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RESEARCH ARTICLE

Boolean lattices in finite alternating and symmetric groups

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Abstract

Given a group G and a subgroup H, we let $\mathcal{O}_G(H)$ denote the lattice of subgroups of G containing H. This article provides a classification of the subgroups H of G such that $\mathcal{O}_G(H)$ is Boolean of rank at least 3 when G is a finite alternating or symmetric group. Besides some sporadic examples and some twisted versions, there are two different types of such lattices. One type arises by taking stabilisers of chains of regular partitions, and the other arises by taking stabilisers of chains of regular product structures. As an application, we prove in this case a conjecture on Boolean overgroup lattices related to the dual Ore's theorem and to a problem of Kenneth Brown.

Contents

1. Introduction

Let G be a finite group, H be a subgroup of G and

$$\mathcal{O}_G(H) := \{K \mid K \text{ subgroup of } G \text{ with } H \leq K\}$$

be the set of subgroups of G containing H. Clearly, $\mathcal{O}_G(H)$ is a lattice under the operations of taking the intersection and taking the subgroup generated; it is called the *overgroup lattice*. The problem of determining whether every finite lattice is isomorphic to some $\mathcal{O}_G(H)$ with G finite arose originally in universal algebra, with the work of Pálfy and Pudlák [25]. In 1938, Ore proved that a finite group is cyclic if and only if its subgroup lattice is distributive [20, Theorem 4], and he extended one way as follows: let G be a finite group and let H be a subgroup of G such that the overgroup lattice $\mathcal{O}_G(H)$ is distributive; then there exists a coset Hg generating G [20, Theorem 7]. Eighty years later, the third author extended Ore's theorem to subfactor planar algebras [23, 24]. Consequently, he obtained a dual version of Ore's theorem in [21]; more precisely, he proved that if $\mathcal{O}_G(H)$ is distributive, then there exists an irreducible complex representation V of G such that $G_{(V^H)} = H$ (where V^H is the H-fixed-points subspace of V, and $G_{(X)}$ is the pointwise stabiliser of X in G). Another way to prove this application (explored with Balodi in [4]) is to show that the dual Euler totient is nonzero.

Let us explain what this means. Let G be a finite group; the Euler totient of G, $\varphi(G)$, is the number of elements g such that $\langle g \rangle = G$. Then $\varphi(G)$ is nonzero if and only if G is cyclic, and when $G = C_n$ is the cyclic group of order n, $\varphi(C_n)$ coincides with the usual Euler's totient function $\varphi(n)$. For a subgroup $H \subset G$, the Euler totient $\varphi(H, G)$ is the number of cosets Hg such that $\langle Hg \rangle = G$. Hall [13] described $\varphi(H, G)$ in terms of the Möbius function μ on the overgroup lattice $\mathcal{O}_G(H)$; precisely he showed that

$$\varphi(H, G) = \sum_{K \in \mathcal{O}_G(H)} \mu(K, G)|K : H|.$$

Note that $\varphi(H, G)$ is nonzero (if and) only if there is a coset Hg generating G. Again, that was extended to subfactor planar algebras [22], and as a consequence it was proved that for any subgroup $H \subset G$, if the *dual Euler totient*

$$\hat{\varphi}(H, G) := \sum_{K \in \mathcal{O}_G(H)} \mu(H, K)|G : K|$$

is nonzero, then there is an irreducible complex representation V such that $G_{(V^H)} = H$ (in particular, if $\hat{\varphi}(G) := \hat{\varphi}(1, G)$ is nonzero, then G is *linearly primitive*, i.e., it admits a faithful irreducible complex representation). So the dual Ore's theorem appears as a natural consequence of the following conjecture:

Conjecture 1.1 (see Conjecture 1.5 in [4]). *If $\mathcal{O}_G(H)$ is Boolean, then $\hat{\varphi}(H, G)$ is nonzero.*

Moreover, in [4, page 58], the authors asked whether the lower bound $\hat{\varphi}(H, G) \geq 2^\ell$ holds when $\mathcal{O}_G(H)$ is Boolean of rank $\ell + 1$. As they pointed out, if the lower bound is correct, then it is optimal, because $\hat{\varphi}(S_1 \times S_2^\ell, S_2 \times S_3^\ell) = 2^\ell$. To highlight previous progress on this context, let us consider the reduced Euler characteristic,

$$\chi(H, G) = -\sum_{K \in \mathcal{O}_G(H)} \mu(K, G)|G : K|,$$

which is an invariant related to $\hat{\varphi}(H, G)$ in the sense that when $K \in \mathcal{O}_G(H)$, and $\mathcal{O}_G(H)$ is Boolean of rank ℓ, then $\mu(K, G) = (-1)^\ell \mu(H, K)$, so that $\chi(H, G) = \pm\hat{\varphi}(H, G)$. It follows that Conjecture 1.1 reduces to investigation of the nonvanishing of $\chi(H, G)$. The problem of studying whether $\chi(G) := \chi(1, G)$ is nonzero for every finite group G is mentioned as open in [32, page 760] and attributed to Brown. It was first approached by Gaschütz, who showed in [11] that $\chi(G) \neq 0$ when G is a solvable group. Patassini proved $\chi(G) \neq 0$ for many almost simple groups G in [26, 27], and obtained further results for some groups with minimal normal subgroups that are products of alternating groups in [28].

Let $\Delta(C(G))$ be the order complex of the poset of all cosets of all proper subgroups of G, ordered by inclusion. It is still unknown whether $\chi(G)$ is nonzero for every finite group G. The weaker question of whether $\Delta(C(G))$ can be contractible, asked by Brown in [7], has a negative answer, as shown by Shareshian and Woodroofe in [32].

A first step to attack Conjecture 1.1 could be to prove the case where G is a finite simple group; hence, as a preliminary aim one should try to classify the inclusions $H \subset G$ with G finite simple and $\mathcal{O}_G(H)$ Boolean. In [4, Example 4.21] it is noticed that if H is the Borel subgroup of a BN-pair structure (of rank ℓ) on G, then $\mathcal{O}_G(H)$ is Boolean (of rank ℓ) and $\chi(H, G)$ is nonzero. Moreover, if G is a finite simple group of Lie type (over a finite field of characteristic p), then its absolute value $\hat{\varphi}(H, G)$ is the p-contribution in the order of G, which is at least $p^{\frac{1}{2}\ell(\ell+1)}$. Does the BN-pair structure cover everything at rank at least 3, or large enough? Shareshian guessed no by suggesting examples of any rank when G is the alternating group, involving stabilisers of nontrivial regular partitions, as shown in [3] for rank 2.

This article proves the existence of these examples for G alternating (or symmetric), but mainly proves that (besides some sporadic cases) there is just one other infinite family of examples, arising from stabilisers of regular product structures. As a consequence, we can prove Conjecture 1.1 in this case (together with the expected lower bound). We consider the case of an almost simple group G with socle an alternating group $\mathrm{Alt}(n)$, for some $n \in \mathbb{N}$. When $n \leq 5$, nothing interesting happens: the largest Boolean lattice of the form $\mathcal{O}_G(H)$ has rank at most 1. Moreover, since the case $n = 6$ is rather special, we deal with it separately. When $G = \mathrm{Alt}(6)$, the largest Boolean lattice has rank 2 and is of the form $(D_4, \mathrm{Sym}(4), \mathrm{Sym}(4))$ or $(D_5, \mathrm{Alt}(5), \mathrm{Alt}(5))$. When G is $\mathrm{PGL}_2(9)$, M_{10} or $\mathrm{P\Gamma L}_2(9)$, the largest Boolean lattice has rank 1. When $G = \mathrm{Sym}(6) \cong \mathrm{P\Sigma L}_2(9)$, the largest Boolean lattice has rank 2 and is of the form $(D_4 \times C_2, 2.\mathrm{Sym}(4), 2.\mathrm{Sym}(4))$ or $(C_5 \rtimes C_4, \mathrm{Sym}(5), \mathrm{Sym}(5))$.

For the rest of the argument, we may suppose $n \neq 6$, and hence for the rest of this article we assume that G is either $\mathrm{Alt}(\Omega)$ or $\mathrm{Sym}(\Omega)$, for some finite set Ω. The following theorem contains terms which are defined in Section 2:

Theorem 1.2. *Let Ω be a finite set, let G be $\mathrm{Alt}(\Omega)$ or $\mathrm{Sym}(\Omega)$, let H be a subgroup of G and suppose that the lattice $\mathcal{O}_G(H) = \{K \mid H \leq K \leq G\}$ is Boolean of rank $\ell \geq 3$. Let $G_1, \dots, G_\ell$ be the maximal elements of $\mathcal{O}_G(H)$. Then one of the following holds:*

1. *For every $i \in \{1, \dots, \ell\}$, there exists a nontrivial regular partition Σ_i with $G_i = \mathbf{N}_G(\Sigma_i)$; moreover, relabelling the index set $\{1, \dots, \ell\}$ if necessary, $\Sigma_1 < \cdots < \Sigma_\ell$.*
2. *$G = \mathrm{Sym}(\Omega)$. Relabelling the index set $\{1, \dots, \ell\}$ if necessary, $G_\ell = \mathrm{Alt}(\Omega)$ and for every $i \in \{1, \dots, \ell - 1\}$, there exists a nontrivial regular partition Σ_i with $G_i = \mathbf{N}_G(\Sigma_i)$; moreover, relabelling the index set $\{1, \dots, \ell - 1\}$ if necessary, $\Sigma_1 < \cdots < \Sigma_{\ell-1}$.*
3. *$|\Omega|$ is odd. For every $i \in \{1, \dots, \ell\}$, there exists a nontrivial regular product structure $\mathcal{F}_i$ with $G_i = \mathbf{N}_G(\mathcal{F}_i)$; moreover, relabelling the index set $\{1, \dots, \ell\}$ if necessary, $\mathcal{F}_1 < \cdots < \mathcal{F}_\ell$.*
4. *$|\Omega|$ is odd and $G = \mathrm{Sym}(\Omega)$. Relabelling the index set $\{1, \dots, \ell\}$ if necessary, $G_\ell = \mathrm{Alt}(\Omega)$ and for every $i \in \{1, \dots, \ell - 1\}$, there exists a nontrivial regular product structure $\mathcal{F}_i$ with $G_i = \mathbf{N}_G(\mathcal{F}_i)$; moreover, relabelling the index set $\{1, \dots, \ell - 1\}$ if necessary, $\mathcal{F}_1 < \cdots < \mathcal{F}_{\ell-1}$.*
5. *$|\Omega|$ is an odd prime power. Relabelling the index set $\{1, \dots, \ell\}$ if necessary, G_ℓ is a maximal subgroup of O'Nan–Scott type HA and for every $i \in \{1, \dots, \ell - 1\}$, there exists a nontrivial regular product structure $\mathcal{F}_i$ with $G_i = \mathbf{N}_G(\mathcal{F}_i)$; moreover, relabelling the index set $\{1, \dots, \ell - 1\}$ if necessary, $\mathcal{F}_1 < \cdots < \mathcal{F}_{\ell-1}$.*
6. *$|\Omega|$ is an odd prime power and $G = \mathrm{Sym}(\Omega)$. Relabelling the index set $\{1, \dots, \ell\}$ if necessary, $G_\ell = \mathrm{Alt}(\Omega)$, $G_{\ell-1}$ is a maximal subgroup of O'Nan-Scott type HA and for every $i \in \{1, \dots, \ell-2\}$, there exists a nontrivial regular product structure $\mathcal{F}_i$ with $G_i = \mathbf{N}_G(\mathcal{F}_i)$; moreover, relabelling the index set $\{1, \dots, \ell - 2\}$ if necessary, $\mathcal{F}_1 < \cdots < \mathcal{F}_{\ell-2}$.*
7. *$\ell = 3$, $G = \mathrm{Sym}(\Omega)$ and, relabelling the index set $\{1, 2, 3\}$ if necessary, G_1 is the stabiliser of a subset Γ of Ω with $1 \leq |\Gamma| < |\Omega|/2$ and G_2 is the stabiliser of a nontrivial regular partition Σ with $\Gamma \in \Sigma$ and $G_3 = \mathrm{Alt}(\Omega)$.*

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8. $\ell = 3$, $G = \mathrm{Sym}(\Omega)$ *and, relabelling the index set* $\{1, 2, 3\}$ *if necessary,* G_1 *is the stabiliser of a subset* Γ *of* Ω *with* $|\Gamma| = 1$, $G_2 \cong \mathrm{PGL}_2(p)$ *for some prime number* p, $|\Omega| = p + 1$ *and* $G_3 = \mathrm{Alt}(\Omega)$.
9. $\ell = 3$, $G = \mathrm{Alt}(\Omega)$, $|\Omega| = 8$ *and the Boolean lattice* $\mathcal{O}_G(H)$ *is shown in Figure 1.*
10. $\ell = 3$, $G = \mathrm{Alt}(\Omega)$, $|\Omega| = 24$ *and, relabelling the index set* $\{1, 2, 3\}$ *if necessary,* G_1 *is the stabiliser of a subset* Γ *of* Ω *with* $|\Gamma| = 1$ *and* $G_2 \cong G_3 \cong M_{24}$.

In Section 8, we show that cases 1 and 2 in Theorem 1.2 do occur for arbitrary values of ℓ. In Section 9, we show that there exist Boolean lattices of arbitary large rank whose maximal elements are stabilisers of regular product structures.

Finally, Section 10 is dedicated to the proof of the following theorem, where 4 is a consequence of Theorem 1.2 and the proof for 5 has already been mentioned:

Theorem 1.3. *Let G be a finite group and H a subgroup such that the overgroup lattice $\mathcal{O}_G(H)$ is Boolean of rank ℓ. Then the lower bound on the dual Euler totient $\hat{\varphi}(H, G) \geq 2^{\ell-1}$ holds in each of the following cases:*

1. $\ell \leq 3$.
2. $\mathcal{O}_G(H)$ *is group-complemented.*
3. G *is solvable.*
4. G *is alternating or symmetric.*
5. G *is of Lie type and H is a Borel subgroup.*

As a consequence, the reduced Euler characteristic $\chi(H, G)$ is nonzero, that is, there is a positive answer to the relative Brown's problem in these cases.

2. Notation, terminology and basic facts

Since we need fundamental results from the work of Aschbacher [1, 2], we follow the notation and the terminology therein. Let G be the finite alternating group $\mathrm{Alt}(\Omega)$ or the finite symmetric group $\mathrm{Sym}(\Omega)$, where Ω is a finite set of cardinality $n \in \mathbb{N}$. Given a subgroup H of G, we write

$$\mathcal{O}_G(H) := \{K \mid H \leq K \leq G\}$$

for the set of subgroups of G containing H. We set

$$\mathcal{O}_G(H)' := \mathcal{O}_G(H) \setminus \{H, G\},$$

that is, $\mathcal{O}_G(H)'$ consists of the lattice $\mathcal{O}_G(H)$ with its minimum and maximum elements removed. (Given a group X, we denote by $\mathbf{F}^*(X)$ the *generalised Fitting subgroup of* X. Observe that when X is a primitive subgroup of $\mathrm{Sym}(\Omega)$, $\mathbf{F}^*(X)$ coincides with the *socle of* X.) We write

$$\mathcal{O}_G(H)'' := \{M \in \mathcal{O}_G(H) \mid \mathbf{F}^*(G) \not\leq M\}$$

and we denote by $\mathcal{M}_G(H)$ *the set of maximal members of* $\mathcal{O}_G(H)''$. We start by familiarising the reader with this terminology.

- When $G = \mathrm{Alt}(\Omega)$, $\mathbf{F}^*(G) = G$, and hence $\mathcal{O}_G(H)''$ is simply $\mathcal{O}_G(H)$ with its maximum element $G = \mathrm{Alt}(\Omega)$ removed. Therefore, $\mathcal{M}_G(H)$ consists of the maximal subgroups of $G = \mathrm{Alt}(\Omega)$ containing H.
- When $G = \mathrm{Sym}(\Omega)$ and $\mathrm{Alt}(\Omega) \not\leq H$, $\mathcal{O}_G(H)''$ is obtained from $\mathcal{O}_G(H)$ by removing $G = \mathrm{Sym}(\Omega)$ only, because if $M \in \mathcal{O}_G(H)$ and $\mathrm{Alt}(\Omega) = \mathbf{F}^*(G) \leq M$, then $\mathrm{Sym}(\Omega) = H\mathbf{F}^*(G) \leq M$ and $M = \mathrm{Sym}(\Omega)$. Therefore, in this case $\mathcal{M}_G(H)$ consists simply of the maximal subgroups of $G = \mathrm{Sym}(\Omega)$ containing H.

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- When $G = \mathrm{Sym}(\Omega)$ and $H \le \mathrm{Alt}(\Omega)$, $\mathcal{O}_G(H)''$ is obtained from $\mathcal{O}_G(H)$ by removing $\mathrm{Sym}(\Omega)$ and $\mathrm{Alt}(\Omega)$. Therefore, $\mathcal{M}_G(H)$ consists of two types of subgroups: the maximal subgroups of $G = \mathrm{Sym}(\Omega)$ containing H and the maximal subgroups M of $\mathrm{Alt}(\Omega)$ containing H that are not contained in any maximal subgroup of $\mathrm{Sym}(\Omega)$ other than $\mathrm{Alt}(\Omega)$. For instance, when $H := M_{12}$ in its transitive action of degree 12, we have $H \le \mathrm{Alt}(12)$, $\mathcal{O}_{\mathrm{Sym}(\Omega)}(M_{12}) = \{M_{12}, \mathrm{Alt}(12), \mathrm{Sym}(12)\}$, $\mathcal{O}_{\mathrm{Sym}(12)}(H)' = \{\mathrm{Alt}(12)\}$, $\mathcal{O}_{\mathrm{Sym}(12)}(M_{12})'' = \{M_{12}\}$ and $\mathcal{M}_{\mathrm{Sym}(12)}(M_{12}) = \{M_{12}\}$.

Some of the material that follows can be traced back to [1, 2] or [16, 29]. However, we prefer to repeat it here because it helps to set some more notation and terminology. Using the action of $\mathrm{Sym}(\Omega)$ on the domain Ω, we can divide the subgroups X of $\mathrm{Sym}(\Omega)$ into three classes:

Intransitive X is *intransitive* on Ω.
Imprimitive X is *imprimitive* on Ω; that is, X is transitive on Ω but not primitive on Ω.
Primitive X is *primitive* on Ω.

In particular, every maximal subgroup M of G can be referred to as intransitive, imprimitive or primitive according to this division.

In what follows we need detailed information on the overgroups of a primitive subgroup of G. This information was obtained independently by Aschbacher [1, 2] and Liebeck, Praeger and Saxl [16, 29]. Both investigations are important in what follows.

2.1. *Intransitive subgroups*

A maximal subgroup M of G is intransitive if and only if M is the stabiliser in G of a subset Γ of Ω with $1 \le |\Gamma| < |\Omega|/2$ (see, e.g., [16]); that is,

$$M = G \cap (\mathrm{Sym}(\Gamma) \times \mathrm{Sym}(\Omega \setminus \Gamma)).$$

Following [1, 2], we let $\mathbf{N}_G(\Gamma)$ denote the setwise stabiliser of Γ in G; that is,

$$\mathbf{N}_G(\Gamma) := \{g \in G \mid \gamma^g \in \Gamma, \forall \gamma \in \Gamma\}.$$

More generally, given a subgroup H of G, we let $\mathbf{N}_H(\Gamma) = \mathbf{N}_G(\Gamma) \cap H$ denote the setwise stabiliser of Γ in H.

The case $|\Gamma| = |\Omega|/2$ is special because $\mathbf{N}_G(\Gamma)$ is not maximal. Indeed, $\mathbf{N}_G(\Gamma)$ is a subgroup of the stabiliser in G of the partition $\{\Gamma, \Omega \setminus \Gamma\}$. This is an imprimitive group, which we analyse in the next subsection.

Summing up, we have the following fact:

Fact 2.1. Let Γ be a subset of Ω with $1 \le |\Gamma| < |\Omega|/2$. Then the intransitive subgroup $\mathbf{N}_G(\Gamma)$ of G is a maximal subgroup of G. Moreover, every intransitive maximal subgroup of G is of this form.

2.2. *Regular partitions and imprimitive subgroups*

The collection of all partitions of Ω is a poset, with the reverse refinement order: given two partitions Σ_1 and Σ_2 of Ω, we say that $\Sigma_1 \le \Sigma_2$ if Σ_2 is a *refinement of* Σ_1; that is, every element in Σ_1 is a union of elements in Σ_2. For instance, when $\Omega := \{1,2,3,4\}$, $\Sigma_1 := \{\{1,3,4\},\{2\}\}$ and $\Sigma_2 := \{\{1\},\{2\},\{3,4\}\}$, we have $\Sigma_1 \le \Sigma_2$.

