A QUASI-LINEAR ELLIPTIC BOUNDARY VALUE PROBLEM

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1. Introduction. Let Ω be a bounded open set in Euclidean n-space, E_n. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be an n-tuple of non-negative integers;

$$|\alpha| = \alpha_1 + \ldots + \alpha_n;$$

and denote by Q_m the set $\{\alpha | 0 \leqslant |\alpha| \leqslant m\}$. Denote by $x = (x_1, \ldots, x_n)$ a typical point in E_n and put

$$D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}, \qquad D_j = \frac{1}{i}\frac{\partial}{\partial x_j} \quad (i^2 = -1).$$

In this paper we establish, under certain circumstances, the existence of weak and classical solutions of the quasi-linear Dirichlet problem

$$(1) \qquad \begin{aligned} Au(x) &= \lambda f[u](x), \qquad x \in \Omega, \\ D^\alpha u(x) &= 0, \qquad \alpha \in Q_{m-1}, \qquad x \in \partial\Omega. \end{aligned}$$

Here A is a linear elliptic partial differential operator of order $2m$ given in the generalized divergence form

$$Au(x) = \sum_{\alpha,\beta \in Q_m} D^\alpha[a_{\alpha\beta}(x) D^\beta u(x)]$$

where the coefficients $a_{\alpha\beta}(x)$ are complex-valued functions on Ω. Also, $f[u](x)$ is a complex-valued function depending on x, $u(x)$, and all the derivatives of $u(x)$ of order not exceeding $m - 1$. We write

$$f[u](x) = f(x, u(x), D^1 u(x), \ldots, D^{m-1} u(x))$$

where D^k represents the vector of all derivatives of order k.

We shall restrict f in such a way that our problem is not a generalization of the linear case $Au(x) = \lambda u(x)$. Among other things our work generalizes results of Duff **(7)** for the equation $\Delta u(x) = -f(x, u(x))$, $f \geqslant \delta > 0$. In particular, it yields suitably normalized eigenfunctions for equations of the form

$$\Delta^m u(x) = f(x, u(x), \ldots, D^{m-1} u(x))$$

for a wide class of functions satisfying, for fixed $\delta > 0$, either $f \geqslant \delta$ or $f \leqslant -\delta$. The conditions on our problem are given in §3 below. The principal results are contained in Theorem 3 of §5 and Theorem 4 of §6.

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2. Sobolev spaces. Let $C_0^\infty(\Omega)$ denote the class of infinitely often continuously differentiable functions with compact support in Ω.

Let $W^{m,p}(\Omega)$ denote, for $1 < p < \infty$, the collection of functions in $L^p(\Omega)$ whose derivatives (in the weak sense—see (**2**, p. 3, Definition 1.5)) of order not exceeding m all belong to $L^p(\Omega)$. This is a separable, reflexive Banach space with respect to the norm

$$||u||_{m,p} = \left\{ \sum_{\alpha \in Q_m} ||D^\alpha u||_{0,p}^p \right\}^{1/p}$$

where $||u||_{0,p}$ is the $L^p(\Omega)$ norm. $W^{m,2}(\Omega)$ is a Hilbert space with respect to the inner product

$$[u, v]_m = \sum_{\alpha \in Q_m} [D^\alpha u, D^\alpha v]$$

where $[u, v]$ is the $L^2(\Omega)$ inner product. Let $H^{m,p}(\Omega)$ be the closed linear subspace of $W^{m,p}(\Omega)$ obtained by taking the closure of the linear manifold $C_0^\infty(\Omega)$ in the norm $||\,.\,||_{m,p}$. The Hilbert space $H^{m,2}(\Omega)$ is the setting of much of our work.

The principal results concerning the Sobolev spaces $W^{m,p}(\Omega)$ and $H^{m,p}(\Omega)$ and their embedding theorems (the Sobolev and Kondrasev theorems) which we shall use may be found in (**2**; **4**; **8**).

3. Conditions on the problem. Throughout this paper we assume that the following conditions are satisfied:

(A) *The functions $a_{\alpha\beta}(x)$ are measurable and uniformly bounded on Ω.*

(B) *There exists a constant $c > 0$ such that the Dirichlet form*

$$a(u, v) = \sum_{\alpha,\beta \in Q_m} \int_\Omega a_{\alpha\beta}(x) D^\beta u(x) \overline{D^\alpha v(x)}\, dx \tag{2}$$

satisfies $|a(u, u)| \geqslant c||u||_{m,2}^2$ for all $u(x) \in C_0^\infty(\Omega)$.

