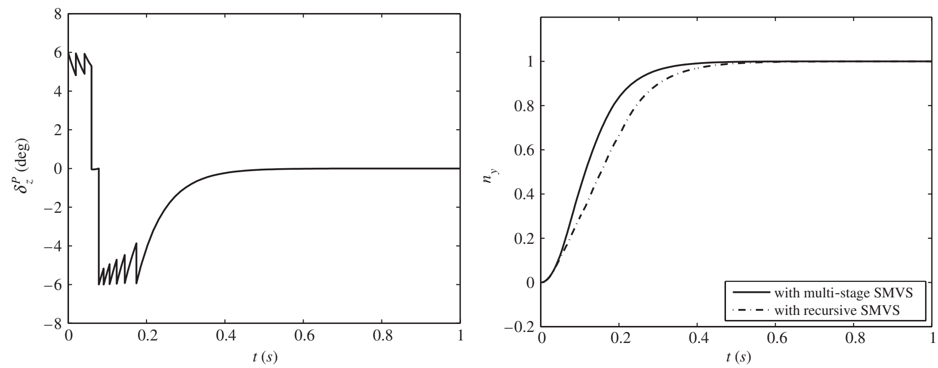

FIGURE 4. The trajectory of (left) the elevator deflection angle δ_z and (right) of the real overload n_y.

For the simulation study, the time constant of the missile, T_M, is 0.1427 s; the relative damping coefficient of the missile, ξ_M, is 0.049; the transfer coefficient of the missile, K_M, is $-0.128\ \mathrm{s}^{-1}$; the flight speed of the missile, v, is 3 Ma; the gravitational acceleration, g, is $9.8\ \mathrm{m\,s}^{-2}$; the chattering eliminating parameter, δ, is 0.1; and the simulation step is taken to be 0.001 s.

6. Conclusions

In this paper, we considered the control of a cruise missile operating in a large air space using an improved SMVS method. The main feature of our method is that the reaching mode segment of the sliding mode variable structure control is decomposed into three segments. Then the task of achieving the optimal time-fuel performance of a cruise missile in a large air space was reformulated as an optimal control problem involving a switched system and subject to a continuous state inequality constraint where the switching times between the subsystems and several parameters are to be determined optimally with reference to a time-fuel performance index. An efficient gradient-based computation method based on the control parametrization, a time scaling transform and the constraint transcription method was developed. The simulation results show that the response speed of the missile is much improved when compared with the result obtained by existing method. The improved variable structure controller obtained is robust against load disturbance and is insensitive to parameter variation. Furthermore, the flight trajectory of the system at the cut-in time is smooth and does not violate the permitted range of the actuator deflection angle.

Acknowledgements

This work was supported by a grant from the national 863 plan-military project (2008AA7060501) and by a grant from Major Program of National Natural Science Foundation of China (No. 60710002).

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