A partition Σ of Ω is said to be *regular* or *uniform* if all parts in Σ have the same cardinality. Following [1, 2], we say that the partition Σ is an (a,b)*-regular partition* if Σ consists of b parts each having cardinality a. In particular, $n = |\Omega| = ab$.

A partition Σ of Ω is said to be *trivial* if Σ equals the universal relation $\Sigma = \{\Omega\}$ or the equality relation $\Sigma = \{\{\omega\} \mid \omega \in \Omega\}$.

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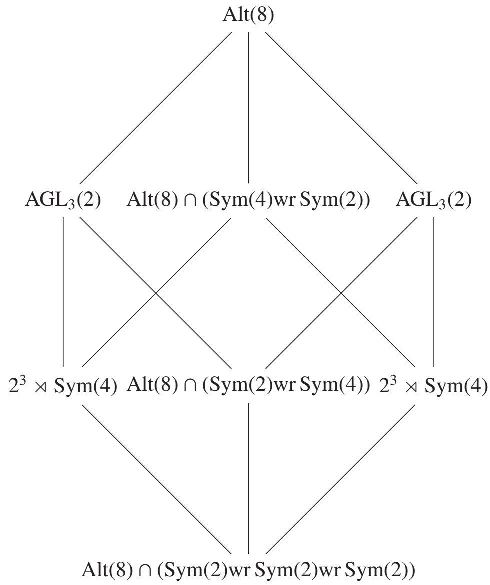

Figure 1. *The Boolean lattice of largest cardinality in* $\mathrm{Alt}(8)$.

We let

$$\mathbf{N}_G(\Sigma) := \{g \in G \mid \Gamma^g \in \Sigma, \forall \Gamma \in \Sigma\}$$

denote the stabiliser in G of the partition Σ. Moreover, when H is a subgroup of G, we write $\mathbf{N}_H(\Sigma) := \mathbf{N}_G(\Gamma) \cap H$.

Let M be a maximal subgroup of G. If M is imprimitive, then M is the stabiliser in G of a nontrivial regular partition. Therefore, there exists an (a, b)-regular partition Σ with $a, b \geq 2$ and with $M = \mathbf{N}_G(\Sigma)$. From [16, 29], we see that when $G = \mathrm{Sym}(\Omega)$ the converse is also true. That is, for every nontrivial (a, b)-regular partition Σ, the subgroup $\mathbf{N}_G(\Sigma)$ is a maximal subgroup of $\mathrm{Sym}(\Omega)$. When $G = \mathrm{Alt}(\Omega)$, the converse is not quite true in general. We summarise what we need in the following fact:

Fact 2.2. Let Σ be a nontrivial regular partition of Ω. Except when $G = \mathrm{Alt}(\Omega)$, $|\Omega| = 8$ and Σ is a $(2, 4)$-regular partition, the imprimitive subgroup $\mathbf{N}_G(\Sigma)$ of G is a maximal subgroup of G.

The case where $G = \mathrm{Alt}(\Omega)$, $|\Omega| = 8$ and Σ is a $(2, 4)$-regular partition is a genuine exception here. Indeed, $\mathbf{N}_G(\Sigma) < \mathrm{AGL}_3(2) < \mathrm{Alt}(\Omega)$, where $\mathrm{AGL}_3(2)$ is the affine general linear group of degree $2^3 = 8$. (This was already observed in [16].) The case $G = \mathrm{Alt}(\Omega)$ and $n = 8$ is combinatorially very interesting: the largest Boolean lattice in $\mathrm{Alt}(8)$ has rank 3 and is shown in Figure 1.

2.3. Regular product structures and primitive subgroups

The modern key for analysing a finite primitive permutation group L is to study the *socle* N of L, that is, the subgroup generated by the minimal normal subgroups of L. The socle of an arbitrary finite group is isomorphic to the nontrivial direct product of simple groups; moreover, for finite primitive groups these simple groups are pairwise isomorphic. The O'Nan–Scott theorem describes in detail the embedding of N in L and collects some useful information about the action of L. In [17, Theorem], five types of primitive groups are defined (depending on the group structure and action structure of the socle) – namely *affine-type* (HA), *almost simple* (AS), *diagonal-type*, *product-type* and the *twisted wreath product* – and it is shown that every primitive group belongs to exactly one of these types. In [29] this division into types was refined further, with the diagonal-type partitioned in *holomorphic simple*

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(HS) and *simple diagonal* (SD), and the product-type into *holomorphic compound* (HC), *compound diagonal* (CD) and *product action* (PA).

It follows from the results in [16, 29] that if M is a maximal subgroup of G and M is primitive, then M has O'Nan–Scott type HA, AS, SD or PA.

Since an overgroup of a primitive group is still primitive, the analogue of Facts 2.1 and 2.2 is obvious:

Fact 2.3. A primitive subgroup M of G is maximal if and only if M is maximal among the primitive subgroups of G.

We recall the definition of a *regular product structure* on Ω from [2, Section 2]. Let m and k be integers with $m \geq 5$ and $k \geq 2$. There are two natural ways to give this definition. First, a regular (m, k)-product structure on Ω is a bijection $f : \Omega \to \Gamma^I$, where $I := \{1, \ldots, k\}$ and Γ is an m-set. The function f consists of a family of functions $(f_i : \Omega \to \Gamma \mid i \in I)$ where $f(\omega) = (f_1(\omega), \ldots, f_k(\omega))$ for each $\omega \in \Omega$. There is a more intrinsic way to define it. Let $\mathcal{F} := \{\Omega_i \mid i \in I\}$ be a set of partitions Ω_i of Ω into m blocks of size m^{k-1}, let $[\omega]_i$ be the block of Ω_i containing the point ω and let $\mathcal{F}(\omega) := \{[\omega]_i \mid i \in I\}$. The set $\mathcal{F}$ is a product structure if, for each pair of distinct points $\omega, \omega' \in \Omega$, we have $\mathcal{F}(\omega) \neq \mathcal{F}(\omega')$. Clearly the two definitions are equivalent. Indeed, given a function $f : \Omega \to \Gamma^I$, we let $\mathcal{F}(f)$ be the set of partitions of Ω defined by f, where the ith partition $\Omega_i := \{f_i^{-1}(\gamma) \mid \gamma \in \Gamma\}$ consists of the fibers of f_i. The product structure $\mathcal{F}$ can also be regarded as a chamber system in the sense of Tits [33].

Following [1], we let $\mathbf{N}_G(\mathcal{F})$ denote the stabiliser of a regular (m, k)-product structure $\mathcal{F} = \{\Omega_1, \ldots, \Omega_k\}$ in G, that is,

$$\mathbf{N}_G(\mathcal{F}) := \{g \in G \mid \Omega_i^g \in \mathcal{F}, \forall i \in \{1, \ldots, k\}\}.$$

(More generally, given a subgroup H of G, we let $\mathbf{N}_H(\mathcal{F}) := \mathbf{N}_G(\mathcal{F}) \cap H$ denote the stabiliser of Γ in H.) Clearly,

$$\mathbf{N}_{\mathrm{Sym}(\Omega)}(\mathcal{F}) \cong \mathrm{Sym}(m) \mathrm{wr} \,\mathrm{Sym}(k),$$

where $\mathrm{Sym}(m)\mathrm{wr}\,\mathrm{Sym}(k)$ is endowed of its primitive product action of degree m^k. Moreover, $\mathbf{N}_{\mathrm{Sym}(\Omega)}(\mathcal{F})$ is a typical primitive maximal subgroup of $\mathrm{Sym}(\Omega)$ of PA type according to the O'Nan–Scott theorem.

Let $\mathcal{F}(\Omega)$ be the set of all regular product structures on Ω. The set $\mathcal{F}(\Omega)$ is endowed of a natural partial order. Let $\mathcal{F} := \{\Omega_i \mid i \in I\}$ and $\tilde{\mathcal{F}} := \{\tilde{\Omega}_j \mid j \in \tilde{I}\}$ be regular (m, k)- and $(\tilde{m}, \tilde{k})$-product structures on Ω, respectively. Set $I := \{1, \ldots, k\}$ and $\tilde{I} := \{1, \ldots, \tilde{k}\}$, and define $\mathcal{F} \leq \tilde{\mathcal{F}}$ if there exists a positive integer s with $\tilde{k} = ks$, and a regular (s, k)-partition $\Sigma = \{\sigma_i \mid i \in I\}$ of $\tilde{I}$, such that for each $i \in I$ and each $j \in \sigma_i$, $\tilde{\Omega}_j \leq \Omega_i$ – that is, the partition Ω_i is a refinement of the partition $\tilde{\Omega}_j$. From [1, (5.1)], the relation $\leq$ is a partial order on $\mathcal{F}(\Omega)$.

We conclude these preliminary observations on regular product structures by recalling [2, (5.10)].

Lemma 2.4. *Let $M = \mathbf{N}_{\mathrm{Sym}(\Omega)}(\mathcal{F})$ be the stabiliser in $\mathrm{Sym}(\Omega)$ of a regular (m, k)-product structure on Ω and let K be the kernel of the action of M on $\mathcal{F}$. Then*

1. *$K \leq \mathrm{Alt}(\Omega)$ if and only if m is even;*
2. *$M \leq \mathrm{Alt}(\Omega)$ if and only if m is even and either $k > 2$ or $k = 2$ and $m \equiv 0 \pmod 4$;*
3. *if $k = 2$ and $m \equiv 2 \pmod 4$, then $M \cap \mathrm{Alt}(\Omega) = K$, so $M \cap \mathrm{Alt}(\Omega)$ is not primitive on Ω (and hence $M \cap \mathrm{Alt}(\Omega)$ is not a maximal subgroup of $\mathrm{Alt}(\Omega)$). Otherwise, $M \cap \mathrm{Alt}(\Omega)$ induces $\mathrm{Sym}(\mathcal{F})$ on $\mathcal{F}$.*

2.4. Preliminary lemmas

A lattice $\mathcal{L}$ is said to be *Boolean* if it is isomorphic to the lattice of subsets of a set X; that is, $\mathcal{L} \cong \mathcal{P}(X)$, where $\mathcal{P}(X) := \{Y \mid Y \subseteq X\}$. We also say that $|X|$ is the *rank* of the Boolean lattice $\mathcal{L}$.

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Lemma 2.5. *Let X be a subgroup of Y. If $\mathcal{O}_Y(X)$ is Boolean of rank ℓ, then every maximal chain from X to Y has length ℓ. In particular, if $|Y : X|$ is divisible by at most ℓ primes (counting these primes with multiplicity), then $\mathcal{O}_Y(X)$ is not Boolean of rank ℓ.*

Proof. This is clear. □

Lemma 2.6. *Let H be a subgroup of G with $\mathcal{O}_G(H)$ Boolean. If every maximal element in $\mathcal{O}_G(H)$ is transitive, then either H is transitive or $\mathcal{O}_G(H)$ contains the stabiliser of a $(|\Omega|/2, 2)$-regular partition.*

Proof. Suppose that H is intransitive and let Γ be an orbit of H of smallest possible cardinality. Assume $1 \leq |\Gamma| < |\Omega|/2$. Then $M := G \cap (\mathrm{Sym}(\Gamma) \times \mathrm{Sym}(\Omega \setminus \Gamma))$ is a maximal element of $\mathcal{O}_G(H)$ and M is intransitive, which is a contradiction. This shows that H has two orbits on Ω, both having cardinality $|\Omega|/2$. In particular, $M := \mathbf{N}_G(\{\Gamma, \Omega \setminus \Gamma\})$ is a member of $\mathcal{O}_G(H)$. □

Lemma 2.7. *Let H be a subgroup of G with $\mathcal{O}_G(H)$ Boolean. If every maximal element in $\mathcal{O}_G(H)$ is primitive, then either H is primitive or $G = \mathrm{Alt}(\Omega)$, $|\Omega| = 8$, $H = \mathbf{N}_G(\Sigma)$ for some $(2, 4)$-regular partition Σ and $\mathcal{O}_G(H)$ has rank 2.*

Proof. From Lemma 2.6, H is transitive. Suppose that H is imprimitive and let Σ be a nontrivial regular partition with $H \leq \mathbf{N}_G(\Sigma)$. If $\mathbf{N}_G(\Sigma)$ is a maximal subgroup of G, we obtain a contradiction. Thus $\mathbf{N}_G(\Sigma)$ is not maximal in G. This implies that $G = \mathrm{Alt}(\Omega)$, $|\Omega| = 8$, Σ is a $(2, 4)$-regular partition and $\mathcal{O}_G(H)$ has rank 2 (see Fact 2.2 and Figure 1). □

Lemma 2.8 is needed in Remark 3.2, and Lemma 2.9 is needed in Theorem 6.2.

Lemma 2.8. *Let Ω be the set of all 2-sets from a finite set Δ. Then in the permutation representation of* $\mathrm{Sym}(\Delta)$ *on* Ω*,* $\mathrm{Sym}(\Delta) \leq \mathrm{Alt}(\Omega)$ *if and only if $|\Delta|$ is even.*

Proof. It is an easy computation to see that if g is a transposition of $\mathrm{Sym}(\Delta)$ (for its action on Δ), then it is an even permutation in its action on Ω if and only if $|\Delta|$ is even. Therefore, the proof follows. □

Lemma 2.9. *Let H be a transitive permutation group on Ω, let $\omega \in \Omega$ and let H_ω be the stabiliser of the point ω in H. Then $\{\omega' \in \Omega \mid \omega'^g = \omega', \forall g \in H_\omega\}$ is a block of imprimitivity for H. In particular, if H is primitive, then either $H_\omega = 1$ or ω is the only point fixed by H_ω.*

Proof. This is an exercise (see [10, Exercise 1.6.5, page 19]). □

3. Results for almost-simple groups

In this section we collect some results from [1, 2] on primitive groups. Our ultimate goal is deducing some structural results on Boolean lattices $\mathcal{O}_G(H)$ when H is an almost-simple primitive group

We start with a rather technical result of Aschbacher on the overgroups of a primitive group which is *product indecomposable* and not *octal*. We prefer to give only a broad description of these concepts here, referring interesteds reader to [1, 2]. These deep results have already played an important role in algebraic combinatorics; for instance, they are the key results for proving that most primitive groups are automorphism groups of edge-transitive hypergraphs [32].

A primitive group $H \leq G$ is said to be *product decomposable* if the domain Ω admits the structure of a Cartesian product (that is, $\Omega \cong \Delta^\ell$ for some finite set Δ and for some $\ell \in \mathbb{N}$ with $\ell \geq 1$) and the group H acts on Ω preserving this Cartesian product structure. We are allowing $\ell = 1$ here, to include the case where H is almost simple. Moreover, for each component L of the socle of H, one of the following holds:

(i) $L \cong \mathrm{Alt}(6)$ and $|\Delta| = 6^2$.
(ii) $L \cong M_{12}$ and $|\Delta| = 12^2$.
(iii) $L \cong \mathrm{Sp}_4(q)$ for some $q > 2$ even and $|\Delta| = (q^2(q^2 - 1)/2)^2$.

We also refer to [30] for a recent thorough investigation on permutation groups admitting Cartesian decompositions, where each of these peculiar examples is thoroughly investigated.

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Following [1, 2], a primitive group H is said to be *octal* if each component L of the socle of H is isomorphic to $\mathrm{PSL}_3(2) \cong \mathrm{PSL}_2(7)$, the orbits of L have order 8 and the action of L on each of its orbits is primitive. For future reference, we report here that a simple computation reveals that when $H = \mathrm{PSL}_3(2)$ is octal, $\mathcal{O}_{\mathrm{Alt}(8)}(H)$ is Boolean of rank 2, whereas $\mathcal{O}_{\mathrm{Sym}(8)}(H)$ is a lattice of size 6.

Theorem 3.1. *[2, Theorem A] Let Ω be a finite set of cardinality n and let H be an almost-simple primitive subgroup of $\mathrm{Sym}(\Omega)$ which is product indecomposable and not octal. Then all members of $\mathcal{O}_{\mathrm{Sym}(\Omega)}(H)$ are almost simple, product indecomposable and not octal, and setting $U := \mathbf{F}^*(H)$, one of the following holds:*

1. $|\mathcal{M}_{\mathrm{Sym}(\Omega)}(H)| = 1$.
2. $U = H$, $|\mathcal{M}_{\mathrm{Sym}(\Omega)}(H)| = 3$, $\mathrm{Aut}(U) \cong \mathbf{N}_{\mathrm{Sym}(\Omega)}(U) \in \mathcal{M}_{\mathrm{Sym}(\Omega)}(U)$, *$\mathbf{N}_{\mathrm{Sym}(\Omega)}(U)$ is transitive on $\mathcal{M}_{\mathrm{Sym}(\Omega)}(H) \setminus \{\mathbf{N}_{\mathrm{Sym}(\Omega)}(U)\}$ and U is maximal in V, where $K \in \mathcal{M}_{\mathrm{Sym}(\Omega)}(H) \setminus \{\mathbf{N}_{\mathrm{Sym}(\Omega)}(U)\}$ and $V = \mathbf{F}^*(K)$. Further, (U, V, n) is one of the following:*
   - ***(a)*** $(HS, \mathrm{Alt}(m), 15, 400)$, *where $m = 176$ and $n = \binom{m}{2}$.*
   - ***(b)*** $(\mathrm{G}_2(3), \Omega_7(3), 3, 159)$.
   - ***(c)*** $(\mathrm{PSL}_2(q), M_n, n)$, *where $q \in \{11, 23\}$, $n = q + 1$ and M_n is the Mathieu group of degree n.*
   - ***(d)*** $(\mathrm{PSL}_2(17), \mathrm{Sp}_8(2), 136)$.
3. $U \cong \mathrm{PSL}_3(4)$, $n = 280$, $|\mathcal{M}_{\mathrm{Sym}(\Omega)}(U)| = 4$, $\mathrm{Aut}(U) \cong \mathbf{N}_{\mathrm{Sym}(\Omega)}(U) \in \mathcal{M}_{\mathrm{Sym}(\Omega)}(U)$, *$\mathbf{N}_{\mathrm{Sym}(\Omega)}(U)$ is transitive on $\mathcal{M}_{\mathrm{Sym}(\Omega)}(U) \setminus \{\mathbf{N}_{\mathrm{Sym}(\Omega)}(U)\}$ and $K \in \mathcal{M}_{\mathrm{Sym}(\Omega)}(H) \setminus \{\mathbf{N}_{\mathrm{Sym}(\Omega)}(U)\}$ is isomorphic to* $\mathrm{Aut}(\mathrm{PSU}_4(3))$.
4. $U \cong \mathrm{Sz}(q)$, $q = 2^k$, $n = q^2(q^2+1)/2$, *$\mathcal{M}_{\mathrm{Sym}(\Omega)}(U) = \{K_1, K_2\}$ where $K_i = \mathbf{N}_{\mathrm{Sym}(\Omega)}(V_i) \cong \mathrm{Aut}(V_i)$, $V_1 \cong \mathrm{Alt}(q^2+1)$, $V_2 \cong \mathrm{Sp}_{4k}(2)$ and $\mathbf{N}_{\mathrm{Sym}(\Omega)}(U) \cong \mathrm{Aut}(U)$ is maximal in V_1.*
5. $H \cong \mathrm{PSL}_2(11)$, $n = 55$, *$\mathrm{PGL}_2(11) \cong \mathbf{N}_{\mathrm{Sym}(\Omega)}(H)$ and $\mathcal{M}_{\mathrm{Sym}(\Omega)}(H) = \{\mathbf{N}_{\mathrm{Sym}(\Omega)}(H), K, K^t\}$, $t \in \mathbf{N}_{\mathrm{Sym}(\Omega)}(H) \setminus H$, where $K \cong \mathrm{Sym}(11)$ and $\mathcal{O}_K(H) = \{H < L < V < K\}$, with $L \cong M_{11}$ and $V \cong \mathrm{Alt}(11)$.*

Remarks 3.2.