(C) *If t is the complex vector (t_α) $(\alpha \in Q_{m-1})$, then $f(x, t)$ is measurable in x for x in Ω and fixed t and is continuous in t.* (Note that $f[u](x) = f(x, t)$ where $t_\alpha = t_\alpha(x) = D^\alpha u(x)$.)

(D) *The growth conditions:*

$$|f(x, t)| \leqslant K + \sum_{k=0}^{m-1} \sum_{\alpha \in Q_k} C_k(|t_\alpha|)$$

where K is a constant and where, if $n < 2m - 2k$, $C_k(r)$ is a non-decreasing function of r for $0 \leqslant r < \infty$. If $n \geqslant 2m - 2k$, then $C_k(r) = B_k r^{\sigma_k}$ where B_k is a constant and

(i) $\quad 1 \leqslant \sigma_k < \dfrac{n + 2m}{n - 2m + 2k} \qquad$ *if* $n > 2m$,

(ii) $\quad 1 \leqslant \sigma_k < \dfrac{2n}{n - 2m + 2k} \qquad$ *if* $2m \geqslant n > 2m - 2k$,

(iii) $\quad 1 \leqslant \sigma_k < \infty \qquad$ *if* $n = 2m - 2k$.

By a *weak solution* of the problem (1) we mean an element $u(x) \in H^{m,2}(\Omega)$ satisfying

$$a(u, v) = \int_\Omega f[u](x)\overline{v(x)}\, dx \tag{3}$$

for all $v(x) \in H^{m,2}(\Omega)$. The weak solution is *non-trivial* provided $||u||_{m,2} > 0$. A *classical solution* is one that is sufficiently differentiable to satisfy the differential equation and boundary conditions in a pointwise sense.

4. The operator equation. We now replace equation (3) above by an operator equation in $H^{m,2}(\Omega)$.

THEOREM 1. *Suppose that the conditions* (A)–(D) *of* §3 *are satisfied. Then there exists a bounded linear operator L mapping* $H^{m,2}(\Omega)$ *onto itself and possessing a bounded inverse* L^{-1}, *and also a completely continuous operator C mapping* $H^{m,2}(\Omega)$ *into itself such that for all* $u(x)$, $v(x) \in H^{m,2}(\Omega)$

$$a(u, v) = [Lu, v]_m, \tag{4}$$

$$\int_\Omega f[u](x)\overline{v(x)}\, dx = [C(u), v]_m. \tag{5}$$

Any weak solution of (1) *is a solution of*

$$u = \lambda L^{-1} C(u) \tag{6}$$

and conversely. Finally, there exists a non-decreasing function $g(r)$ *for* $0 \leqslant r < \infty$ *such that* $||C(u)||_{m,2} \leqslant g(||u||_{m,2})$ *for all* $u \in H^{m,2}(\Omega)$.

Proof. The existence of L satisfying (4) is an immediate consequence of condition (A) and the Riesz representation theorem. The invertibility of L on $H^{m,2}(\Omega)$ is a consequence of condition (B), the fact that $C_0^\infty(\Omega)$ is dense in $H^{m,2}(\Omega)$, and the Lax–Milgram representation theorem (**2**, p. 99).

The left side of (5) is a conjugate linear functional of v on $H^{m,2}(\Omega)$. That it is bounded follows from condition (D) and Sobolev's embedding theorem. For example, suppose $n > 2m$. Then $H^{m,2}(\Omega)$ is embedded continuously in $L^r(\Omega)$ where $r = 2n(n-2m)^{-1}$. If $r^{-1} + s^{-1} = 1$, then

$$\sigma_k\, s < 2n(n - 2m + 2k)^{-1}$$

and so $H^{m-k,2}(\Omega)$ is embedded continuously in $L^p(\Omega)$ where $p = \sigma_k\, s$. Thus if $|\alpha| = k$,

$$\begin{aligned}\int_\Omega |D^\alpha u(x)|^{\sigma_k} |v(x)|\, dx &\leqslant ||D^\alpha u||_{0,s\sigma_k}^{\sigma_k} ||v||_{0,r} \\ &\leqslant \text{const. } ||D^\alpha u||_{m-k,2}^{\sigma_k} ||v||_{m,2} \\ &\leqslant \text{const. } ||u||_{m,2}^{\sigma_k} ||v||_{m,2}.\end{aligned}$$

