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ON CONWAY'S CONJECTURE FOR INTEGER SETS

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Let A be a finite set of integers $\{a_i\}$ and $A+A$ denote $\{a_i+a_j\}$ with p different members and $A-A$ denote $\{a_i-a_j\}$ with m different members, the interesting conjecture of Conway [1] states that $p<m$, unless A is symmetric.

Marica [2] proves:

Theorem A: $p=m$ if A is symmetric.

Theorem B: $p=m=15$ for the nonsymmetric $A=\{0, 1, 3, 4, 5, 8\}$, counter to Conway's Conjecture.

Theorem C: $p>m$ for $A=\{1, 2, 3, 5, 8, 9, 13, 15, 16\}$ to further violate the Conjecture. Here $p=30$, $m=29$.

We present several related conjectures based on numerical evidence. First, note that the a_i may be taken as distinct, since we allow $i=j$ in forming $A+A$ and $A-A$. If A consists of the $n+1$ integers $a_0<a_1<\cdots<a_n$, all numbers may be shifted by a constant without changing the values of p and m. Thus the set may be normalized by taking $a_0=0$. Then the set A can be characterized by the n positive differences $\{d_i\}$ where $d_i=a_i-a_{i-1}$. We note further that the values of p and m are unchanged if the d_i are multiplied by a constant or are reversed.

Conjecture 1. For nonsymmetric A, $p<m$ for $n<4$.

EXAMPLE. The set with differences 1, 1, 2, 1 has $p=m=11$ for $n=4$.

Conjecture 2. $p\le m$ for $n<8$.

EXAMPLE. The counterexample in Theorem C has 8 differences 1, 1, 2, 3, 1, 4, 2, 1.

Conjecture 3. The smallest value of p for which $p>m$ is 28.

EXAMPLE. The differences 1, 1, 2, 1, 4, 3, 1, 1 yield $p=28$, $m=27$.

Conjecture 4. $p\le m$ if the differences are restricted to 1's, 2's, and 3's.

Conjecture 5. A block of differences beginning and ending with 1 may be repeated any number of times yielding the same value of $p-m$.

EXAMPLE. The block 1, 1, 2, 1, 4, 3, 1, 1 yields for (p, m) on repetition the values (28, 27), (56, 55), (84, 83), ...

COROLLARY. *$p>m$ infinitely often.*

Conjecture 6. The repetition of certain interior blocks can cause p to be increased by a greater constant than that by which m is increased.

QUESTION. Must the interior block contain at least 3 elements?

EXAMPLE 1. For the block 1, 1, 2, 1, 4, 3, 2, 2, 1 repetition of the interior block 1, 4, 3 gives (p, m) the values (34, 33), (50, 47), (66, 61), ... The limiting value of p/m is 8/7.

COROLLARY. *$p-m$ can be arbitrarily large.*

EXAMPLE 2. For the block 1, 1, 3, 1, 2, 1, 5, 7, 1, 1, 3, 1, 2, 1 repetition of the interior block 1, 5, 7, 1 gives (p, m) the values (61, 57), (89, 81), (117, 105), ... The limiting value of p/m is 7/6.

EXAMPLE 3. For the block 1, 1, 2, 1, 4, 3, 1, 4, 4, 3, 1, 4, 4, 3, 1, 4, 3, 2, 2, 1 repetition of the interior block 4, 4, 3, 1, 4 gives (p, m) the values (94, 87), (126, 113), (158, 139), ... The limiting value of p/m is 16/13. From this Example, on repetition of the interior block 652 times or more, follows the

THEOREM. *There exist sets for which $p/m > 1.23$.*

REFERENCES

1. J. H. Conway, Problem 7 of Section VI of H. T. Croft's *Research problems*, mimeographed notes, Cambridge, August, 1967.

2. J. Marica, *On a conjecture of Conway*, Canad. Math. Bull. **12** (1969), 233–234.

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