1. In case 1, since $\mathcal{M}_G(H)$ contains only one element, we deduce that the lattice $\mathcal{O}_{\mathrm{Sym}(\Omega)}(H)$ is not Boolean unless it has rank 1.
2. In case 2(a), all elements in $\mathcal{M}_{\mathrm{Sym}(\Omega)}(H)$ are maximal subgroups of $\mathrm{Alt}(\Omega)$. This is because the permutation representations of $\mathrm{Aut}(HS) = HS.2$ and of $\mathrm{Sym}(m)$ of degree $\binom{m}{2}$ are the natural permutation representations on the 2-sets of a set of cardinality m. Since $m = 176$ is even, these permutation representations embed in $\mathrm{Alt}(\binom{m}{2}) = \mathrm{Alt}(\Omega)$ (see Lemma 2.8). From this, we deduce that $\mathcal{O}_{\mathrm{Sym}(\Omega)}(H)$ is not Boolean, because $\mathrm{Alt}(\Omega)$ is the only maximal element of $\mathcal{O}_{\mathrm{Sym}(\Omega)}(H)$. When $G = \mathrm{Alt}(\Omega)$, $\mathcal{O}_G(M)$ has three maximal elements, one of which is $\mathrm{Aut}(H) \cong HS.2$. If $\mathcal{O}_G(H)$ is Boolean, then it has rank 3 and hence $\mathcal{O}_{HS.2}(HS)$ is Boolean of rank 2; however, this is a contradiction, because $|\mathrm{Aut}(HS) : HS| = |HS.2 : HS| = 2$ (see Lemma 2.5).

   In case 2(b), the group $\mathrm{Aut}(\Omega_7(3)) \cong \Omega_7(3).2$ has no faithful permutation representations of degree 3,159. Since $|\mathcal{M}_{\mathrm{Sym}(\Omega)}(H)| = 3$, we deduce that $\mathcal{M}_{\mathrm{Sym}(\Omega)}(H)$ contains two subgroups isomorphic to $\Omega_7(3)$ which are contained in $\mathrm{Alt}(\Omega)$ and $\mathrm{Aut}(U) \cong \mathrm{G}_2(3).2$, which is not contained in $\mathrm{Alt}(\Omega)$ (the fact that $\mathrm{G}_2(3).2 \not\leq \mathrm{Alt}(\Omega)$ can be easily verified with the computer algebra system `magma` [6]). When $G = \mathrm{Sym}(\Omega)$, we obtain the result that $\mathcal{O}_G(H)$ is not Boolean. When $G = \mathrm{Alt}(\Omega)$, we are not able to determine whether $\mathcal{O}_G(H)$ is Boolean, but if it is, then it has rank 2 and maximal elements that are two subgroups isomorphic to $\Omega_7(3)$.

   In case 2(c) with $n = 12$, we see that $M_{12}.2$ does not admit a permutation representation of degree 12. Therefore, as before, since $|\mathcal{M}_{\mathrm{Sym}(\Omega)}(H)| = 3$, we deduce that $\mathcal{M}_{\mathrm{Sym}(\Omega)}(H)$ contains two subgroups isomorphic to M_{12} which are contained in $\mathrm{Alt}(\Omega)$ and $\mathrm{Aut}(U) \cong \mathrm{PGL}_2(11)$, which is not contained in $\mathrm{Alt}(\Omega)$. Therefore, $\mathcal{O}_{\mathrm{Sym}(\Omega)}(H)$ is not Boolean. When $G = \mathrm{Alt}(\Omega)$, we have verified with the help of a computer that $\mathcal{O}_G(H)$ is indeed Boolean of rank 2. In case 2(c) with $n = 24$, we see that $\mathrm{Aut}(M_{24}) = M_{24}$. Therefore, since $|\mathcal{M}_{\mathrm{Sym}(\Omega)}(H)| = 3$, we deduce that $\mathcal{M}_{\mathrm{Sym}(\Omega)}(H)$ contains two subgroups isomorphic to M_{24} which are contained in $\mathrm{Alt}(\Omega)$ and $\mathrm{Aut}(U) \cong \mathrm{PGL}_2(23)$, which is not contained in $\mathrm{Alt}(\Omega)$. Therefore, $\mathcal{O}_{\mathrm{Sym}(\Omega)}(H)$ is not Boolean. When $G = \mathrm{Alt}(\Omega)$, we have verified with the help of a computer that $\mathcal{O}_G(H)$ is indeed Boolean of rank 2.

In case 2(d), we see that $\mathrm{Aut}(\mathrm{Sp}_8(2)) = \mathrm{Sp}_8(2)$. Thus, as $|\mathcal{M}_{\mathrm{Sym}(\Omega)}(H)| = 3$, we deduce that $\mathcal{M}_{\mathrm{Sym}(\Omega)}(H)$ contains two subgroups isomorphic to $\mathrm{Sp}_8(2)$ which are contained in $\mathrm{Alt}(\Omega)$ and $\mathrm{Aut}(U) \cong \mathrm{PGL}_2(17)$, which is not contained in $\mathrm{Alt}(\Omega)$. Therefore, $\mathcal{O}_{\mathrm{Sym}(\Omega)}(H)$ is not Boolean. When $G = \mathrm{Alt}(\Omega)$, we are not able to determine whether $\mathcal{O}_G(H)$ is Boolean, but if it is, then it has rank 2.

3. We use a computer to deal with case 3. None of the four elements in $\mathcal{M}_{\mathrm{Sym}(\Omega)}(U)$ is contained in $\mathrm{Alt}(\Omega)$. Therefore, if $\mathcal{O}_{\mathrm{Sym}(\Omega)}(H)$ is Boolean, then it has rank 4. Moreover, the intersection of these four subgroups is H, and we see that $|H : U| = 2$. As $|\mathrm{Aut}(\mathrm{PSL}_3(4)) : \mathrm{PSL}_3(4)| = 12$, we deduce that $|\mathbf{N}_{\mathrm{Sym}(\Omega)}(U) : H| = 6 = 2 \cdot 3$. Therefore, $\mathcal{O}_{\mathbf{N}_{\mathrm{Sym}(\Omega)}(H)}(H)$ cannot be a rank 3 Boolean lattice (see Lemma 2.5), contradicting our assumption that $\mathcal{O}_{\mathrm{Sym}(\Omega)}(H)$ is Boolean. Assume then $G = \mathrm{Alt}(\Omega)$. Define $M_0 := \mathbf{N}_{\mathrm{Alt}(\Omega)}(U)$ and let M_1, M_2, M_3 be the intersections with $\mathrm{Alt}(\Omega)$ of the three maximal subgroups of $\mathrm{Sym}(\Omega)$ isomorphic to $\mathrm{Aut}(\mathrm{PSU}_4(3))$. Assume that $\mathcal{O}_{\mathrm{Alt}(\Omega)}(H)$ is Boolean. If $H < M_0$, then $\mathcal{O}_{\mathrm{Alt}(\Omega)}(H)$ is Boolean of rank 4, and hence $\mathcal{O}_{M_0}(H)$ is Boolean of rank 3. However, this is impossible, because $|M_0 : U| = 6 = 2 \cdot 3$. Therefore $H = M_0 = \mathbf{N}_{\mathrm{Alt}(\Omega)}(U)$. But this is another contradiction, because M_0 is maximal in $\mathrm{Alt}(\Omega)$.
4. In case 4, k is odd and hence H is a subgroup of $\mathrm{Alt}(\Omega)$. The action under consideration arises using the standard 2-transitive action of $\mathrm{Sz}(q)$ of degree $q^2 + 1$. Here, the action of degree $q^2(q^2 + 1)/2$ is the action on the 2-sets of the set $\{1, \ldots, q^2 + 1\}$. Here $K_1 \not\leq \mathrm{Alt}(\Omega)$, because $q^2 + 1$ is odd (see Lemma 2.8). Moreover, $\mathrm{Aut}(\mathrm{Sp}_{4k}(2)) = \mathrm{Sp}_{2k}(2)$ and $K_2 = V_2$, and hence $V_2 \leq \mathrm{Alt}(\Omega)$. From this we deduce that the maximal elements in $\mathcal{O}_{\mathrm{Sym}(\Omega)}(H)$ are $K_1 \cong \mathrm{Sym}(q^2 + 1)$ and $\mathrm{Alt}(\Omega)$. However, this lattice is not Boolean, because $H \neq K_1 \cap \mathrm{Alt}(\Omega) = V_1 \cong \mathrm{Alt}(q^2 + 1)$. When $G = \mathrm{Alt}(\Omega)$, the maximal elements in $\mathcal{O}_G(H)$ are $V_1 \cong \mathrm{Alt}(q^2 + 1)$ and $V_2 \cong \mathrm{Sp}_{4k}(2)$. Therefore, if $\mathcal{O}_G(H)$ is Boolean, then its rank is 2.
5. In case 5, $\mathcal{O}_{\mathrm{Sym}(\Omega)}(H)$ is not Boolean because $\mathcal{O}_K(H)$ is not Boolean. When $G = \mathrm{Alt}(\Omega)$, $\mathcal{O}_{\mathrm{Alt}(\Omega)}(H)$ contains two maximal elements V and V^t both isomorphic to $\mathrm{Alt}(11)$. Therefore, if $\mathcal{O}_{\mathrm{Alt}(11)}(H)$ were Boolean, $\mathcal{O}_{\mathrm{Alt}(\Omega)}(H)$ would have rank 2. But this is not the case, because $\mathcal{O}_V(H) = \{H < M < V\}$ and $\mathcal{O}_{V^t} = \{H < M^t < V^t\}$ with $M \cong M^t \cong M_{11}$. Therefore, $\mathcal{O}_{\mathrm{Alt}(\Omega)}(H)$ is not Boolean.

Corollary 3.3. *Let H be an almost-simple primitive subgroup of G which is product indecomposable and not octal. If $\mathcal{O}_G(H)$ is Boolean, then it has rank at most* 2.

Proof. This follows from Theorem 3.1 and Remarks 3.2. □

4. Boolean intervals $\mathcal{O}_G(H)$ with H primitive

Lemma 4.1. *Let M be a maximal subgroup of G of O'Nan–Scott type* SD *and let H be a maximal subgroup of M acting primitively on Ω. Then M and H have the same socle.*

Proof. This follows from [29, Theorem] (using the notation there, applied with $G_1 := M$; see also [29, Proposition 8.1]). □

Lemma 4.2. *Let H be a primitive subgroup of G with $\mathcal{O}_G(H)$ Boolean of rank ℓ. Suppose that there exists a maximal element $M \in \mathcal{O}_G(H)$ of O'Nan–Scott type* SD*. Then $\ell \leq 2$.*

Proof. Let V be the socle of M. From the structure of primitive groups of type SD, we deduce $V \cong T^\kappa$ and $|\Omega| = |T|^{\kappa-1}$ for some nonabelian simple group T and for some integer $\kappa \geq 2$.

If $\ell = 1$, then we have nothing to prove; therefore we suppose $\ell \geq 2$ and we let $M' \in \mathcal{O}_G(H)$ be a maximal element with $M' \neq M$. Set $H' := M \cap M'$. Since $\mathcal{O}_G(H)$ is Boolean, H' is maximal in M, and since $H \leq H'$, H' acts primitively on Ω. From Lemma 4.1, with H there replaced by H' here, we obtain that H' has socle V. From the O'Nan–Scott theorem and in particular from the structure of the socles of

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primitive groups, we deduce that H' has type HS or SD, where type HS can arise only when $\kappa = 2$. Now, from [29, Proposition 8.1], we obtain that either M' is a primitive group of SD type having socle V or $M' = \mathrm{Alt}(\Omega)$. In the first case, $M' = \mathbf{N}_G(V) = M$, which is a contradiction. Therefore $M' = \mathrm{Alt}(\Omega)$. Thus $G = \mathrm{Sym}(\Omega)$, and $\mathrm{Alt}(\Omega)$ and M are the only maximal members in $\mathcal{O}_G(H)$. This gives $\ell = 2$. □

Lemma 4.3. *Let M be a maximal subgroup of G of O'Nan–Scott type* HA *with socle V, and let H be a maximal subgroup of M acting primitively on Ω. Then either*

1. *$V \leq H$ or*
2. *$|\Omega| = 8$, $G = \mathrm{Alt}(\Omega)$, $H \cong \mathrm{PSL}_2(7)$ and $M \cong \mathrm{AGL}_3(2)$.*

Proof. Here, $n = |\Omega| = p^d$ for some prime number p and some positive integer d. The result is clear when $n \leq 4$, and hence we suppose $n \geq 5$. In what follows, we assume $V \nleq H$ and we show that $n = 8$, $G = \mathrm{Alt}(\Omega)$, $H \cong \mathrm{PSL}_2(7)$ and $M \cong \mathrm{AGL}_3(2)$.

The maximality of H in M yields $VH = M$. Since $V \cap H \trianglelefteq \langle V, H\rangle = M$, we deduce $V \cap H = 1$, that is, H is a complement of V in M, and hence $H \cong M/V$. Since $\mathbf{N}_{\mathrm{Sym}(n)}(V) \cong \mathrm{AGL}_d(p)$, we deduce that M/V and H are isomorphic to $\mathrm{GL}_d(p)$ or to an index 2 subgroup of $\mathrm{GL}_d(p)$.

Since H acts primitively on Ω, we deduce that $\mathbf{Z}(H) = 1$ or $\mathbf{Z}(H) = H$. Clearly the second case cannot arise here, because M/V is non-abelian, being $n \geq 5$. Suppose then $\mathbf{Z}(H) = 1$.

If $G = \mathrm{Sym}(\Omega)$, then $M/V \cong \mathrm{GL}_d(p)$ has a trivial centre only when $p = 2$. It is easy to verify (using the fact that $\mathrm{GL}_d(2)$ is generated by transvections) that $\mathrm{AGL}_d(2)$ is contained in $\mathrm{Alt}(\Omega)$ when $d \geq 3$. Thus $M < \mathrm{Alt}(\Omega) < G$, contradicting the hypothesis that M is maximal in G. This shows that $G = \mathrm{Alt}(\Omega)$. In particular, when $p = 2$ we have $M/V \cong \mathrm{GL}_d(2)$, and when $p > 2$, M/V is isomorphic to a subgroup of $\mathrm{GL}_d(p)$ having index 2.

Since $\mathrm{GL}_d(p)$ has a centre of order $p-1$ and since $\mathbf{Z}(H) = 1$, we deduce that either $p = 2$ or $(p-1)/2 = 1$, that is, $p \in \{2, 3\}$. In both cases, a simple computation reveals that $M = \mathrm{ASL}_d(p)$, and hence $H \cong M/V \cong \mathrm{SL}_d(p)$. Observe that when $p = 3$, d is odd because $1 = |\mathbf{Z}(H)| = |\mathbf{Z}(\mathrm{SL}_d(3))| = \gcd(d, 2)$. In particular, in both cases, $H \cong M/V \cong \mathrm{SL}_d(p) \cong \mathrm{PSL}_d(p)$ is a non-abelian simple group. Given $\omega \in \Omega$, $|H : H_\omega| = p^d$ is a power of the prime p, and hence from [12, (3.1)] we deduce that $(d, p) = (3, 2)$. Thus $n = p^d = 8$, $H \cong \mathrm{SL}_3(2) \cong \mathrm{PSL}_2(7)$. □

Lemma 4.4. *Let H be a primitive subgroup of G with $\mathcal{O}_G(H)$ Boolean of rank ℓ. Suppose that there exists a maximal element $M \in \mathcal{O}_G(H)$ of O'Nan–Scott type* HA. *Then every maximal element M' in $\mathcal{O}_G(H)$ with $M' \neq M$ is either* $\mathrm{Alt}(\Omega)$ *or the stabiliser in G of a regular product structure on Ω.*

Proof. If $\ell = 1$, then we have nothing to prove; therefore we suppose that $\ell \geq 2$ and we let $M' \in \mathcal{O}_G(H)$ be a maximal element of $\mathcal{O}_G(H)$ with $M' \neq M$. Set $H' := M \cap M'$. Since $\mathcal{O}_G(H)$ is Boolean, H' is maximal in M, and since $H \leq H'$, H' acts primitively on Ω. From Lemma 4.3, with H there replaced by H' here, either H' contains the socle V of M or $n = 8$, $G = \mathrm{Alt}(\Omega)$, $H' \cong \mathrm{PSL}_2(7)$ and $M \cong \mathrm{AGL}_3(2)$. In the second case, a computer computation reveals that the largest Boolean lattice $\mathcal{O}_{\mathrm{Alt}(8)}(H)$ with H primitive has rank 2. Therefore, for the rest of the proof, we suppose $V \leq M'$. In particular, M' is a primitive permutation group containg an abelian regular subgroup. Thus M' is one of the groups classified in [15, Theorem 1.1]. We apply that classification here and the notation therein.

Assume M' is as in [15, Theorem 1.1 (1)], that is, M' is a maximal primitive subgroup of G of O'Nan–Scott type HA. Let V' be the socle of M'. From Lemma 4.3, we deduce that $V' \leq M$ and hence $VV' \leq H'$. Since $V \trianglelefteq M$ and $V' \trianglelefteq M'$, we deduce that $VV' \trianglelefteq H'$. As H' acts primitively on Ω, we deduce that VV' is the socle of H' and hence $|VV'| = |V|$. Therefore $V = V'$. Thus $M' = \mathbf{N}_G(V) = M$, which is a contradiction. Therefore M' is one of the groups listed in [15, Theorem 1.1 (2)].

Suppose first that $l = 1$ (the positive integer l is defined in [15, Theorem 1.1]). An inspection of the list in [15, Theorem 1.1 (2)] (using the maximality of M' in G) yields one of the following:

1. $M' \cong M_{11}$, $n = 11$ and $G = \mathrm{Alt}(\Omega)$.
2. $M' \cong M_{23}$, $n = 23$ and $G = \mathrm{Alt}(\Omega)$.

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3. $M' \cong \mathbf{N}_G(\mathrm{PSL}_{d'}(q'))$ for some integer $d' \geq 2$ and some prime power q', with $n = p = (q'^{d'} - 1)/(q' - 1)$.
4. $M' = \mathrm{Alt}(\Omega)$ and $G = \mathrm{Sym}(\Omega)$.

A computer computation shows that in 1 and 2, $M = \mathbf{N}_G(V) \leq M'$, which is a contradiction. Assume that M' is as in 3. Write $q' = r'^{\kappa'}$ for some prime number r' and some positive integer κ'. Then V is a Singer cycle in $\mathrm{PGL}_{d'}(q')$. As $H' = M \cap M' = \mathbf{N}_G(V) \cap M' = \mathbf{N}_{M'}(V)$, we obtain

$$|H' : V| = \begin{cases} d'\kappa', & \text{when } \mathbf{N}_{\mathrm{Sym}(p)}(\mathrm{PGL}_{d'}(q')) \leq G, \\ d'\kappa'/2, & \text{when } \mathbf{N}_{\mathrm{Sym}(p)}(\mathrm{PGL}_{d'}(q')) \nleq G. \end{cases}$$

We claim that d' is prime. If d' is not prime, then $d' = d_1 d_2$ for some positive integers $d_1, d_2 > 1$. Thus $H' < \mathbf{N}_G(\mathrm{PSL}_{d_1}(q'^{d_2})) < M'$, contradicting the fact that H' is maximal in M'. Therefore, d' is a prime number. Moreover, since H' is maximal in M and M/V is cyclic (of order $p - 1$ or $(p - 1)/2$), we deduce that $s' := |M : H'|$ is a prime number.

Let M'' be a maximal element in $\mathcal{O}_G(H)$ with $M \neq M'' \neq M'$, and let $H'' := M \cap M''$. Arguing as in the previous paragraph (with M' replaced by M''), M'' cannot be as in 1 or 2. Suppose that M'' is as in 3. Then $M'' \cong \mathbf{N}_G(\mathrm{PSL}_{d''}(q''))$ for some integer $d'' \geq 2$ and some prime power q'' with $n = p = (q''^{d''} - 1)/(q'' - 1)$. Write $q'' = r''^{\kappa''}$ for some prime number r'' and some positive integer κ''. Arguing as in the previous paragraph, we obtain that d'' and $s'' := |M : H''|$ are prime numbers. Now $M' \cap M''$ acts primitively on Ω with $n = |\Omega| = p$ prime, and hence, from a result of Burnside, $M' \cap M''$ is either solvable (and $V \trianglelefteq M' \cap M''$) or 2-transitive. In the first case, $M \cap M' = \mathbf{N}_{M'}(V) \geq M' \cap M''$; however, this contradicts the fact that $\mathcal{O}_G(H)$ is Boolean. Therefore $M' \cap M''$ is 2-transitive and nonsolvable. From [14, Theorem 3], we deduce that one of the following holds:

1. $M' \cap M'' = \mathrm{PSL}_2(11)$ and $n = p = 11$.
2. $M' \cap M'' = M_{11}$ and $n = p = 11$.
3. $M' \cap M'' = M_{23}$ and $n = p = 23$.
4. $M' \cap M'' \trianglelefteq M'$ and $M' \cap M'' \trianglelefteq M''$.