Condition (D) now gives

$$(7) \qquad \left| \int_\Omega f[u](x)\overline{v(x)}\, dx \right| \leqslant \text{const.} \left\{ 1 + \sum_{k=0}^{m-1} \sum_{\alpha \in Q_k} ||u||_{m,2}^{\sigma k} \right\} ||v||_{m,2}.$$

The other cases follow similarly. The existence of C satisfying (5) follows by the Riesz representation theorem. The function $g(||u||_{m,2})$ is given by the factor multiplying $||v||_{m,2}$ on the right side of (7) (for the case $n > 2m$).

In proving the complete continuity of C we again consider only the case $n > 2m$, the other cases being similar though more tedious. (A detailed proof for all cases may be found in **(1)**.) Since $\sigma_k < (n + 2m)(n - 2m + 2k)^{-1}$, there exist constants $\epsilon_k > 0$ such that $\sigma_k(1 + \epsilon_k) \leqslant (n + 2m)(n - 2m + 2k)^{-1}$. Define $q_k = 2n(n - 2m + 2k)^{-1}(1 + \epsilon_k)^{-1}$ choosing ϵ_k smaller if necessary so that $q_k \geqslant 1$. By the Kondrasev theorem (also called Rellich's lemma) the embedding map

$$J_k: H^{m-k,2}(\Omega) \to L^{q_k}(\Omega)$$

is completely continuous. Thus, so is the product mapping

$$J : \prod_{k=0}^{m-1} \prod_{\alpha \in Q_k} H^{m-k,2}(\Omega) \to \prod_{k=0}^{m-1} \prod_{\alpha \in Q_k} L^{q_k}(\Omega),$$

$$J(\ldots, u_j, \ldots) = (\ldots, J_k u_j, \ldots), u_j \in H^{m-k,2}(\Omega).$$

Here the product spaces are normed by the Pythagorean formula. We define the operator

$$B: \prod_{k=0}^{m-1} \prod_{\alpha \in Q_k} L^{q_k}(\Omega) \to L^p(\Omega)$$

where $p = 2n(n + 2m)^{-1}$ by the formula

$$B(t(x)) = f(x, t(x)).$$

If we put $p_k = q_k p^{-1}$, we have $p_k \geqslant \sigma_k$ and so by condition (D)

$$|B(t(x))| \leqslant K + \sum_{k=0}^{m-1} \sum_{\alpha \in Q_k} B_k |t_\alpha(x)|^{p_k}.$$

By an extension of a theorem of M. M. Vainberg **(10**, p. 253**)** which can be proved using the method of **(10)** (the proof can be found in **(1)**) it follows that B is a continuous mapping. Hence the mapping

$$BJ : \prod_{k=0}^{m-1} \prod_{\alpha \in Q_k} H^{m-k,2}(\Omega) \to L^p(\Omega)$$

is completely continuous. Now is $u_n \to u$ weakly in $H^{m,2}(\Omega)$, it is easily verified that $D^\alpha u_n \to D^\alpha u$ weakly in $H^{m-k,2}(\Omega)$. By Hölder's inequality and Sobolev's theorem, we obtain, using (5),

$$\begin{aligned} ||C(u_n) - C(u)||_{m,2} &= \sup_{||v||_{m,2}=1} |[C(u_n) - C(u), v]_m| \\ &\leqslant \text{const.} ||f[u_n] - f[u]||_{0,p} \\ &\leqslant \text{const.} ||BJ((D^\alpha u_n)_{\alpha \in Q_{m-1}}) - BJ((D^\alpha u)_{\alpha \in Q_{m-1}})||_{0,p}, \end{aligned}$$

which tends to 0 as n tends to infinity. This completes the proof, the rest of the theorem being obvious.

5. Existence theory. Using various well-known fixed-point theorems, we demonstrate the existence of a solution of equation (6). In order to be sure that the solution is non-trivial we assume that $C(0) \neq 0$ in $H^{m,2}(\Omega)$, or equivalently, $||f(x, 0)||_{0,2} \neq 0$.