The last case cannot arise, because $M' \cap M'' \trianglelefteq \langle M', M'' \rangle = G$ implies $M' \cap M'' = 1$, which is a contradiction. Also, none of the first three cases can arise here, because p is not of the form $(q'^{d'} - 1)/(q' - 1)$. This final contradiction shows that if M'' is a maximal element of $\mathcal{O}_G(H)$ with $M'' \notin \{M, M'\}$, then $M'' = \mathrm{Alt}(\Omega)$. Thus $\ell = 3$, $|\Omega| = p$, $G = \mathrm{Sym}(\Omega)$ and the maximal elements in $\mathcal{O}_G(H)$ are $\mathrm{Alt}(\Omega)$, $\mathrm{AGL}_1(p)$ and $\mathrm{P\Gamma L}_{d'}(q')$.

Since $M \cong \mathrm{AGL}_1(p)$, $|H' : V| = d'\kappa'$ and $|M : H'| = s'$ is prime, we obtain

$$q' \frac{q'^{d'-1} - 1}{q' - 1} = p - 1 = |M : V| = |M : H'||H' : V| = s'd'\kappa'. \tag{1}$$

Suppose first that $d' = 2$ and hence $p = q' + 1 = r'^{\kappa'} + 1$. We get the equation $r'^{\kappa'} = 2s'\kappa'$ and hence $r' = 2$. Therefore, $2^{\kappa'-1} = s'\kappa'$ and $s' = 2$, and hence $2^{\kappa'-2} = \kappa'$. Thus $\kappa' = 4$ and $n = p = 17$. A computer computation shows that this case does not arise, because $\mathrm{Alt}(17) \cap \mathrm{AGL}_1(17) = \mathrm{AGL}_1(17) \cap \mathrm{P\Gamma L}_2(16)$. Suppose now that $d' > 2$.

Assume $\kappa' = 1$. Then equation (1) yields $s' = 2$, because $p - 1$ is even. A computation shows that the equation

$$q' \frac{q'^{d'-1} - 1}{q' - 1} = 2d'$$

has a solution only when $d' = 3$ and $q' = 2$. Thus $n = p = 7$. A computer computation shows that this case does not arise, because $\mathrm{Alt}(7) \cap \mathrm{AGL}_1(7) \leq \mathrm{Alt}(7) \cap \mathrm{PGL}_2(7)$. Therefore $\kappa' > 1$.

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Now we first show $d' \neq r'$. To this end, we argue by contradiction, supposing $d' = r'$. Then equation (1) yields

$$\frac{q'}{r'}\frac{q'^{d'-1}-1}{q'-1} = s'\kappa'. \tag{2}$$

Since $q'/r' = r'^{\kappa'-1}$ and $(q'^{d'-1}-1)/(q'-1)$ are relatively prime and s' is prime, we have either $s' = r'$ or s' divides $(q'^{d'-1}-1)/(q'-1)$. In the first case,

$$\frac{q'}{r'^2}\frac{q'^{d'-1}-1}{q'-1} = \kappa',$$

and hence, for $d' > 3$, $\kappa' \geq (q'^{d'-1}-1)/(q'-1) \geq q'^2 = r'^{2\kappa'}$, which is impossible. It is not difficult to observe that equation (2) is also not satisfied for $d' = 3$. In the second case, $\kappa' \geq q'/r' = r'^{\kappa'-1}$, which is possible only when $\kappa' = 2$. When $\kappa' = 2$, equation (2) becomes

$$r'\frac{(r'^{d-1}-1)(r'^{d-1}+1)}{r'^2-1} = 2s',$$

which has no solution with s' prime. Therefore $d' \neq r'$.

Since d' is a prime number and $d' \neq r'$, from Fermat's little theorem we have $q'^{d'-1} \equiv 1 \pmod{d'}$, that is, d' divides $q'^{d'-1}-1$. If $q' \equiv 1 \pmod{d'}$, then

$$p = \frac{q'^{d'}-1}{q'-1} = q'^{d'-1} + q'^{d'-2} + \cdots + \cdots + q' + 1 \equiv 0 \pmod{d'},$$

and hence $p = d'$; however, this is clearly a contradiction, because $p > d'$. Thus d' does not divide $q'-1$. This proves that d' divides $(q'^{d'-1}-1)/(q'-1)$ and hence that $(q'^{d'-1}-1)/d'(q'-1)$ is an integer. From equation (1), we get

$$q'\frac{q'^{d'-1}-1}{d'(q'-1)} = \kappa' s'.$$

Since s' is prime and $q' > \kappa'$, this equality might admit a solution only when $(q'^{d'-1}-1)/(d'(q'-1)) = 1$, that is, $q'^{d'-1}-1 = d'(q'-1)$. This happens only when $q' = 2$ and $d' = 3$, but this contradicts $\kappa' > 1$.

For the rest of the argument we may suppose $l \geq 2$. In particular, from [15, Theorem 1.1] we obtain either that $M' = \mathrm{Alt}(\Omega)$ or that M' is the stabiliser in G of a regular product structure on Ω. Since this argument does not depend upon M', the result follows. □

Lemma 4.5. *Let M be a maximal subgroup of G of O'Nan–Scott type* AS *with $M \neq \mathrm{Alt}(\Omega)$ and let H be a maximal subgroup of M acting primitively on Ω. Then*

1. *M and H have the same socle, or*
2. *H has O'Nan–Scott type* AS *and the pair (H, M) appears in Tables 3–6 of [16] or*
3. *H has O'Nan–Scott type* HA *and the pair (H, M) appears in Table 2 of [29].*

Proof. Suppose that H and M do not have the same socle. It follows from [29, Proposition 6.2] that either H has O'Nan–Scott type AS and the pair (H, M) appears in Tables 3–6 of [16] or it has O'Nan–Scott type HA and the pair (H, M) appears in Table 2 of [29]. □

Lemma 4.6. *Let H be a primitive subgroup of G with $\mathcal{O}_G(H)$ Boolean of rank ℓ. Suppose that there exists a maximal element $M \in \mathcal{O}_G(H)$ of O'Nan–Scott type* AS *with $M \neq \mathrm{Alt}(\Omega)$. Then $\ell \leq 2$.*

Proof. If $\ell \leq 2$, we have nothing to prove; therefore we suppose $\ell \geq 3$. Since M is a maximal element in $\mathcal{O}_G(H)$ of O'Nan–Scott type AS and $M \neq \mathrm{Alt}(\Omega)$, from Lemma 4.4 we deduce that no maximal element in $\mathcal{O}_G(H)$ is of O'Nan–Scott type HA. Similarly, from Lemma 4.2, no maximal element in

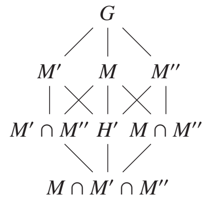

Figure 2. *The Boolean lattice in the proof of Lemma 4.6.*

$\mathcal{O}_G(H)$ is of O'Nan–Scott type SD. As H acts primitively on Ω, all elements in $\mathcal{O}_G(H)$ are primitive, and hence the maximal elements in $\mathcal{O}_G(H)$ have O'Nan–Scott type AS or PA. Since $\ell \geq 3$, we let $M' \in \mathcal{O}_G(H)$ be a maximal element with $\mathrm{Alt}(\Omega) \neq M' \neq M$. Moreover, we let M'' be any maximal element in $\mathcal{O}_G(H)$ with $M \neq M'' \neq M'$. Set $H' := M \cap M'$ and $H'' := M \cap M' \cap M''$ (see Figure 2).

Since $\mathcal{O}_G(H)$ is Boolean, H' is maximal in M, and hence we are in the position to apply Lemma 4.5 with H there replaced by H' here. We discuss the three possibilities in turn.

Suppose first that H' has O'Nan–Scott type HA and let V' be the socle of H'. Since in $\mathcal{O}_G(H)$ there are no maximal members of type HA, $\mathbf{N}_G(V')$ is not a maximal subgroup of G. It follows from [16, Theorem] that $n \in \{7, 11, 17, 23\}$ and $G = \mathrm{Alt}(\Omega)$. A computer computation shows that none of these cases gives rise to a Boolean lattice of rank 3 or larger.

Suppose now that H' and M have the same socle, or that the pair (H', M) appears in Tables 3–6 of [16]. In these cases, H' has O'Nan–Scott type AS. Since $\mathcal{O}_G(H)$ is Boolean, H'' is maximal in H' and hence, from Lemma 4.5, either

- H'' and H' have the same socle, or
- H'' has O'Nan–Scott type AS and the pair (H'', H') appears in Tables 3–6 of [16] or
- H'' has O'Nan–Scott type HA and the pair (H'', H') appears in Table 2 of [29].

Suppose first that H'' has O'Nan–Scott type HA and let V'' be the socle of H''. Since in $\mathcal{O}_G(H)$ there are no maximal members of type HA, $\mathbf{N}_G(V'')$ is not a maximal subgroup of G, as before. It follows from [16, Theorem] that $n \in \{7, 11, 17, 23\}$ and $G = \mathrm{Alt}(\Omega)$. The same computer computation as before shows that none of these cases gives rise to a Boolean lattice of rank 3 or larger. Therefore, H'' has O'Nan–Scott type AS.

As $\mathcal{O}_G(H'')$ has rank 3 and H'' has type AS, Corollary 3.3 implies that H'' is either product decomposable or octal. If H'' is octal, then $n = 8$ and $H'' \cong \mathrm{PSL}_2(7)$; however, the largest Boolean lattice containing H'' has rank 2. Thus H'' is product decomposable.

From [16, Table II], one of the following holds:

1. $n = 36$ and $H'' = \mathrm{Alt}(6).2$.
2. $n = 144$ and $H'' = M_{12}.2$.
3. $n = q^4(q^2-1)^2/4$ and $\mathbf{F}^*(H'') = \mathrm{Sp}_4(q)$, where $q > 2$ is even.

When $n = 144$ and $H'' = M_{12}.2$, we see that H' cannot have the same socle as H'' because $H'' \cong \mathrm{Aut}(M_{12})$, and hence (H'', H') is one of the pairs in Tables 3–6 of [16]. However, there is no such pair satisfying $n = 144$ and $\mathbf{F}^*(H'') \cong M_{12}$. When $n = 36$ and $H'' = \mathrm{Alt}(6).2$, we see with a computer computation that $\mathbf{N}_{\mathrm{Sym}(36)}(H'') = H''$, and hence H' cannot have the same socle as H''. Therefore (H'', H') is one of the pairs in Tables 3–6 of [16]. However, there is no such pair satisfying $n = 36$ and $\mathbf{F}^*(H'') \cong \mathrm{Alt}(6)$. Finally, suppose $n = q^4(q^2-1)^2/4$ and $\mathbf{F}^*(H'') = \mathrm{Sp}_4(q)$, where $q > 2$ is even. Since there is no pair (H'', H') in Tables 3–6 of [16] satisfying these conditions for n and $\mathbf{F}^*(H'')$, we deduce that H'' and H' have the same socle. Therefore $\mathbf{F}^*(H') = \mathrm{Sp}_4(q)$, with $q > 2$ even.

Summing up, we have two inclusions $H' \leq M$ and $H' \leq M'$, with H' maximal in both M and M', with $\mathbf{F}^*(H') = \mathrm{Sp}_4(q)$ and with $n = q^4(q^2-1)^2/4$. Again using Tables 3–6 of [16], we deduce that both M and M' must have the same socle of H'. However, this is a contradiction, because $G = \langle M, M' \rangle \leq \mathbf{N}_G(\mathbf{F}^*(H'))$. □

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Corollary 4.7. *Let H be a primitive subgroup of G with $\mathcal{O}_G(H)$ Boolean of rank $\ell \geq 3$ and let $G_1, \ldots, G_\ell$ be the maximal members in $\mathcal{O}_G(H)$. Then one of the following holds:*

1. *$n = |\Omega|$ is odd. For every $i \in \{1, \ldots, \ell\}$, there exists a nontrivial regular product structure $\mathcal{F}_i$ with $G_i = \mathbf{N}_G(\mathcal{F}_i)$; moreover, relabelling the index set $\{1, \ldots, \ell\}$ if necessary, $\mathcal{F}_1 < \cdots < \mathcal{F}_\ell$.*
2. *$n = |\Omega|$ is odd and $G = \mathrm{Sym}(\Omega)$. Relabelling the index set $\{1, \ldots, \ell\}$ if necessary, $G_\ell = \mathrm{Alt}(\Omega)$ and for every $i \in \{1, \ldots, \ell - 1\}$, there exists a nontrivial regular product structure $\mathcal{F}_i$ with $G_i = \mathbf{N}_G(\mathcal{F}_i)$; moreover, relabelling the index set $\{1, \ldots, \ell - 1\}$ if necessary, $\mathcal{F}_1 < \cdots < \mathcal{F}_{\ell-1}$.*
3. *$n = |\Omega|$ is an odd prime power. Relabelling the index set $\{1, \ldots, \ell\}$ if necessary, G_ℓ is maximal subgroup of O'Nan–Scott type HA and for every $i \in \{1, \ldots, \ell - 1\}$, there exists a nontrivial regular product structure $\mathcal{F}_i$ with $G_i = \mathbf{N}_G(\mathcal{F}_i)$; moreover, relabelling the index set $\{1, \ldots, \ell - 1\}$ if necessary, $\mathcal{F}_1 < \cdots < \mathcal{F}_{\ell-1}$.*
4. *$n = |\Omega|$ is an odd prime power and $G = \mathrm{Sym}(\Omega)$. Relabelling the index set $\{1, \ldots, \ell\}$ if necessary, $G_\ell = \mathrm{Alt}(\Omega)$, $G_{\ell-1}$ is a maximal subgroup of O'Nan–Scott type HA and for every $i \in \{1, \ldots, \ell - 2\}$, there exists a nontrivial regular product structure $\mathcal{F}_i$ with $G_i = \mathbf{N}_G(\mathcal{F}_i)$; moreover, relabelling the index set $\{1, \ldots, \ell - 2\}$ if necessary, $\mathcal{F}_1 < \cdots < \mathcal{F}_{\ell-2}$.*

Proof. As $\ell \geq 3$, from Lemmas 4.2, 4.4 and 4.6 all the elements in $\{G_1, \ldots, G_\ell\}$ are stabilisers of regular product structures, except possibly that one of these elements might be $\mathrm{Alt}(\Omega)$ or a maximal subgroup of type HA. Relabelling the index set $\{1, \ldots, \ell\}$, suppose that $\{G_1, \ldots, G_\kappa\}$ are stabilisers of regular product structures, that is, $G_i := \mathbf{N}_G(\mathcal{F}_i)$. Thus $\kappa \geq \ell - 2$.

Observe that for every $i, j \in \{1, \ldots, \kappa\}$ with $i \neq j$, $G_i \cap G_j$ is a maximal subgroup of both G_i and G_j. It follows from [2, Section 5] that either $\mathcal{F}_i < \mathcal{F}_j$ or $\mathcal{F}_j < \mathcal{F}_i$. Therefore $\{\mathcal{F}_1, \ldots, \mathcal{F}_\kappa\}$ forms a chain. Relabelling the index set $\{1, \ldots, \kappa\}$ if necessary, we may suppose $\mathcal{F}_1 < \mathcal{F}_2 < \cdots < \mathcal{F}_\kappa$.

Assume that $\mathcal{F}_i$ is a regular (m_i, k_i)-product structure. Since $\mathcal{F}_i \leq \mathcal{F}_{i+1}$, there exists an integer $s_i > 1$ with $m_i = m_{i+1}^{s_i}$ and $k_{i+1} = k_i s_i$. From [2, (5.12)], $\mathcal{O}_G(\mathbf{N}_G(\mathcal{F}_i) \cap \mathbf{N}_G(\mathcal{F}_{i+1}))$ is Boolean of rank 2 only when

$$m_{i+1} \text{ is odd, or } s_i = 2 \text{ and } m_{i+1} \equiv 2 \pmod 4. \tag{3}$$

Suppose that $\kappa \geq 3$. Applying the previous paragraph with $i := \kappa - 1$, we deduce that if m_κ is even, then $s_{\kappa-1} = 2$ and $m_\kappa \equiv 2 \pmod 4$. In turn, since $m_{\kappa-1} = m_\kappa^{s_{\kappa-1}}$ is even, we have $s_{\kappa-2} = 2$ and $m_{\kappa-1} \equiv 2 \pmod 4$. However, $m_{\kappa-1} = m_\kappa^{s_{\kappa-1}} \equiv 0 \pmod 4$, contradicting the fact that $m_{\kappa-1} \equiv 2 \pmod 4$. Therefore, when $\kappa \geq 3$, m_i is odd for every $i \in \{1, \ldots, \kappa\}$ – that is, $n = |\Omega|$ is odd. In particular, when $\kappa = \ell$, we obtain part 1.

Suppose that $G = \mathrm{Sym}(\Omega)$, $\kappa = \ell - 1$ and $G_\ell = \mathrm{Alt}(\Omega)$. If $|\Omega|$ is odd, we obtain part 2. Suppose then that $n = |\Omega|$ is even. In particular, $\kappa = \ell - 1 \leq 2$ and hence $\ell = 3$. Clearly, m_2 is even and hence condition (3) applied with $i = 1$ yields $s_1 = 2$. Thus $m_1 = m_2^{s_1} = m_2^2 \equiv 0 \pmod 4$. Lemma 2.4(2) yields $G_1 \leq \mathrm{Alt}(\Omega) = G_3$, which is a contradiction.

Suppose that $\kappa = \ell - 1$ and G_ℓ is a primitive group of type HA. If $|\Omega|$ is odd, we obtain part 3. Suppose then that $n = |\Omega|$ is even, that is, $n = 2^d$ for some positive integer $d \geq 3$. In particular, $\kappa = \ell - 1 \leq 2$ and hence $\ell = 3$. Clearly, m_2 is even, and hence condition (3) applied with $i = 1$ yields $m_2 \equiv 2 \pmod 4$. Therefore $m_2 = 2$, but this contradicts the fact that in a regular (m, k)-product structure we must have $m \geq 5$.

Finally, suppose that $\kappa = \ell - 2$, $G = \mathrm{Sym}(\Omega)$, $G_\ell = \mathrm{Alt}(\Omega)$ and $G_{\ell-1}$ is a primitive group of type HA. If $|\Omega|$ is even, then $|\Omega| = 2^d$ for some $d \geq 3$. As $G_2 \cong \mathrm{AGL}_d(2) \leq \mathrm{Alt}(\Omega) = G_3$, we obtain a contradiction. Therefore $|\Omega|$ is odd and we obtain 4. □

5. Boolean intervals containing a maximal imprimitive subgroup

The scope of this section is to gather some information on Boolean lattices $\mathcal{O}_G(H)$ containing a maximal element that is imprimitive. Our main tool in this task is a result of Aschbacher and Shareshian [3, Theorem 5.2].

Hypothesis 5.1. Let G be either $\mathrm{Sym}(\Omega)$ or $\mathrm{Alt}(\Omega)$ with $n := |\Omega|$, let Σ be a nontrivial regular partition, let $G_1 := \mathbf{N}_G(\Sigma)$, let G_2 be a maximal subgroup of G distinct from $\mathrm{Alt}(\Omega)$ and let $H := G_1 \cap G_2$. Assume that

- $\mathcal{O}_G(H)$ is a Boolean lattice of rank 2 with maximal elements M_1 and M_2 and
- H acts transitively on Ω.

Theorem 5.2. *[3, Theorem 5.2] Assume Hypothesis 5.1. Then one of the following holds:*

1. *For every $i \in \{1, 2\}$, there exists a nontrivial regular partition Σ_i with $G_i = \mathbf{N}_G(\Sigma_i)$; moreover, for some $i \in \{1, 2\}$, $\Sigma_i < \Sigma_{3-i}$. Further, $n \geq 8$ and, if $n = 8$, then $G = \mathrm{Sym}(\Omega)$.*
2. *$G = \mathrm{Alt}(\Omega)$, $n = 2^{a+1}$ for some positive integer $a > 1$, G_2 is an affine primitive group, $V = \mathbf{F}^*(G_2) \leq H$, V_Σ is a hyperplane of V, the elements of Σ are the two orbits of V_Σ on Ω and $H = \mathbf{N}_{G_2}(V_\Sigma)$.*
3. *$G = \mathrm{Alt}(\Omega)$, $n \equiv 0 \pmod 4$, $n > 8$ and, for every $i \in \{1, 2\}$, there exists a nontrivial regular partition Σ_i such that*
   - *(a) $G_i = \mathbf{N}_G(\Sigma_i)$,*
   - *(b) Σ_1 and Σ_2 are lattice complements in the poset of partitions of Ω and*
   - *(c) one of Σ_1, Σ_2 is $(2, n/2)$-regular and the other is $(n/2, 2)$-regular.*

(Observe that two partitions Σ_1 and Σ_2 of Ω are lattice complements if the smallest partition Σ of Ω with $\Sigma_1 \leq \Sigma$ and $\Sigma_2 \leq \Sigma$ and the largest partition Σ' of Ω with $\Sigma' \leq \Sigma_1$ and $\Sigma' \leq \Sigma_2$ are the two trivial partitions of Ω. Futher, V_Σ denotes the pointwise stabiliser of Σ in V.)

Hypothesis 5.3. Let G be either $\mathrm{Sym}(\Omega)$ or $\mathrm{Alt}(\Omega)$ with $n := |\Omega|$, let Σ be a nontrivial regular partition, let $G_1 := \mathbf{N}_G(\Sigma)$, let G_2 and G_3 be maximal subgroups of G and let $H := G_1 \cap G_2 \cap G_3$. Assume that

- $\mathcal{O}_G(H)$ is a Boolean lattice of rank 3 with maximal elements G_1, G_2 and G_3 and
- H acts transitively on Ω.

Theorem 5.4. *Assume Hypothesis 5.3. Then one of the following holds:*

1. *For every $i \in \{1, 2, 3\}$, there exists a nontrivial regular partition Σ_i with $G_i = \mathbf{N}_G(\Sigma_i)$; moreover, relabelling the index set $\{1, 2, 3\}$ if necessary, $\Sigma_1 < \Sigma_2 < \Sigma_3$.*
2. *$G = \mathrm{Sym}(\Omega)$. Relabelling the index set $\{1, 2, 3\}$ if necessary, $G_3 = \mathrm{Alt}(\Omega)$ and, for every $i \in \{1, 2\}$, there exists a nontrivial regular partition Σ_i with $G_i = \mathbf{N}_G(\Sigma_i)$; moreover, for some $i \in \{1, 2\}$, $\Sigma_i < \Sigma_{3-i}$.*
3. *$G = \mathrm{Alt}(\Omega)$, $|\Omega| = 8$ and the Boolean lattice $\mathcal{O}_G(H)$ is shown in Figure 1.*

Proof. If none of G_1, G_2 and G_3 is $\mathrm{Alt}(\Omega)$ and if $G = \mathrm{Sym}(\Omega)$, then the result follows directly from Theorem 5.2 and we obtain 1. Suppose $G = \mathrm{Sym}(\Omega)$ and one of G_2 or G_3 is $\mathrm{Alt}(\Omega)$. Without loss of generality we may assume that $G_3 = \mathrm{Alt}(\Omega)$. Now the result follows directly from Theorem 5.2 applied to $\{G_1, G_2\}$; we obtain 2.

It remains to consider the case where $G = \mathrm{Alt}(\Omega)$. In particular, we can apply Theorem 5.2 to the pairs $\{G_1, G_2\}$ and $\{G_1, G_3\}$. Relabelling the index set $\{1, 2, 3\}$ if necessary, we have to consider in turn the following cases:

A Theorem 5.2 part 1 holds for both pairs $\{G_1, G_2\}$ and $\{G_1, G_3\}$.
B Theorem 5.2 part 1 holds for $\{G_1, G_2\}$ and Theorem 5.2 part 2 holds for $\{G_1, G_3\}$.
C Theorem 5.2 part 1 holds for $\{G_1, G_2\}$ and Theorem 5.2 part 3 holds for $\{G_1, G_3\}$.
D Theorem 5.2 part 2 holds for both pairs $\{G_1, G_2\}$ and $\{G_1, G_3\}$.
E Theorem 5.2 part 2 holds for $\{G_1, G_2\}$ and Theorem 5.2 part 3 holds for $\{G_1, G_3\}$.
F Theorem 5.2 part 3 holds for both pairs $\{G_1, G_2\}$ and $\{G_1, G_3\}$.

Case A: In particular, G_2 and G_3 are stabilisers of nontrivial regular partitions, and hence we are in the position to apply Theorem 5.2 also to the pair $\{G_2, G_3\}$. It is not hard to see that Theorem 5.2 part 1 holds for $\{G_2, G_3\}$ and that conclusion 1 of Theorem 5.4 holds.

Case B: Since G_1 is the stabiliser of a nontrivial regular partition and $\{G_1, G_3\}$ satisfies Theorem 5.2 part 2, we deduce that G_3 is an affine primitive group and Σ_1 is an $(n/2, 2)$-regular partition.

Since G_2 is the stabiliser of the nontrivial regular partition Σ_2, we deduce that we may apply Theorem 5.2 to the pair $\{G_2, G_3\}$. In particular, as G_3 is primitive, Theorem 5.2 part 2 must hold for $\{G_2, G_3\}$, and hence G_2 is the stabiliser of an $(n/2, 2)$-regular partition. However, this contradicts the fact that $\{G_1, G_2\}$ satisfies Theorem 5.2 part 1 – that is, $\Sigma_1 < \Sigma_2$ or $\Sigma_2 < \Sigma_1$.

Case C: We have either

(a) $\Sigma_1 < \Sigma_2$, Σ_1 is a $(2, n/2)$-regular partition, Σ_3 is an $(n/2, 2)$-regular partition and Σ_1, Σ_3 are lattice complements or
(b) $\Sigma_2 < \Sigma_1$, Σ_1 is an $(n/2, 2)$-regular partition, Σ_3 is a $(2, n/2)$-regular partition and Σ_1, Σ_3 are lattice complements.

In case **(b)**, $\Sigma_2 < \Sigma_1$, and hence Σ_1 is a refinement of Σ_2; however, as Σ_1 is an $(n/2, 2)$-regular partition, this is not possible. Therefore, case **(b)** does not arise. As G_2 and G_3 are stabilisers of nontrivial regular partitions of Ω, we are in the position to apply Theorem 5.2 also to the pair $\{G_2, G_3\}$. If Theorem 5.2 part 1 holds for $\{G_2, G_3\}$, then either $\Sigma_2 < \Sigma_3$ or $\Sigma_3 < \Sigma_2$. However, both possibilities lead to a contradiction. Indeed, if $\Sigma_2 < \Sigma_3$ and **(a)** holds, then $\Sigma_1 < \Sigma_2 < \Sigma_3$, contradicting the fact that Σ_1 and Σ_3 are lattice complements. The argument when $\Sigma_3 < \Sigma_2$ is analogous. Similarly, if Theorem 5.2 part 3 holds for $\{G_2, G_3\}$, then Σ_2 and Σ_3 are lattice complements and either

(a)′ Σ_2 is a $(2, n/2)$-regular partition and Σ_3 is an $(n/2, 2)$-regular partition or
(b)′ Σ_2 is an $(n/2, 2)$-regular partition and Σ_3 is a $(2, n/2)$-regular partition.

However, an easy case-by-case analysis shows that **(a)**′ and **(b)**′ are incompatible with **(a)**.

Case D: In particular, G_2 and G_3 are both primitive groups of affine type. Let V_2 be the socle of G_2 and let V_3 be the socle of G_3. From Lemma 2.7 applied to $\mathcal{O}_G(G_2 \cap G_3)$, we deduce that either $G_2 \cap G_3$ is primitive or $G = \mathrm{Alt}(\Omega)$, $|\Omega| = 8$ and $G_2 \cap G_3$ is the stabiliser of a $(2, 4)$-regular partition. In the latter case, we see with a direct computation that part 3 holds. Suppose then that $G_2 \cap G_3$ is primitive. From Lemma 4.3 applied to the inclusions $G_2 \cap G_3 < G_2$ and $G_2 \cap G_3 < G_3$, we deduce that either

(a)″ $G_2 \cap G_3$, G_2 and G_3 have the same socle or
(b)″ $n = 8$, $G_2 \cap G_3 \cong \mathrm{PSL}_2(7)$ and $G_2 \cong G_3 \cong \mathrm{AGL}_3(2)$.

In the former case, we have $V_2 = V_3$ and hence $G_2 = \mathbf{N}_G(V_2) = \mathbf{N}_G(V_3) = G_3$, contradicting the fact that $G_2 \neq G_3$. In the latter case, we have checked with the invaluable help of the computer algebra system `magma` [6] that $\mathcal{O}_{\mathrm{Alt}(8)}(\mathrm{PSL}_2(7)) = \{\mathrm{PSL}_2(7) < \mathrm{AGL}_3(2) < \mathrm{Alt}(8)\}$, contradicting the fact that it is a Boolean lattice.

Case E: In this case, Σ_1 is an $(n/2, 2)$-regular partition, Σ_3 is a $(2, n/2)$-regular partition and Σ_1, Σ_3 are lattice complements. As G_3 is the stabiliser of a nontrivial regular partition, we are in the position to apply Theorem 5.2 to the pair $\{G_2, G_3\}$. As G_2 is primitive, we see that Theorem 5.2 part 2 holds for $\{G_2, G_3\}$, and hence Σ_3 is an $(n/2, 2)$-regular partition, which implies $(n/2, 2) = (2, n/2)$, that is, $n = 4$. However this contradicts $a > 1$ in Theorem 5.2 part 2.

Case F: In particular, both Σ_2 and Σ_3 are either $(n/2, 2)$-regular partitions or $(2, n/2)$-regular partitions. As G_2 and G_3 are stabilisers of nontrivial regular partitions, we may apply Theorem 5.2 also to the pair $\{G_2, G_3\}$. Clearly, none of parts 1, 2 or 3 in Theorem 5.2 holds for $\{G_2, G_3\}$, which is a contradiction. □

Corollary 5.5. *Let H be a transitive subgroup of G and suppose that $\mathcal{O}_G(H)$ is Boolean of rank $\ell \geq 3$ and that $\mathcal{O}_G(H)$ contains a maximal element which is imprimitive. Let $\{G_1, \ldots, G_\ell\}$ be the maximal elements of $\mathcal{O}_G(H)$. Then one of the following holds:*

1. *For every $i \in \{1, \ldots, \ell\}$, there exists a nontrivial regular partition Σ_i with $G_i = \mathbf{N}_G(\Sigma_i)$; moreover, relabelling the index set $\{1, \ldots, \ell\}$ if necessary, $\Sigma_1 < \cdots < \Sigma_\ell$.*

2. $G = \mathrm{Sym}(\Omega)$. *Relabelling the indexed set* $\{1,\ldots,\ell\}$ *if necessary,* $G_\ell = \mathrm{Alt}(\Omega)$ *and for every* $i \in \{1,\ldots,\ell-1\}$, *there exists a nontrivial regular partition* Σ_i *with* $G_i = \mathbf{N}_G(\Sigma_i)$ *and* $\Sigma_1 < \cdots < \Sigma_{\ell-1}$.
3. $G = \mathrm{Alt}(\Omega)$, $|\Omega| = 8$, $\ell = 3$ *and the Boolean lattice* $\mathcal{O}_G(H)$ *is shown in Figure 1.*

Proof. This follows arguing inductively on ℓ; the base case $\ell = 3$ is Theorem 5.4. □

6. Boolean intervals containing a maximal intransitive subgroup

The scope of this section is to gather some information on Boolean lattices $\mathcal{O}_G(H)$ containing a maximal element that is intransitive. Some of the material in this section can be also traced back to [5].

Hypothesis 6.1. Let G be either $\mathrm{Sym}(\Omega)$ or $\mathrm{Alt}(\Omega)$ with $n := |\Omega|$, let Γ be a subset of Ω with $1 \leq |\Gamma| < |\Omega|/2$, let $G_1 := \mathbf{N}_G(\Gamma)$, let G_2 be a maximal subgroup of G and let $H := G_1 \cap G_2$. Assume that $\mathcal{O}_G(H)$ is Boolean of rank 2 with maximal elements G_1 and G_2.

Theorem 6.2. *Assume Hypothesis 6.1. Then one of the following holds:*

1. $G = \mathrm{Sym}(\Omega)$ *and* $G_2 = \mathrm{Alt}(\Omega)$.
2. G_2 *is an imprimitive subgroup having* Γ *as a block of imprimitivity.*
3. $G = \mathrm{Alt}(\Omega)$, $n = 7$, $|\Gamma| = 3$ *and* $G_2 \cong \mathrm{SL}_3(2)$ *acts primitively on* Ω.
4. $|\Gamma| = 1$ *and one of the following holds:*
   - *(a)* $G = \mathrm{Alt}(\Omega)$ *and* $G_2 \cong \mathrm{AGL}_d(2)$ *with* $d \geq 3$.
   - *(b)* $G = \mathrm{Alt}(\Omega)$, $G_2 \cong \mathrm{Sp}_{2m}(2)$ *and* $|\Omega| \in \{2^{m-1}(2^m+1), 2^{m-1}(2^m-1)\}$.
   - *(c)* $G = \mathrm{Alt}(\Omega)$, $G_2 \cong HS$ *and* $|\Omega| = 176$.
   - *(d)* $G = \mathrm{Alt}(\Omega)$, $G_2 \cong Co_3$ *and* $|\Omega| = 276$.
   - *(e)* $G = \mathrm{Alt}(\Omega)$, $G_2 \cong M_{12}$ *and* $|\Omega| = 12$.
   - *(f)* $G = \mathrm{Alt}(\Omega)$, $G_2 \cong M_{24}$ *and* $|\Omega| = 24$.
   - *(g)* $G = \mathrm{Sym}(\Omega)$, $G_2 \cong \mathrm{PGL}_2(p)$ *with* p *prime and* $|\Omega| = p+1$.
   - *(h)* $G = \mathrm{Alt}(\Omega)$, $G_2 \cong \mathrm{PSL}_2(p)$ *with* p *prime and* $|\Omega| = p+1$.

Proof. Suppose that G_2 is intransitive. Thus $G_2 = G \cap (\mathrm{Sym}(\Gamma') \times \mathrm{Sym}(\Omega \setminus \Gamma'))$, for some subset $\Gamma' \subseteq \Omega$ with $1 \leq |\Gamma'| < |\Omega|/2$. In particular,

$$H = G_1 \cap G_2 = G \cap (\mathrm{Sym}(\Gamma \cap \Gamma') \times \mathrm{Sym}(\Gamma \setminus \Gamma') \times \mathrm{Sym}(\Gamma' \setminus \Gamma) \times \mathrm{Sym}(\Omega \cup (\Gamma \cup \Gamma'))).$$

Thus H is contained in

- $G \cap (\mathrm{Sym}(\Gamma \cap \Gamma') \times \mathrm{Sym}(\Omega \setminus (\Gamma \cap \Gamma')))$,
- $G \cap (\mathrm{Sym}(\Gamma \setminus \Gamma') \times \mathrm{Sym}(\Omega \setminus (\Gamma \setminus \Gamma')))$,
- $G \cap (\mathrm{Sym}(\Gamma' \setminus \Gamma) \times \mathrm{Sym}(\Omega \setminus (\Gamma' \setminus \Gamma)))$,
- $G \cap (\mathrm{Sym}(\Gamma \cup \Gamma') \times \mathrm{Sym}(\Omega \setminus (\Gamma \cup \Gamma')))$.

Since the only overgroups of H are H, G_1, G_2 and G, each of the previous four subgroups must be one of H, G_1, G_2 and G. This immediately implies $G = G \cap (\mathrm{Sym}(\Gamma \cap \Gamma') \times \mathrm{Sym}(\Omega \setminus (\Gamma \cap \Gamma')))$, that is, $\Gamma \cap \Gamma' = \emptyset$. However, $G \cap (\mathrm{Sym}(\Gamma \cup \Gamma') \times \mathrm{Sym}(\Omega \setminus (\Gamma \cup \Gamma')))$ is neither H nor G_1 nor G_2 nor G, because $1 \leq |\Gamma|, |\Gamma'| < |\Omega|/2$.

Suppose that G_2 is imprimitive. In particular, G_2 is the stabiliser of a nontrivial (a,b)-regular partition of Ω – that is, G_2 is the stabiliser of a partition $\Sigma_2 := \{X_1,\ldots,X_b\}$ of the set Ω into b parts each having cardinality a, for some positive integers a and b with $a, b \geq 2$. Thus

$$G_2 = \mathbf{N}_G(\Sigma_2) \text{ and } \mathbf{N}_{\mathrm{Sym}(\Omega)}(\Sigma_2) \cong \mathrm{Sym}(a) \mathrm{wr}\, \mathrm{Sym}(b).$$

The group $H = G_1 \cap G_2$ is intransitive. Since G_1 is the only proper overgroup of H that is intransitive, we deduce that H has only two orbits on Ω, namely Γ and $\Omega \setminus \Gamma$. From this it follows that for every

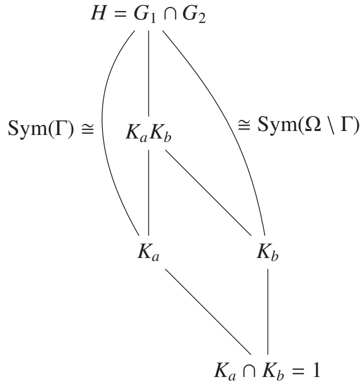

Figure 3. *Structure of $H = G_1 \cap G_2$.*

$i \in \{1, \dots, b\}$, either $X_i \subseteq \Gamma$ or $X_i \subseteq \Omega \setminus \Gamma$. Let $\Sigma_2' := \{X \in \Sigma_2 \mid X \subseteq \Gamma\}$ and $\Sigma_2'' := \{X \in \Sigma_2 \mid X \subseteq \Omega \setminus \Gamma\}$. Therefore,

$$\begin{aligned} H = G_1 \cap G_2 &= G \cap (\mathbf{N}_{\mathrm{Sym}(\Gamma)}(\Sigma_2') \times \mathbf{N}_{\mathrm{Sym}(\Omega\setminus\Gamma)}(\Sigma_2'')),\\ \mathbf{N}_{\mathrm{Sym}(\Gamma)}(\Sigma_2') &\cong \mathrm{Sym}(a) \wr \mathrm{Sym}(b_1),\\ \mathbf{N}_{\mathrm{Sym}(\Omega\setminus\Gamma)}(\Sigma_2'') &\cong \mathrm{Sym}(a) \wr \mathrm{Sym}(b_2), \end{aligned}$$

where b_1 is the number of parts in Σ_2' and b_2 is the number of parts in Σ_2''. Therefore, H is contained in subgroups isomorphic to

$$G \cap (\mathrm{Sym}(\Gamma) \times \mathbf{N}_{\mathrm{Sym}(\Omega\setminus\Gamma)}(\Sigma_2'')) \quad \text{and} \quad G \cap (\mathbf{N}_{\mathrm{Sym}(\Gamma)}(\Sigma_2') \times \mathrm{Sym}(\Omega \setminus \Gamma)).$$

Since H and G_1 are the only intransitive overgroup of H, we deduce that these two subgroups are H or G_1. However this happens if and only if $b_1 = 1$. In other words, this happens if and only if $\Gamma \in \Sigma_2$, and we obtain part 2.

Suppose that G_2 is primitive. We divide our analysis into various cases.

Case 1: $|\Gamma| \geq 3$, or $|\Gamma| = 2$ and $G = \mathrm{Sym}(\Omega)$.

Now $H = G_1 \cap G_2$ is a maximal subgroup of G_1. Moreover, $G_1 = \mathrm{Sym}(\Gamma) \times \mathrm{Sym}(\Omega \setminus \Gamma)$ when $G = \mathrm{Sym}(\Omega)$ and $G_1 = \mathrm{Alt}(\Omega) \cap (\mathrm{Sym}(\Gamma) \times \mathrm{Sym}(\Omega \setminus \Gamma))$ when $G = \mathrm{Alt}(\Omega)$. Consider $\pi_a : G_1 \to \mathrm{Sym}(\Gamma)$ and $\pi_b : G_1 \to \mathrm{Sym}(\Omega \setminus \Gamma)$ the natural projections. Oberve that these projections are surjective.

Assume that $\pi_a(G_1 \cap G_2)$ is a proper subgroup of $\mathrm{Sym}(\Gamma)$. Then, from the maximality of $G_1 \cap G_2$ in G_1, we have

$$G_1 \cap G_2 = G \cap (\pi_a(G_1 \cap G_2) \times \mathrm{Sym}(\Omega \setminus \Gamma)).$$

As $|\Omega \setminus \Gamma| \geq 3$, we deduce that $G_1 \cap G_2$ contains a 2-cycle or a 3-cycle. In particular, the primitive group G_2 contains a 2-cycle or a 3-cycle. By a celebrated result of Jordan [10, Theorem 3.3 A], we obtain $\mathrm{Alt}(\Omega) \leq G_2$. Thus $G = \mathrm{Sym}(\Omega)$ and $G_2 = \mathrm{Alt}(\Omega)$, and we obtain part 1.