THEOREM 2 (Schauder, Schaefer, Birkhoff–Kellogg). *Let B_r be the ball of radius r centred at the origin in the separable Hilbert space H, and let S_r be its surface. Let T be a completely continuous operator in H. Then*

(a) *if T maps B_r into B_r, it has a fixed point in B_r;*

(b) *if T maps H into H and $\lambda_0 > 0$, either there exists $u \in H$ such that $u = \lambda_0 T(u)$ or for any $r > 0$ there exists $u \in S_r$ and λ with $0 < \lambda < \lambda_0$ such that $u = \lambda T(u)$;*

(c) *if T maps S_r into S_r, it has a fixed point on S_r.*

The proofs of these results may be found in Cronin **(6)**.

THEOREM 3. *Suppose the conditions* (A)–(D) *of* §3 *are satisfied. Let B_r be the ball of radius r centred at the origin in $H^{m,2}(\Omega)$, and let S_r be its surface. Then we have the following:*

(a) *If $||C(0)||_{m,2} > 0$, then for any $r > 0$ there exists $\lambda_0 > 0$ such that if $0 < \lambda \leqslant \lambda_0$ then there exists a non-trivial solution of* (6) *in B_r.*

(b) *Given $\lambda_0 > 0$, either there exists $u \in H^{m,2}(\Omega)$ such that $u = \lambda_0 L^{-1}C(u)$ or for any $r > 0$ there exists a solution of* (5) *on S_r for some $\lambda < \lambda_0$.*

(c) *If for some $r > 0$, C satisfies*

$$\inf_{u \in S_r} ||C(u)||_{m,2} = \theta > 0, \tag{2}$$

then there exists a solution of (6) *on S_r for some λ satisfying*

$$\frac{R}{||L^{-1}||g(R)} \leqslant \lambda \leqslant \frac{R||L||}{\theta}.$$

Proof. Part (a) follows from Theorem 2(a) if we put

$$T(u) = \lambda L^{-1}C(u), \qquad \lambda_0 = \frac{R}{||L^{-1}||g(R)}.$$

Part (b) is an immediate consequence of Theorem 2(b).

Part (c) follows from Theorem 2(c) if we put

$$T(u) = R\frac{L^{-1}C(u)}{||L^{-1}C(u)||_{m,2}},$$

for since $||L^{-1}C(u)||_{m,2} \geqslant \theta(||L||)^{-1}$ for $u \in S_r$, it follows that $L^{-1}C(S_r)$ is bounded away from the origin. But then the projection P from the origin of $H^{m,2}(\Omega)$ onto S_r is continuous on $L^{-1}C(S_r)$ and so $T = PL^{-1}C$ is completely continuous.

Remark. Condition (8) may be put in the equivalent form

$$(8') \qquad \inf_{u \in S_r} \sup_{v \in S_1} \left| \int_\Omega f[u](x)\overline{v(x)}\, dx \right| = \theta > 0.$$

The existence of a $\theta > 0$ satisfying (8) is equivalent to the condition $||C(u)||_{m,2} > 0$ for all $u \in B_r$ because each element of B_r is the weak limit of a sequence on S_r and $||C(u)||_{m,2}$, being a weakly continuous functional, takes on its infimum on the weakly compact set B_r.

6. Regularity theory. We denote by $C^m(\Omega)$ (by $C^m(\bar{\Omega})$), the class of functions which together with all their derivatives of order not exceeding m are continuous in Ω (are uniformly continuous on the closure $\bar{\Omega}$ of Ω). $C^{m,r}(\Omega)$ is the class of functions in $C^m(\bar{\Omega})$ which together with all their derivatives of order not exceeding m satisfy a Hölder condition of exponent r in Ω.

We place the following regularity conditions on our problem:

(E) *A is uniformly elliptic and if $n = 2$ it satisfies the "roots condition"* **(3**, p. 57**)**. *Also*, $a_{\alpha\beta}(x) \in C^{2m}(\bar{\Omega})$.

(F) Ω *is of class C^{2m} in E_n* (cf. definition in **(2**, p. 128**)**).