Suppose then $\pi_a(G_1 \cap G_2) = \mathrm{Sym}(\Gamma)$ and let $K_a := \mathrm{Ker}(\pi_a) \cap G_1 \cap G_2$. If $\pi_b(G_1 \cap G_2)$ is a proper subgroup of $\mathrm{Sym}(\Omega \setminus \Gamma)$, using the same argument as the previous paragraph we obtain part 1.

Suppose then $\pi_b(G_1 \cap G_2) = \mathrm{Sym}(\Omega \setminus \Gamma)$ and let $K_b := \mathrm{Ker}(\pi_b) \cap G_1 \cap G_2$. In the rest of the proof of this case, the reader might find it useful to refer to Figure 3.

Now, K_aK_b is a subgroup of $G_1 \cap G_2$; moreover $(G_1 \cap G_2)/(K_aK_b)$ is an epimorphic image of both $\mathrm{Sym}(\Gamma)$ and $\mathrm{Sym}(\Omega \setminus \Gamma)$. Assume $|\Omega \setminus \Gamma| \geq 5$. Then the only epimorphic image of both $\mathrm{Sym}(\Gamma)$ and $\mathrm{Sym}(\Omega \setminus \Gamma)$ is either the identity group or the cyclic group of order 2. Therefore, $|G_1 \cap G_2 : K_aK_b| \leq 2$. Moreover, $K_aK_b/K_b \cong K_a/(K_a \cap K_b) = K_a$ is isomorphic to either $\mathrm{Alt}(\Omega \setminus \Gamma)$ or $\mathrm{Sym}(\Omega \setminus \Gamma)$. In both cases, $\mathrm{Alt}(\Omega \setminus \Gamma) \le K_a \le G_2$ and hence G_2 contains a 3-cycle. As before, this implies $G = \mathrm{Sym}(\Omega)$ and $G_2 = \mathrm{Alt}(\Omega)$, and part 1 holds. Assume $|\Omega \setminus \Gamma| \le 4$. As $1 \le |\Gamma| < |\Omega|/2$, we deduce $|\Omega| \le 7$. When $|\Gamma| = 3$, we obtain $|\Omega| = 7$, and we can verify with a direct analysis that part 1 holds when $G = \mathrm{Sym}(\Omega)$ and part 3 holds when $G = \mathrm{Alt}(\Omega)$. Finally, if $|\Gamma| = 2$, we have $|\Omega| \in \{5, 6\}$ and $G = \mathrm{Sym}(\Omega)$. A direct inspection in each of these cases reveals that every maximal subgroup of G_1 contains either a 2-cycle or a 3-cycle. Therefore $G_2 = \mathrm{Alt}(\Omega)$ and part 1 holds.

Case 2: $|\Gamma| = 2$ and $G = \mathrm{Alt}(\Omega)$.

In this case, $G_1 = \mathrm{Alt}(\Omega) \cap (\mathrm{Sym}(\Gamma) \times \mathrm{Sym}(\Omega \setminus \Gamma)) \cong \mathrm{Sym}(\Omega \setminus \Gamma)$.

Assume that $H = G_1 \cap G_2$ acts intransitively on $\Omega \setminus \Gamma$ and let Δ be one of its smallest orbits. In particular, H fixes setwise Γ, Δ and $\Omega \setminus (\Gamma \cup \Delta)$. Now, $\mathrm{Alt}(\Omega) \cap (\mathrm{Sym}(\Gamma \cup \Delta) \times \mathrm{Sym}(\Omega \setminus (\Gamma \cup \Delta)))$ is a proper overgroup of H that is intransitive and different from G_1, which is a contradiction. Therefore, H acts transitively on $\Omega \setminus \Gamma$. Suppose that H acts imprimitively on $\Omega \setminus \Gamma$. Since H is maximal in $G_1 \cong \mathrm{Sym}(\Omega \setminus \Gamma)$, we deduce $H = \mathbf{N}_{G_1}(\Sigma)$, where Σ is a nontrivial (a, b)-regular partition of $\Omega \setminus \Gamma$. If $a \ge 3$, then H contains a 3-cycle and hence so does G_2. Since G_2 is primitive, we deduce from [10, Theorem 3.3 A] that $G_2 = \mathrm{Alt}(\Omega) = G$, which is a contradiction. If $a = 2$, then H contains a permutation that is the product of two disjoint transpositions. Since G_2 is primitive, we deduce from [10, Theorem 3.3 D and Example 3.3.1] that either $G_2 = \mathrm{Alt}(\Omega) = G$ or $|\Omega| \le 8$. The first possibility is clearly impossible, and hence $|\Omega| \in \{6, 8\}$. However, a computation in $\mathrm{Alt}(6)$ and in $\mathrm{Alt}(8)$ reveals that no case arises. Therefore H acts primitively on $\Omega \setminus \Gamma$.

Let $\Gamma = \{\gamma, \gamma'\}$. As $|\Gamma| = 2$, the group $(G_1 \cap G_2)_\gamma = H_\gamma$ has index at most 2 in $G_1 \cap G_2 = H$, and hence $H_\gamma \trianglelefteq H$. Since H acts primitively on $\Omega \setminus \Gamma$ and $H_\gamma \trianglelefteq H$, H_γ acts transitively on $\Omega \setminus \Gamma$ or is trivial. The second possibility is clearly a contradiction, because it implies $|H| = 2$ and hence $|\Omega| = 4$. Thus H_γ acts transitively on $\Omega \setminus \Gamma$ and the orbits of H_γ on Ω are $\{\gamma\}, \{\gamma'\}, \Omega \setminus \Gamma$, with cardinality $1, 1, |\Omega| - 2$. Since G_2 is primitive and not regular, from Lemma 2.9 we deduce that γ is the only fixed point of $(G_2)_\gamma$. Since H_γ is a subgroup of $(G_2)_\gamma$, from the cardinality of the orbits of H_γ we deduce that $(G_2)_\gamma$ acts transitively on $\Omega \setminus \{\gamma\}$, that is, G_2 is 2-transitive. Similarly, since $H_\gamma \le (G_2)_\gamma \cap (G_2)_{\gamma'}$, we deduce also that G_2 is 3-transitive.

From the classification of the finite 3-transitive groups, we deduce that

1. G_2 equals the Mathieu group M_n and $n = |\Omega| \in \{11, 12, 22, 23, 24\}$ or
2. $G_2 = M_{11}$ and $|\Omega| = 12$ or
3. $\mathbf{F}^*(G_2) = \mathrm{PSL}_2(q)$ and $|\Omega| = q + 1$.

Using this information, a computation with the computer algebra system `magma` shows that cases 1 and 2 do not arise, because $\mathcal{O}_G(H)$ is not Boolean of rank 2. In case 3, from the structure of $\mathrm{PSL}_2(q)$ we deduce that $G_1 \cap G_2$ is solvable, and hence $G_1 \cap G_2$ is a solvable group acting primitively on $|\Omega| - 2$ points. This yields the result that $q - 1$ is a prime power, say $q - 1 = x^y$, for some prime x and for some positive integer y. Write $q = p^f$ for some prime power p and some positive integer f. Since $p^f - 1$ is a power of a prime, we deduce that $p^f - 1$ has no primitive prime divisors. From a famous result of Zsigmondy [34], this yields

(a) $f = 1$, $x = 2$ and $q - 1 = 2^y$ or
(b) $q = 9$, $x = 2$ and $y = 3$ or
(c) $p = 2$, f is prime and $q - 1 = 2^f - 1 = x$ is a prime.

We can now refine further our argument. Recall that $G_1 \cap G_2$ is a maximal subgroup of $G_1 \cong \mathrm{Sym}(\Omega \setminus \Gamma)$. Since $G_1 \cap G_2$ is solvable, we deduce that $G_1 \cap G_2$ is isomorphic to the general linear group $\mathrm{AGL}_y(x)$ and hence $|G_1 \cap G_2| = x^y |\mathrm{GL}_y(x)| = (q - 1)|\mathrm{GL}_y(x)|$. Since $G_2 = \mathbf{N}_{\mathrm{Alt}(q+1)}(\mathrm{PSL}_2(q))$ and $|\mathrm{Aut}(\mathrm{PSL}_2(q))| = fq(q^2 - 1)$, we deduce that $|G_1 \cap G_2|$ divides $2f(q - 1)$. Therefore $|\mathrm{GL}_y(x)|$ divides $2f$. Cases **(a)** and **(b)** are readily seen to be impossible, and in case **(c)** we have that $|\mathrm{GL}_1(x)| = 2^f - 2 = 2(2^{f-1} - 1)$ divides $2f$, which is possible only when $f = 3$. A computation reveals that in this latter case, $\mathcal{O}_G(H)$ has five elements and hence is not Boolean.

Case 3: $|\Gamma| = 1$.

We assume that the conclusion in part 1 of this theorem does not hold, and hence G_2 is a primitive subgroup of G with $\mathrm{Alt}(\Omega) \not\leq G_2$.

Assume that $H = G_1 \cap G_2$ acts intransitively on $\Omega \setminus \Gamma$ and let Δ be one of its smallest orbits. In particular, H fixes setwise Γ, Δ and $\Omega \setminus (\Gamma \cup \Delta)$. Now, $\mathrm{Alt}(\Omega) \cap (\mathrm{Sym}(\Gamma \cup \Delta) \times \mathrm{Sym}(\Omega \setminus (\Gamma \cup \Delta)))$ is a proper overgroup of H that is intransitive and different from G_1, which is a contradiction. Therefore H acts transitively on $\Omega \setminus \Gamma$. Suppose that H acts imprimitively on $\Omega \setminus \Gamma$. Since H is maximal in $G_1 \cong \mathrm{Sym}(\Omega \setminus \Gamma)$, we deduce $H = \mathbf{N}_{G_1}(\Sigma)$, where Σ is a nontrivial (a, b)-regular partition of $\Omega \setminus \Gamma$. If $a \geq 3$, then H contains a 3-cycle and hence so does G_2. Since G_2 is primitive, we deduce from [10, Theorem 3.3 A] that $\mathrm{Alt}(\Omega) \leq G_2$, which is a contradiction. If $a = 2$, then H contains a permutation that is the product of two disjoint transpositions. Since G_2 is primitive, we deduce from [10, Theorem 3.3 D and Example 3.3.1] that either $\mathrm{Alt}(\Omega) \leq G_2$ or $|\Omega| \leq 8$. The first possibility is clearly impossible. In the second case, as $a = 2$, we have that $|\Omega \setminus \Gamma|$ is even and hence $|\Omega| \in \{5, 7\}$. However, a computation in $\mathrm{Alt}(5)$, $\mathrm{Sym}(5)$, $\mathrm{Alt}(7)$ and $\mathrm{Sym}(7)$ reveals that no case arises. Therefore

$$H \text{ acts primitively on } \Omega \setminus \Gamma.$$

In particular, G_2 is 2-transitive on Ω. One of the first main applications of the classification of finite simple groups is the classification of the finite 2-transitive groups (see [8]). These groups are either affine or almost simple. For the rest of the proof we go through this classification to investigate G_2 further; we assume that the reader is broadly familiar with this classification and refer the reader to [10, Section 7.7].

Case 3A: G_2 is affine.

Since G_2 is a maximal subgroup of G, we deduce that $G_2 \cong G \cap \mathrm{AGL}_d(p)$ for some prime number p and some positive integer d. Now, $G_1 \cap G_2 \cong G \cap \mathrm{GL}_d(p)$ and the action of $G_1 \cap G_2$ on $\Omega \setminus \Gamma$ is permutation isomorphic to the natural action of a certain subgroup of index at most 2 of the linear group $\mathrm{GL}_d(p)$ acting on the nonzero vectors of a d-dimensional vector space over the field with p elements. Clearly, this action is primitive if and only if $d = 1$ and $p - 1$ is prime or $p = 2$. Indeed, if V is the d-dimensional vector space over the field $\mathbb{F}_p$ with p elements, then $\mathrm{GL}_d(p)$ preserves the partition $\{\{av \mid a \in \mathbb{F}_p, a \neq 0\} \mid v \in V, v \neq 0\}$ of $V \setminus \{0\}$. This partition is the trivial partition only when $p = 2$ or $d = 1$. When $d = 1$, the group $\mathrm{GL}_1(p)$ is cyclic of order $p - 1$ and acts primitively on $V \setminus \{0\}$ if and only if $p - 1$ is a prime number. Since the only two consecutive primes are 2 and 3, in the latter case we obtain $|\Omega| = 3$ and no case arises here. Thus $p = 2$.

If $d \leq 2$, then $\mathrm{Alt}(\Omega) \leq G_2$, which is a contradiction. Therefore $d \geq 3$. With a computation (using the fact that $\mathrm{GL}_d(2)$ is generated by transvections, for example) we see that when $d \geq 3$, the group $\mathrm{AGL}_d(2)$ consists of even permutations and hence $\mathrm{AGL}_d(2) \leq \mathrm{Alt}(\Omega)$. This implies $G = \mathrm{Alt}(\Omega)$, and we obtain one of the examples stated in the theorem, namely part 4**(a)**.

Case 3B: $G_2 \cong \mathrm{Sp}_{2m}(2)$ and $|\Omega| = 2^{m-1}(2^m + 1)$ or $|\Omega| = 2^{m-1}(2^m - 1)$.

The group $G_1 \cap G_2$ is isomorphic to either $\mathrm{O}^+_{2m}(2)$ or $\mathrm{O}^-_{2m}(2)$, depending on whether $|\Omega| = 2^{m-1}(2^m + 1)$ or $|\Omega| = 2^{m-1}(2^m - 1)$. Since G_2 is a simple group, we deduce $G_2 \leq \mathrm{Alt}(\Omega)$ and hence $G = \mathrm{Alt}(\Omega)$. We obtain part 4**(b)**.

Case 3C: $\mathbf{F}^*(G_2) \cong \mathrm{PSU}_3(q)$ and $|\Omega| = q^3 + 1$.

Let $q = p^f$ for some prime number p and some positive integer f. Observe that $G_1 \cap G_2$ is solvable, is a maximal subgroup of G_1 and acts primitively on $\Omega \setminus \Gamma$. From this we deduce that $G_1 \cap G_2$ is isomorphic to $G \cap \mathrm{AGL}_{3f}(p)$. Since $|\mathrm{Aut}(\mathrm{PSU}_3(q))| = 2f(q^3 + 1)q^3(q^2 - 1)$ and $|\Omega| = q^3 + 1$, we deduce that the order of $G_1 \cap G_2$ is a divisor of $2fq^3(q^2 - 1)$. Therefore $|\mathrm{AGL}_{3f}(p)| = q^3|\mathrm{GL}_{3f}(p)|$ divides $4fq^3(q^2 - 1)$. (The extra 2 multplied into $2fq^3(q^2 - 1)$ takes into account the case where $G = \mathrm{Alt}(\Omega)$ and $G \cap \mathrm{AGL}_{3f}(p)$ has index 2 in $\mathrm{AGL}_{3f}(p)$.) Therefore $|\mathrm{GL}_{3f}(p)|$ divides $4f(q^2 - 1)$. However, the inequality $|\mathrm{GL}_{3f}(p)| \leq 4f(p^{2f} - 1)$ is never satisfied.

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Case 3D: $\mathbf{F}^*(G_2) \cong \mathrm{Sz}(q)$, $q = 2^f$ for some odd positive integer $f \geq 3$ and $|\Omega| = q^2 + 1$.

Since $\mathrm{Aut}(\mathrm{Sz}(q)) \cong \mathrm{Sz}(q).f$ and f is odd, we deduce $G_2 \leq \mathrm{Alt}(\Omega)$. In particular, $G = \mathrm{Alt}(\Omega)$. As in the previous case, $G_1 \cap G_2$ is solvable, is a maximal subgroup of G_1 and acts primitively on $\Omega \setminus \Gamma$. From this we deduce that $G_1 \cap G_2$ is isomorphic to $G \cap \mathrm{AGL}_{2f}(2)$. Since $|\mathrm{Aut}(\mathrm{Sz}(q))| = f(q^2+1)q^2(q-1)$ and $|\Omega| = q^2+1$, we deduce that the order of $G_1 \cap G_2$ is a divisor of $fq^2(q-1)$. Therefore $|\mathrm{AGL}_{2f}(2)| = q^2|\mathrm{GL}_{2f}(2)|$ divides $4fq^2(q-1)$ and $|\mathrm{GL}_{2f}(2)|$ divides $4f(q-1)$. However, the inequality $|\mathrm{GL}_{2f}(2)| \leq 4f(2^f-1)$ is never satisfied.

Case 3E: $\mathbf{F}^*(G_2) \cong \mathrm{Ree}(q)$, $q = 3^f$ for some odd positive integer $f \geq 1$ and $|\Omega| = q^3 + 1$.

Since $\mathrm{Aut}(\mathrm{Ree}(q)) \cong \mathrm{Ree}(q).f$ and f is odd, we deduce $G_2 \leq \mathrm{Alt}(\Omega)$. In particular, $G = \mathrm{Alt}(\Omega)$. As in the previous cases, $G_1 \cap G_2$ is solvable, is a maximal subgroup of G_1 and acts primitively on $\Omega \setminus \Gamma$. From this we deduce that $G_1 \cap G_2$ is isomorphic to $G \cap \mathrm{AGL}_{3f}(3)$. Since $|\mathrm{Aut}(\mathrm{Ree}(q))| = f(q^3+1)q^3(q-1)$ and $|\Omega| = q^3+1$, we deduce that the order of $G_1 \cap G_2$ is a divisor of $fq^3(q-1)$. Therefore $|\mathrm{AGL}_{3f}(3)| = q^3|\mathrm{GL}_{3f}(3)|$ divides $4fq^3(q-1)$ and $|\mathrm{GL}_{3f}(3)|$ divides $4f(q-1)$. However, the inequality $|\mathrm{GL}_{3f}(3)| \leq 4f(3^f-1)$ is never satisfied.

Case 3F: $(G_2, |\Omega|) \in \{(HS, 176), (Co_3, 276), (\mathrm{Alt}(7), 15), (\mathrm{PSL}_2(11), 11), (M_{11}, 12)\}$.

Since $\mathrm{PSL}_2(11) < M_{11}$ in their degree 11 actions, $\mathrm{Alt}(7) < \mathrm{PSL}_4(2)$ in their degree 15 actions and $M_{11} < M_{12}$ in their degree 12 actions, we see that $\mathrm{PSL}_2(11)$, $\mathrm{Alt}(7)$ and M_{12} are not maximal in G and hence cannot be G_2. Therefore, we are left with $(G_2, |\Omega|) \in \{(HS, 176), (Co_3, 276)\}$. We obtain part 4**(c)** and **(d)**.

Case 3G: $(G_2, |\Omega|) \in \{(M_{11}, 11), (M_{12}, 12), (M_{22}, 22), (M_{22}.2, 22), (M_{23}, 23), (M_{24}, 24)\}$.

With a computer computation we see that when $G_2 \cong M_{11}$, the lattice $\mathcal{O}_G(H)$ is not Boolean. The cases M_{22} and $M_{22}.2$ do not arise, because in these two cases $G_1 \cap G_2$ is isomorphic to either $\mathrm{PSL}_3(4)$ (when $G = \mathrm{Alt}(\Omega)$) or $\mathrm{P\Sigma L}_3(4)$ (when $G = \mathrm{Sym}(\Omega)$). However, these two groups are not maximal subgroups of G_1, because they are contained respectively in $\mathrm{PGL}_3(4)$ and in $\mathrm{P\Gamma L}_3(4)$. Therefore, we are left with $(G_2, |\Omega|) \in \{(M_{12}, 12), (M_{23}, 23), (M_{24}, 24)\}$. The case $(G_2, |\Omega|) = (M_{23}, 23)$ also does not arise, because with a computation we see that $\mathcal{O}_G(H)$ consists of five elements. Thus we are left with only part 4**(e)** and **(f)**.

Case 3H: $\mathbf{F}^*(G_2) \cong \mathrm{PSL}_d(q)$ for some prime power q and some positive integer $d \geq 2$ and $|\Omega| = (q^d-1)/(q-1)$.