(G) *$f(x, t)$ satisfies a local Lipschitz condition in each component of t, and also a Hölder condition of exponent r* $(0 < r < 1)$ *in x.*

Lemma. *Suppose that conditions* (A)–(G) *are satisfied and let $u(x)$ be a weak solution of problem* (1). *Then $u(x) \in W^{2m,p}(\Omega)$ for any $p < \infty$.*

Proof. Again we consider only the case $n > 2m$. The other cases are similar but more complicated and are treated in detail in **(1)**. Since $u(x) \in H^{m,2}(\Omega)$, it follows from Sobolev's embedding theorem that

$$D^\alpha u(x) \in L^{r_{1k}}(\Omega), \qquad 0 \leqslant |\alpha| = k \leqslant m - 1,$$

where $r_{1k} = 2n(n - 2m + 2k)^{-1}$. Defining ϵ_k as in the proof of Theorem 1, we put $\epsilon = \min \epsilon_k > 0$. It follows from condition (D) that

$$|D^\alpha u(x)|^{\sigma k} \in L^{p_{1k}}(\Omega)$$

where $p_{1k} = r_{1k}(\sigma_k)^{-1} \geqslant 2n(1 + \epsilon)(n + 2m)^{-1} \equiv p_1 > 1$. Hence, if we put $f(x) = f[u](x)$,

$$f(x) \in L^{p_1}(\Omega).$$

It follows by a theorem of Agmon **(3**, p. 88, Theorem 8.2**)**, the conditions of which are fulfilled under our stated conditions, that

$$u(x) \in W^{2m,p_1}(\Omega).$$

From this point, we may repeat the above argument and obtain

$$D^r u(x) \in L^{r_{2k}}(\Omega), \qquad 0 \leqslant |\alpha| = k \leqslant m - 1,$$

where, if $n \leqslant (2m - k)p_1$, r_{2k} is any finite number greater than unity, or, if $n > (2m - k)p_1$,

$$r_{2k} = \frac{np_1}{n - (2m - k)p_1} \leqslant \frac{2n}{n - 2m + 2k}(1 + \epsilon).$$

It follows that

$$|D^\alpha u|^{\sigma_k} \in L^{p_{2k}}(\Omega)$$

where $p_{2k} = r_{2k}(\sigma_k)^{-1} \geqslant 2n(1 + \epsilon)^2(n + 2m)^{-1} = (1 + \epsilon)p_1 \equiv p_2$. Again by the theorem of Agmon referred to above we have

$$u(x) \in W^{2m,p_2}(\Omega).$$

This "boot-strapping" procedure can be continued to produce

$$u(x) \in W^{2m,p_s}(\Omega)$$

for a sequence of values $p_s = (1 + \epsilon)^{s-1}p_1$ tending to infinity. This completes the proof.

COROLLARY. $u(x) \in C^{2m}(\Omega) \cap C^{m-1}(\Omega)$.

Proof. For p large enough ($p > n(m - r)^{-1}$), any function in $W^{2m,p}(\Omega)$ belongs, after possible redefinition on a subset of Ω of measure zero, to $C^{m,r}(\Omega)$, by Sobolev's embedding theorem. In particular, $u(x) \in C^{m-1}(\bar{\Omega})$. Since, for $0 \leqslant |\alpha| \leqslant m - 1$, $D^\alpha u(x)$ is uniformly bounded on Ω, and since $f(x, t)$ satisfies condition (G), it follows that $f(x) = f[u](x)$ belongs to $C^{0,r}(\Omega)$. By a theorem of Browder (**5**, Theorem 1(iii)), the conditions of which are satisfied under our stated conditions, it follows that $u(x) \in C^{2m,r}(\Omega')$ for any compact subdomain $\Omega' \subset \Omega$. In particular, $u(x) \in C^{2m}(\Omega)$.

THEOREM 4. *Under conditions* (A)–(G) *any weak solution* $u(x)$ *of* (1) *is a classical solution.*

Proof. Since $u(x) \in C^{m-1}(\bar{\Omega}) \cap H^{m,2}(\Omega)$, it follows for $\alpha \in Q_{m-1}$ that $D^\alpha u(x) \in C^0(\bar{\Omega}) \cap H^{1,2}(\Omega)$, and so be a lemma of Nirenberg (**9**, §2, Lemma 8) $D^\alpha u(x) = 0$ in a pointwise sense on the boundary of Ω. Since $u(x)$ is a weak solution of (1), we obtain from (3), by integration by parts in (2), that

$$\int_\Omega [Au(x) - \lambda f[u](x)]\overline{v(x)}\, dx = 0$$

for all functions $v(x) \in C_0^\infty(\Omega)$. Since the term in the square brackets is continuous, it vanishes identically in Ω. Thus $u(x)$ is a classical solution of (1).

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