Since the group G_2 is acting on the points of a $(d-1)$-dimensional projective space, we deduce that $G_1 \cap G_2$ acts primitively on $\Omega \setminus \Gamma$ only when G_2 is acting on the projective line – that is, $d = 2$. (Indeed, consider the action of $X := \mathrm{P\Gamma L}_d(q)$ on the points of the projective space $\mathcal{P}$, consider a point p of $\mathcal{P}$ and consider the stabiliser Y of the point p in X. Then Y preserves a natural partition on $\mathcal{P} \setminus \{p\}$, where two points p_1 and p_2 are declared to be in the same part if the lines $\langle p, p_1\rangle$ and $\langle p, p_2\rangle$ are equal. This partition is trivial only when $\mathcal{P}$ is a line, that is, $d = 2$.) Let $q = p^f$ for some prime number p and some positive integer f. Observe that $G_1 \cap G_2$ is solvable, is a maximal subgroup of G_1 and acts primitively on $\Omega \setminus \Gamma$. From this we deduce that $G_1 \cap G_2$ is isomorphic to $G \cap \mathrm{AGL}_f(p)$. Since $|\mathrm{Aut}(\mathrm{PSL}_2(q))| = f(q^2-1)q$ and $|\Omega| = q+1$, we deduce that the order of $G_1 \cap G_2$ is a divisor of $f(q-1)q$. Therefore $|\mathrm{AGL}_f(p)| = q|\mathrm{GL}_f(p)|$ divides $2f(q-1)q$. (The extra 2 in front of $f(q-1)q$ takes in account the case where $G = \mathrm{Alt}(\Omega)$ and $G \cap \mathrm{AGL}_f(p)$ has index 2 in $\mathrm{AGL}_f(p)$). Therefore $|\mathrm{GL}_f(p)|$ divides $2f(q-1)$. The inequality $|\mathrm{GL}_f(p)| \leq 2f(p^f-1)$ is satisfied only when $f = 1$ or $p = f = 2$. When $p = f = 2$, we have $|\Omega| = 5$ and hence $G_2 = \mathrm{Alt}(\Omega)$, which is not the case. Thus $q = p$ and $f = 1$. In particular, $\mathbf{F}^*(G_2) = \mathrm{PSL}_2(p)$ for some prime number p and we obtain part 4**(g)** or **(h)** depending on whether $G = \mathrm{Sym}(\Omega)$ or $G = \mathrm{Alt}(\Omega)$. □

Hypothesis 6.3. Let G be either $\mathrm{Sym}(\Omega)$ or $\mathrm{Alt}(\Omega)$, let Γ be a subset of Ω with $1 \leq |\Gamma| < |\Omega|/2$, let $G_1 := \mathbf{N}_G(\Gamma)$, let G_2 and G_3 be maximal subgroups of G and let $H := G_1 \cap G_2 \cap G_3$. Assume that $\mathcal{O}_G(H)$ is Boolean of rank 3 with maximal elements G_1, G_2 and G_3.

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Theorem 6.4. *Assume Hypothesis 6.3. Then, relabelling the indexed set $\{1,2,3\}$ if necessary, one of the following holds:*

1. $G = \mathrm{Sym}(\Omega)$*, G_2 is an imprimitive group having Γ as a block of imprimitivity and* $G_3 = \mathrm{Alt}(\Omega)$.
2. $G = \mathrm{Sym}(\Omega)$*,* $|\Gamma| = 1$*,* $G_3 = \mathrm{Alt}(\Omega)$*,* $G_2 \cong \mathrm{PGL}_2(p)$ *for some prime p and* $|\Omega| = p+1$.
3. $G = \mathrm{Alt}(\Omega)$*,* $|\Gamma| = 1$*,* $G_2 \cong G_3 \cong M_{24}$ *and* $|\Omega| = 24$.

Proof. A computation shows that the largest Boolean lattice in $\mathrm{Alt}(\Omega)$ when $|\Omega| = 7$ has rank 2. Hence, in the rest of our argument we suppose that $|\Omega| \neq 7$; in particular, Theorem 6.2 part 3 does not arise.

We apply Theorem 6.2 to the pairs $\{G_1, G_2\}$ and $\{G_1, G_3\}$. Relabelling the indexed set $\{2,3\}$ if necessary, we have to consider in turn each of the following cases:

A G_2 and G_3 are imprimitive (hence G_2 and G_3 are stabilisers of nontrivial regular partitions having Γ as one block).
B G_2 is imprimitive and G_3 is primitive.
C G_2 and G_3 are primitive.

Case A: Since $\mathcal{O}_G(G_2 \cap G_3)$ is Boolean of rank 2, from Lemma 2.6 we deduce that either $G_2 \cap G_3$ is transitive or G_2 or G_3 is the stabiliser of an $(|\Omega|/2, 2)$-regular partition. As $|\Gamma| \neq |\Omega|/2$, we deduce that $G_2 \cap G_3$ is transitive. Therefore, we are in the position to apply Theorem 5.2 to the pair $\{G_2, G_3\}$. However, none of the possibilities there can arise here, because both G_2 and G_3 have Γ as a block of imprimitivity and $1 \leq |\Gamma| < |\Omega|/2$.

Case B: From Theorem 6.2, we have that Γ is a block of imprimitivity for G_2. If $G_3 = \mathrm{Alt}(\Omega)$, then we obtain part 1. Suppose then $G_3 \neq \mathrm{Alt}(\Omega)$. As $|\Gamma| \neq |\Omega|/2$, Lemma 2.6 implies that $G_2 \cap G_3$ is transitive and hence we can apply Theorem 5.2 to the pair $\{G_2, G_3\}$. In particular, Theorem 5.2 part 2 holds, and hence G_3 is an affine primitive group and G_2 is the stabiliser of an $(n/2, 2)$-regular partition. Thus $|\Gamma| = |\Omega|/2$, which is a contradiction.

Case C: Suppose that either G_2 or G_3 equals $\mathrm{Alt}(\Omega)$. Relabelling the indexed set $\{2,3\}$ if necessary, we may suppose that $G_3 = \mathrm{Alt}(\Omega)$. In particular, $G = \mathrm{Sym}(\Omega)$. Now Theorem 6.2 implies that $|\Gamma| = 1$, $G_2 \cong \mathrm{PGL}_2(p)$ for some prime p and $|\Omega| = p+1$. Therefore, we obtain part 2.

It remains to consider the case where G_2 and G_3 are both primitive and both different from $\mathrm{Alt}(\Omega)$. As $|\Omega| \neq 7$, Theorem 6.2 implies that $|\Gamma| = 1$ and G_2 and G_3 are among the groups described in part 4. Now, $G_1 \cong \mathrm{Sym}(\Omega \setminus \Gamma)$ or $G_1 \cong \mathrm{Alt}(\Omega \setminus \Gamma)$, depending on whether $G = \mathrm{Sym}(\Omega)$ or $G = \mathrm{Alt}(\Omega)$. Moreover, $\mathcal{O}_G(G_2 \cap G_3)$ is a Boolean lattice of rank 2 having G_2 and G_3 as maximal elements. From Lemma 2.7, we deduce that either $G_2 \cap G_3$ acts primitively on Ω or $G = \mathrm{Alt}(\Omega)$, $G_2 \cap G_3 = \mathbf{N}_G(\Sigma)$ for some $(2,4)$-regular partition Σ. In the latter case, we see with a computation that the lattice $\mathcal{O}_G(G_1 \cap G_2 \cap G_3)$ is not Boolean (see also Figure 1). Therefore

$$G_2 \cap G_3 \text{ acts primitively on } \Omega.$$

Consider then $H := G_1 \cap G_2 \cap G_3$ and suppose that H is intransitive on $\Omega \setminus \Gamma$. Since $|\Omega \setminus \Gamma| = |\Omega| - 1$, H has an orbit $\Delta \subseteq \Omega \setminus \Gamma$ with $1 \leq |\Delta| < |\Omega|/2$. Then $\mathbf{N}_G(\Delta) \in \mathcal{O}_G(H)$ and $\mathbf{N}_G(\Delta)$ is a maximal element of $\mathcal{O}_G(H)$, contradicting the fact that G_1 is the only intransitive element in $\mathcal{O}_G(H)$. Thus H is transitive on $\Omega \setminus \Gamma$. Therefore

$$G_2 \cap G_3 \text{ acts 2-transitively on } \Omega. \tag{1}$$

Suppose that G_2 is as in Theorem 6.2 part 4**(a)** – that is, $G_2 \cong \mathrm{AGL}_d(2)$ for some $d \geq 3$. Let V_2 be the socle of G_2. From Lemma 4.3 applied with H there replaced by $G_2 \cap G_3$ here, we have either $V_2 \leq G_2 \cap G_3$ or $|\Omega| = 8$, $G = \mathrm{Alt}(\Omega)$ and $G_2 \cap G_3 \cong \mathrm{PSL}_2(7)$. In the second case, $G_1 \cap G_2 \cap G_3 \cong C_7 \rtimes C_3$; however, a computation yields that $\mathcal{O}_{\mathrm{Alt}(8)}(C_7 \rtimes C_3)$ is not Boolean of rank 3. Therefore $V_2 \leq G_2 \cap G_3$. The only primitive groups in Theorem 6.2 part 4 with $|\Omega|$ a power of a prime are $\mathrm{AGL}_d(2)$ or $\mathrm{PSL}_2(p)$ when $p + 1 = 2^d$. In particular, either $G_3 \cong \mathrm{AGL}_d(2)$ or $G_3 \cong \mathrm{PSL}_2(p)$ and $p + 1 = 2^d$. In the second case, since the elementary abelian 2-group V_2 is contained in $G_2 \cap G_3$, we deduce that $\mathrm{PSL}_2(p)$ contains an elementary abelian 2-group of order 2^d, which is impossible. Therefore, $G_3 \cong \mathrm{AGL}_d(2)$. Let V_3 be the socle of G_3. From Lemma 4.3, we deduce that $V_3 \leq G_2 \cap G_3$. In particular, $V_2 \trianglelefteq G_2 \cap G_3$ and $V_3 \trianglelefteq G_2 \cap G_3$. Since $G_2 \cap G_3$ is primitive, we infer $V_2 = V_3$ and hence $G_2 = \mathbf{N}_G(V_2) = \mathbf{N}_G(V_3) = G_3$, which is a contradiction.

Suppose that G_2 is as in Theorem 6.2 part 4**(b)** – that is, $G_2 \cong \mathrm{Sp}_{2m}(2)$. To deal with both actions simultaneously we set $\Omega^+ := \Omega$ when $|\Omega| = 2^{m-1}(2^m + 1)$ and $\Omega^- := \Omega$ when $|\Omega| = 2^{m-1}(2^m - 1)$. We can read off from [18, Table 1] the maximal subgroups of G_2 transitive on either Ω^+ or Ω^- (this is our putative $G_2 \cap G_3$). Comparing these candidates with the list of 2-transitive groups, we see that none of these groups is 2-transitive, contradicting statement (1).

Suppose that G_2 is as in Theorem 6.2 part 4**(c)** – that is, $G_2 \cong HS$. The only maximal subgroup of G_2 primitive on Ω is M_{22} in its degree 176 action. Thus $G_2 \cap G_3 \cong M_{22}$ in its degree 176 action. However, this action is not 2-transitive, contradicting statement (1).

Suppose that G_2 is as in Theorem 6.2 part 4**(d)** – that is, $G_2 \cong Co_3$. From [18, Table 6], we see that Co_3 has no proper subgroup acting primitively on Ω. Therefore this case does not arise in our investigation.

Suppose that G_2 is as in Theorem 6.2 part 4**(e)** – that is, $G_2 \cong M_{12}$. In particular, $G_1 \cap G_2 \cong M_{11}$. Up to conjugacy, there are five maximal subgroups of M_{11} (see [9]): one of them is our putative $G_1 \cap G_2 \cap G_3$. For each of these five subgroups, with the help of a computer we have computed the orbits on Ω. Observe that one of these orbits is Γ. If $G_1 \cap G_2 \cap G_3$ is intransitive on $\Omega \setminus \Gamma$, then $\mathcal{O}_G(H)$ contains a maximal intransitive subgroup which is not G_1, contradicting our assumptions. Among the five choices, there is only one (isomorphic to $\mathrm{PSL}_2(11)$) which is transitive on $\Omega \setminus \Gamma$. Thus $G_1 \cap G_2 \cap G_3 \cong \mathrm{PSL}_2(11)$. Next, we computed $\mathcal{O}_{\mathrm{Alt}(12)}(\mathrm{PSL}_2(11))$ and we checked that it is not Boolean (it is a lattice of size 6).

Suppose that G_2 is as in Theorem 6.2 part 4**(f)** – that is, $G_2 \cong M_{24}$. The only maximal subgroup of M_{24} acting primitively is $\mathrm{PSL}_2(23)$. Thus $G_2 \cap G_3 \cong \mathrm{PSL}_2(23)$, and $G_1 \cap G_2 \cap G_3 \cong C_{23} \rtimes C_{11}$. Now, $\mathcal{O}_{G_1}(G_1 \cap G_2 \cap G_3) \cong \mathcal{O}_{\mathrm{Alt}(23)}(C_{23} \rtimes C_{11})$. Since $\mathcal{O}_{G_1}(G_1 \cap G_2 \cap G_3)$ is Boolean of rank 2, so is $\mathcal{O}_{\mathrm{Alt}(23)}(C_{23} \rtimes C_{11})$. We checked with the help with a computer that $\mathcal{O}_{\mathrm{Alt}(24)}(C_{23} \rtimes C_{11})$ is Boolean of rank 3, and this gives rise to the marvellous example in Theorem 6.4 part 3.

Using the subgroup structure of $\mathrm{PSL}_2(p)$ and $\mathrm{PGL}_2(p)$ with p prime, we see that $\mathrm{PSL}_2(p)$ does not contain a proper subgroup acting primitively on the $p + 1$ points of the projective line, whereas the only proper primitive subgroup of $\mathrm{PGL}_2(p)$ acting primitively on the projective line is $\mathrm{PSL}_2(p)$. Thus Theorem 6.2 part 4**(h)** does not arise, and if part 4**(g)** arises, then $G_2 \cap G_3 \cong \mathrm{PSL}_2(p)$. However this is impossible, because it implies that $G_2 \cap G_3 \leq \mathrm{Alt}(\Omega)$ and hence $\mathrm{Alt}(\Omega)$ must be a maximal element of $\mathcal{O}_G(H)$. We have dealt with this situation already. □

Corollary 6.5. *Let H be a subgroup of G and suppose that $\mathcal{O}_G(H)$ is Boolean of rank $\ell \geq 3$ and that $\mathcal{O}_G(H)$ contains a maximal element which is intransitive. Then $\ell = 3$; moreover, relabelling the indexed set $\{1, 2, 3\}$ if necessary, $G_1 = \mathbf{N}_G(\Gamma)$ for some $\Gamma \subseteq \Omega$ with $1 \leq |\Gamma| < |\Omega|/2$ and one of the following holds:*

1. *$G = \mathrm{Sym}(\Omega)$, G_2 is an imprimitive group having Γ as a block of imprimitivity and $G_3 = \mathrm{Alt}(\Omega)$.*
2. *$G = \mathrm{Sym}(\Omega)$, $|\Gamma| = 1$, $G_3 = \mathrm{Alt}(\Omega)$ and $G_2 \cong \mathrm{PGL}_2(p)$ for some prime p and $|\Omega| = p + 1$.*
3. *$G = \mathrm{Alt}(\Omega)$, $|\Gamma| = 1$, $G_2 \cong G_3 \cong M_{24}$ and $|\Omega| = 24$.*

Proof. Let $G_1, G_2, \ldots, G_\ell$ be the maximal elements of $\mathcal{O}_G(H)$. Relabelling the indexed set if necessary, we may suppose that $G_1 = \mathbf{N}_G(\Gamma)$ for some $\Gamma \subseteq \Omega$ with $1 \leq |\Gamma| < |\Omega|/2$. From Theorem 6.4 applied to $\mathcal{O}_G(G_1 \cap G_2 \cap G_3)$, we obtain that G_1, G_2, G_3 satisfy one of the cases listed there. We consider these cases in turn. Suppose $G_3 = \mathrm{Alt}(\Omega)$ and G_2 is an imprimitive group having Γ as a block of imprimitivity. If $\ell \geq 4$, then we can apply Theorem 6.4 to $\{G_1, G_2, G_4\}$ and deduce that $G_4 = \mathrm{Alt}(\Omega) = G_3$, which is a contradiction. Suppose then $|\Gamma| = 1$, $G_3 = \mathrm{Alt}(\Omega)$, $G_2 \cong \mathrm{PGL}_2(p)$ for some prime p and $|\Omega| = p + 1$. If $\ell \geq 4$, then we can apply Theorem 6.4 to $\{G_1, G_2, G_4\}$ and deduce that $G_4 = \mathrm{Alt}(\Omega) = G_3$, which is a contradiction.

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Finally, suppose that $G = \mathrm{Alt}(\Omega)$, $|\Omega| = 24$, $|\Gamma| = 1$ and $G_2 \cong G_2 \cong M_{24}$. If $\ell \geq 4$, then we can apply Theorem 6.4 to $\{G_1, G_2, G_4\}$ and deduce that $G_4 \cong M_{24}$. In particular, $\mathcal{O}_{G_1}(G_1 \cap G_2 \cap G_3 \cap G_4)$ is a Boolean lattice of rank 3 having three maximal subgroups $G_1 \cap G_2, G_1 \cap G_3, G_1 \cap G_4$ all isomorphic to M_{23}. Arguing as usual, $G_1 \cap G_2 \cap G_3 \cap G_4$ acts transitively on $\Omega \setminus \Gamma$. Therefore, M_{23} has a chain $M_{23} > A > B > C$ with C maximal in B, B maximal in A and A maximal in M_{23}, with C transitive. However, there is no such chain. □

7. Proof of Theorem 1.2

We use the notation and terminology from the statement of Theorem 1.2. If, for some $i \in \{1, \ldots, \ell\}$, G_i is intransitive, then the proof follows from Corollary 6.5. In particular, we may assume that G_i is transitive for every $i \in \{1, \ldots, \ell\}$. If, for some $i \in \{1, \ldots, \ell\}$, G_i is imprimitive, then the proof follows from Corollary 5.5. In particular, we may assume that G_i is primitive for every $i \in \{1, \ldots, \ell\}$. Now the proof follows from Corollary 4.7.

8. Large Boolean lattices arising from imprimitive maximal subgroups

In this section, we prove that G admits Boolean lattices $\mathcal{O}_G(H)$ of arbitrarily large rank, arising from Theorem 1.2 part 1. Let ℓ be a positive integer with $\ell \geq 2$ and let $\Sigma_1, \ldots, \Sigma_\ell$ be a family of nontrivial regular partitions of Ω with

$$\Sigma_1 < \Sigma_2 < \cdots < \Sigma_\ell.$$

For each $i \in \{1, \ldots, \ell\}$, we let

$$M_i := \mathbf{N}_G(\Sigma_i) = \{g \in G \mid X^g \in \Sigma_i, \forall X \in \Sigma_i\}$$

be the stabiliser of the partition Σ_i in G. More generally, for every $I \subseteq \{1, \ldots, \ell\}$, we let

$$M_I := \bigcap_{i \in I} M_i.$$

When $I = \{i\}$, we have $M_{\{i\}} = M_i$. Moreover, when $I = \emptyset$, we are implicitly setting $G = M_\emptyset$. We let $H := M_{\{1,\ldots,\ell\}}$.

Here we show that except when $|\Omega| = 8$ and $G = \mathrm{Alt}(\Omega)$,

$$\mathcal{O}_G(H) = \{M_I \mid I \subseteq \{1, \ldots, \ell\}\} \tag{1}$$

and hence $\mathcal{O}_G(H)$ is isomorphic to the Boolean lattice of rank ℓ. As usual, the case $|\Omega| = 8$ and $G = \mathrm{Alt}(\Omega)$ is exceptional because of Figure 1. To prove equation (1), it suffices to show that if $M \in \mathcal{O}_G(H)$, then there exists $I \subseteq \{1, \ldots, \ell\}$ with $M = M_I$.

We start by describing the structure of the groups M_I for each $I \subseteq \{1, \ldots, \ell\}$. Let $i \in \{1, \ldots, \ell\}$. Since M_i is the stabiliser of a nontrivial regular partition Σ_i, we have

$$M_i \cong G \cap (\mathrm{Sym}(n/n_i) \mathrm{wr}\, \mathrm{Sym}(n_i)),$$

where Σ_i is an $(n/n_i, n_i)$-regular partition. (Strictly speaking, we are abusing our notation in this equation: indeed, the group $\mathrm{Sym}(n/n_i)wr\, \mathrm{Sym}(n_i)$ is defined only as an abstract group, and not as a subgroup of $\mathrm{Sym}(\Omega)$. In order to be mathematically rigorous, we pay the price of having to use cumbersome notation. Therefore, for this proof and for the rest of the article, we have adopted a less precise notation when it should not cause any misunderstanding or confusion.) Since $\{\Sigma_i\}_{i=1}^{\ell}$ forms a chain, we deduce that n_i divides n_{i+1} for each $i \in \{1, \ldots, \ell - 1\}$. Now set $i, j \in \{1, \ldots, \ell\}$ with $i < j$. The group $M_{\{i,j\}} = \mathbf{N}_G(\Sigma_i) \cap \mathbf{N}_G(\Sigma_j)$ is the stabiliser in G of Σ_i and Σ_j. Since $\Sigma_i < \Sigma_j$, we deduce that

$$M_{\{i,j\}} \cong G \cap \left(\mathrm{Sym}(n/n_j) \mathrm{wr}\, \mathrm{Sym}(n_j/n_i) \mathrm{wr}\, \mathrm{Sym}(n_i)\right).$$

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The structure of an arbitrary element M_I is analogous. Let $I = \{i_1, \ldots, i_\kappa\}$ be a subset of I with $i_1 < i_2 < \ldots < i_\kappa$. Since $\Sigma_{i_1} < \Sigma_{i_2} < \cdots < \Sigma_\kappa$, we deduce that

$$M_I \cong G \cap \left(\mathrm{Sym}(n/n_{i_\kappa}) \mathrm{wr}\, \mathrm{Sym}(n_{i_\kappa}/n_{i_{\kappa-1}}) \mathrm{wr} \cdots \mathrm{wr}\, \mathrm{Sym}(n_{i_2}/n_{i_1}) \mathrm{wr}\, \mathrm{Sym}(n_{i_1})\right).$$

In particular,

$$H \cong G \cap (\mathrm{Sym}(n/n_\ell) \mathrm{wr}\, \mathrm{Sym}(n_\ell/n_{\ell-1}) \mathrm{wr} \cdots \mathrm{wr}\, \mathrm{Sym}(n_2/n_1) \mathrm{wr}\, \mathrm{Sym}(n_1)).$$

Before proceeding with our general argument we prove a preliminary lemma.

Lemma 8.1. *The only nontrivial systems of imprimitivity for H are $\Sigma_1, \ldots, \Sigma_\ell$ or $G = \mathrm{Alt}(\Omega)$ and $|\Omega| = 4$.*

Proof. Let $\Sigma := \{X_1, \ldots, X_\kappa\}$ be a nontrivial system of imprimitivity for H. Set $\Sigma_\ell = \{Y_1, \ldots, Y_\iota\}$. From the structure of H, it is clear that the action induced by $\mathbf{N}_H(Y_i)$ on Y_i is that of $\mathrm{Sym}(Y_i)$, for each $i \in \{1, \ldots, \iota\}$. Set $i \in \{1, \ldots, \iota\}$ and $j \in \{1, \ldots, \kappa\}$ with $Y_i \cap X_j \neq \emptyset$. Since Σ and Σ_ℓ are H-invariant, we have $|X_j \cap Y_i| = 1$ or $Y_i \subseteq X_j$. We investigate the first alternative a little further. Since Σ is nontrivial and $|Y_i| \geq 2$, there exists $j' \in \{1, \ldots, \kappa\} \setminus \{j\}$ with $X_{j'} \cap Y_i \neq \emptyset$. Therefore, we again have the two alternatives $|X_{j'} \cap Y_i| = 1$ or $Y_i \subseteq X_{j'}$. Suppose that $Y_i \subseteq X_{j'}$. It is readily seen from the structure of H that $\mathbf{N}_H(X_j) \cap \mathbf{N}_H(X_{j'})$ acts transitively on X_j. However, since Σ is H-invariant and $Y_i \subseteq X_{j'}$, we deduce that $\mathbf{N}_H(X_j) \cap \mathbf{N}_H(X_{j'})$ fixes setwise Y_i. Therefore, $\mathbf{N}_H(X_j) \cap \mathbf{N}_H(X_{j'})$ fixes the singleton $X_j \cap Y_i$, contradicting the fact that $\mathbf{N}_H(X_j) \cap \mathbf{N}_H(X_{j'})$ is transitive on X_j or the fact that Σ_ℓ is nontrivial.

Therefore, $|X_{j'} \cap Y_i| = 1$. Write $X_j \cap Y_i = \{x\}$. Now let $(\mathbf{N}_H(X_j))_x$ be the stabiliser of the point x in $\mathbf{N}_H(X_j)$. If $G = \mathrm{Sym}(\Omega)$, or $\kappa \geq 3$, or $|X_j| \geq 3$, then from the structure of H we deduce that $(\mathbf{N}_H(X_j))_x$ is transitive on $X_{j'}$. However, since Σ_ℓ is H-invariant, $x \in Y_i \in \Sigma_\ell$, we deduce that $(\mathbf{N}_H(X_j))_x$ fixes setwise Y_j, contradicting the fact that $|X_{j'} \cap Y_i| = 1$. Therefore, $G = \mathrm{Alt}(\Omega)$, $\iota = 2$ and $|X_j| = 2$ – that is, $|\Omega| = 4$ – and we have the first possibility in the statement of this lemma.

The previous paragraph shows that for every $j \in \{1, \ldots, \kappa\}$ and every $i \in \{1, \ldots, \iota\}$ with $X_j \cap Y_i \neq \emptyset$, we have $Y_i \subseteq X_j$. That is, $\Sigma \leq \Sigma_\ell$. The proof follows by induction on ℓ, replacing Ω with Σ_ℓ, G with $\mathrm{Sym}(\Sigma_\ell)$ and H with the permutation group induced by H on Σ_ℓ. □

We now continue with our construction and show equation (1), arguing by induction on ℓ. When $\ell = 1$, $H = M_1 = \mathbf{N}_G(\Sigma_1)$ and $\mathcal{O}_G(H) = \{H, G\}$, because H is a maximal subgroup of G by Fact 2.2 (recall that we are excluding the case $G = \mathrm{Alt}(\Omega)$ and $|\Omega| = 8$ in the discussion here). For the rest of the proof, we suppose $|\Omega| > 4$ and $\ell \geq 2$.

Let $M \in \mathcal{O}_G(H)$. Suppose M is primitive. As $H \leq M$, we deduce that M contains a 2-cycle or a 3-cycle (when $G = \mathrm{Sym}(\Omega)$ or when $n/n_1 \geq 3$), or a product of two transpositions (when $G = \mathrm{Alt}(\Omega)$ and $n/n_1 = 2$). From [10, Theorem 3.3 D and Example 3.3.1], either $\mathrm{Alt}(\Omega) \leq M$ or $|\Omega| \leq 8$. In the first case, $M = M_\emptyset$. When $|\Omega| \in \{6, 8\}$, we see with a direct inspection that no exception arises (recall that we are excluding the case $G = \mathrm{Alt}(\Omega)$ and $|\Omega| = 8$ in the discussion here). Therefore, M is not primitive.

Since M is imprimitive, $H \leq M$ and $\Sigma_1, \ldots, \Sigma_\ell$ are the only systems of imprimitivity left invariant by H, we deduce that M leaves invariant one of these systems of imprimitivity. Let $i \in \{1, \ldots, \ell\}$ be maximum, such that M leaves invariant Σ_i – that is, $M \leq M_i$. Fix $X \in \Sigma_i$ and consider $\mathbf{N}_M(X) = \{g \in M \mid X^g = X\}$. Consider also the natural projection

$$\pi : \mathbf{N}_{M_i}(X) \to \mathrm{Sym}(X) \cong \mathrm{Sym}(n/n_i).$$

This projection is surjective. For each $j \in \{1, \ldots, \ell\}$ with $i < j$, consider $\Sigma'_j := \{Y \in \Sigma_j \mid Y \subseteq X\}$. By construction, Σ'_j is a nontrivial regular partition of X and

$$\Sigma'_{i+1} < \Sigma'_{i+2} < \cdots < \Sigma'_\ell.$$

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Moreover,

$$\pi(\mathbf{N}_{M_j}(X)) = \mathbf{N}_{\mathrm{Sym}(X)}(\Sigma'_j).$$

In particular, as $\cap_{j=i+1}^{\ell}\mathbf{N}_{\mathrm{Sym}(X)}(\Sigma'_j) = \pi(\mathbf{N}_H(X)) \leq \pi(\mathbf{N}_M(X))$, by induction on ℓ,

$$\pi(\mathbf{N}_M(X)) = \bigcap_{j\in I'}\mathbf{N}_{\mathrm{Sym}(X)}(\Sigma'_j)$$

for some $I' \subseteq \{i+1,\dots,\ell\}$. If $I' \neq \emptyset$, then the action of $\mathbf{N}_M(X)$ on X leaves invariant some Σ'_j for some $j \in I'$. Since $\Sigma_i < \Sigma_j$ and M leaves invariant Σ_i, it is not hard to see that M leaves invariant Σ_j. However, as $i < j$, we contradict the maximality of i. Therefore $I' = \emptyset$ and hence

$$\pi(\mathbf{N}_M(X)) = \mathrm{Sym}(X).$$

Let $H_{(\Omega\setminus X)}$ and $M_{(\Omega\setminus X)}$ be the pointwise stabiliser of $\Omega \setminus X$ in H and in M, respectively. Thus $H_{(\Omega\setminus X)} \leq M_{(\Omega\setminus X)} \leq \mathrm{Sym}(X)$. From the definition of H and the fact that X is a block of Σ_i, we deduce that $H_{(\Omega\setminus X)}$ is isomorphic to

$$\begin{cases} \mathrm{Sym}(n/n_\ell)\,\mathrm{wr}\,\mathrm{Sym}(n_\ell/n_{\ell-1})\,\mathrm{wr}\cdots\mathrm{wr}\,\mathrm{Sym}(n_{i+1}/n_i) & \text{when } G = \mathrm{Sym}(\Omega),\\ \mathrm{Alt}(n/n_i) \cap (\mathrm{Sym}(n/n_\ell)\,\mathrm{wr}\,\mathrm{Sym}(n_\ell/n_{\ell-1})\,\mathrm{wr}\cdots\mathrm{wr}\,\mathrm{Sym}(n_{i+1}/n_i)) & \text{when } G = \mathrm{Alt}(\Omega).\end{cases}$$

We claim that

$$\mathrm{Alt}(X) \leq M_{(\Omega\setminus X)}. \tag{2}$$

When $i = \ell$, this is clear, because in this case $\mathrm{Alt}(X) \leq H_{(\Omega\setminus X)}$ from the structure of $H_{(\Omega\setminus X)}$. Suppose then $i \leq \ell - 1$. Assume first that either $n/n_\ell \geq 3$ or $n/n_i = |X| \geq 5$. From the description of $H_{(\Omega\setminus X)}$ and from $i \leq \ell - 1$, it is clear that $H_{(\Omega\setminus X)}$ contains a permutation g which is either a cycle of length 3 or the product of two transpositions. Define $V := \langle g^m \mid m \in \mathbf{N}_M(X)\rangle$. As $H \leq M$, we deduce that $g \in M_{(\Omega\setminus X)}$ and hence $V \leq M_{(\Omega\setminus X)}$. Since $\pi(\mathbf{N}_M(X)) = \mathrm{Sym}(X)$, we get $V \trianglelefteq \mathrm{Sym}(X)$ and hence $V = \mathrm{Alt}(X)$. In particular, our claim is proved in this case.

It remains to consider the case where $n/n_\ell = 2$ and $|X| < 5$. As $i \leq \ell - 1$, this yields $i = \ell - 1$, $n/n_\ell = n_\ell/n_{\ell-1} = 2$ and $|X| = 4$. Observe that in this case, the group V has order 4 and is the Klein subgroup of $\mathrm{Alt}(X)$. When $G = \mathrm{Sym}(\Omega)$, $H_{(\Omega\setminus X)}$ contains a transposition and hence we can repeat this argument, replacing g with this transposition; in this case, we deduce $M_{(\Omega\setminus X)} = \mathrm{Sym}(X)$ and hence our claim is proved. Assume then $G = \mathrm{Alt}(\Omega)$, $i = \ell - 1$, $n/n_\ell = n_\ell/n_{\ell-1} = 2$ and $|X| = 4$. Among all elements $h \in \mathbf{N}_M(X)$ with $\pi(h)$ a cycle of length 3, choose h with the maximum number of fixed points on Ω. Assume that h fixes pointwise some $X' \in \Sigma_i$. From the structure of H, we see that H contains a permutation g normalising both X and X', acting on both sets as a transposition and fixing pointwise $\Omega \setminus (X \cup X')$. Now a computation shows that $g^{-1}h^{-1}gh$ acts as a cycle of length 3 on X and fixes pointwise $\Omega \setminus X$ – that is, $g^{-1}h^{-1}gh \in M_{(\Omega\setminus X)}$. In particular, $\mathrm{Alt}(X) \leq M_{(\Omega\setminus X)}$ in this case. Therefore, we may suppose that h fixes pointwise no block $X' \in \Sigma_i$. Assume that h acts as a cycle of length 3 on three blocks $X_1, X_2, X_3 \in \Sigma_i$ – that is, $X_1^h = X_2$, $X_2^h = X_3$ and $X_3^h = X_1$. From the structure of H, we see that H contains a permutation g normalising both X and X_1, acting on both sets as a transposition and fixing pointwise $\Omega \setminus (X \cup X_1)$. Now a computation shows that $g^{-1}h^{-1}gh$ acts as a cycle of length 3 on X and as a transposition on X_1, and fixes pointwise $\Omega \setminus (X \cup X_1)$. In particular, $(g^{-1}h^{-1}gh)^2$ acts as a cycle of length 3 and fixes pointwise $\Omega \setminus X$. Thus $(g^{-1}h^{-1}gh)^2 \in M_{(\Omega\setminus X)}$ and $\mathrm{Alt}(X) \leq M_{(\Omega\setminus X)}$ again.

Finally, suppose that h fixes setwise but not pointwise each block in Σ_i. In particular, for each $X' \in \mathbf{N}_M(X)$ we have $X'^h = X'$ and h acts as a cycle of length 3 on X'. Let $X' \in \Sigma_i$ with $X' \neq X$. From the structure of H, we see that H contains a permutation g normalising both X and X', acting on both sets as a transposition and fixing pointwise $\Omega \setminus (X \cup X')$. Now a computation shows that $g^{-1}h^{-1}gh$ acts as a cycle of length 3 on X and on X' and fixes pointwise $\Omega \setminus (X \cup X')$. As h was choosen with the maximum number of fixed points, with $\pi(h)$ having order 3, we deduce that $\Omega = X \cup X'$, that is, $n = 8$. In particular, we end up with the exceptional case in Figure 1, which we are excluding in our discussion. Therefore, equation (2) is now proved.

Let K_i be the kernel of the action of M_i on Σ_i. Thus

$$K_i = G \cap \prod_{X \in \Sigma_i} \mathrm{Sym}(X).$$

From equation (2), we deduce

$$\mathrm{Alt}(n/n_i)^{n_i} \cong \prod_{X \in \Sigma_i} \mathrm{Alt}(X) \leq M.$$

As $H \leq M$, we obtain $K_i = H(\prod_{X \in \Sigma_i} \mathrm{Alt}(X)) \leq M$.

Since $\Sigma_1 < \Sigma_2 < \cdots < \Sigma_i$, for every $j \in \{1, \ldots, i\}$ we may consider Σ_j as a regular partition of Σ_i. More formally, define $\Omega'' := \Sigma_i$ and define $\Sigma_j'' := \{\{Y \in \Sigma_i \mid Y \subseteq Z\} \mid Z \in \Sigma_j\}$. Thus Σ_j'' is the quotient partition of Σ_j via Σ_i. Clearly, $M_j/K_i = \mathbf{N}_{M_i}(\Sigma_j'')$. Applying our induction hypothesis to the chain $\Sigma_1'' < \cdots < \Sigma_i''$, we have $M/K_i = M_I/K_i$ for some subset I of $\{1, \ldots, i\}$. Since $K_i \leq M$, we deduce $M = M_I$.

9. Large Boolean lattices arising from primitive maximal subgroups

Lemma 9.1. *Let Σ be a (c, d)-regular partition of Ω. Given a transitive subgroup U of* $\mathrm{Sym}(d)$*, we identify the group $X = \mathrm{Sym}(c) \mathrm{wr}\, U$ with a subgroup of $\mathbf{N}_{\mathrm{Sym}(\Omega)}(\Sigma)$. If X normalises a regular partition $\tilde{\Sigma}$ of Ω, then $\tilde{\Sigma} \leq \Sigma$.*

Proof. Let A and $\tilde{A}$ be blocks, respectively, of Σ and $\tilde{\Sigma}$ with $A \cap \tilde{A} \neq \emptyset$ and let $a \in A \cap \tilde{A}$. Then for every $z \in A \setminus \{a\}$, the transposition $(a, z) \in X$ fixes at least one element of $\tilde{A}$ and therefore (a, z) normalises $\tilde{A}$ and consequently $z \in \tilde{A}$. Therefore, either $A \subseteq \tilde{A}$ or $\tilde{A} \subseteq A$. From this, it follows that either $\Sigma \leq \tilde{\Sigma}$ or $\tilde{\Sigma} \leq \Sigma$. We can exclude the first possibility, because $\mathbf{N}_X(A)$ acts on A as the symmetric group $\mathrm{Sym}(A)$. □

Since we aim to prove that there exist Boolean lattices of arbitrarily large rank of the type described in Thereom 1.2 part 3, we suppose $n = |\Omega|$ is odd. Let ℓ be an integer with $\ell \geq 3$ and let

$$\mathcal{F}_1 < \cdots < \mathcal{F}_\ell$$

be a chain of regular product structures on Ω. In particular, $\mathcal{F}_\ell$ is a regular (a, b)-product structure for some integers $a \geq 5$ and $b \geq 2$ with a odd and $n = a^b$. From the partial order in the poset of regular product structures, we deduce that we may write $b = b_1 \cdots b_\ell$ such that, if we set $d_i := b_i \cdots b_\ell$ and $c_i := b/d_i$, then $\mathcal{F}_{\ell+1-i}$ is a regular (a^{c_i}, d_i)-product structure for every $i \in \{1, \ldots, \ell\}$.

Let $M_i := \mathbf{N}_{\mathrm{Sym}(\Omega)}(\mathcal{F}_i) \cong \mathrm{Sym}(a^{c_i}) \mathrm{wr}\, \mathrm{Sym}(d_i)$ and let $H := M_1 \cap \cdots \cap M_\ell$. We have

$$H := \mathrm{Sym}(a) \mathrm{wr}\, \mathrm{Sym}(b_1) \mathrm{wr}\, \mathrm{Sym}(b_2) \mathrm{wr} \cdots \mathrm{wr}\, \mathrm{Sym}(b_\ell)$$

as a permutation group of degree n. Moreover, if I is a subset of $\{1, \ldots, \ell\}$, we let $M_I := \cap_{i \in I} M_i$, where we are implicitly setting $M_\emptyset = \mathrm{Sym}(n)$. In particular, if $I = \{r_1, \ldots, r_s\}$, then M_I is isomorphic to

$$\mathrm{Sym}(a^{b_1 \cdots b_{r_1 - 1}}) \mathrm{wr}\, \mathrm{Sym}(b_{r_1} \cdots b_{r_2 - 1}) \mathrm{wr} \cdots \mathrm{wr}\, \mathrm{Sym}(b_{r_s} \cdots b_\ell).$$

To prove that $\mathcal{O}_G(H)$ is Boolean of rank ℓ, we need to show that for every $K \in \mathcal{O}_G(H)$, there exists $I \subseteq \{1, \ldots, \ell\}$ with $K = M_I$.

We may identity H with the wreath product $\mathrm{Sym}(a)\mathrm{wr}\, X$ with

$$X = \mathrm{Sym}(b_1)\mathrm{wr}\ \mathrm{Sym}(b_2)\mathrm{wr} \cdots \mathrm{wr}\ \mathrm{Sym}(b_\ell),$$

where X has degree b and is endowed of the imprimitive action of the iterated wreath product and $\mathrm{Sym}(a)\mathrm{wr}\, X$ is primitive of degree $n = a^b$ and endowed of the primitive action of the wreath product.

Lemma 9.2. *If H normalises a regular product structure $\mathcal{F}$, then $\mathcal{F} \in \{\mathcal{F}_1, \dots, \mathcal{F}_\ell\}$.*

Proof. The group $H = \mathrm{Sym}(a)\mathrm{wr}\, X$ is semisimple and not almost simple. Since the components of H are isomorphic to $\mathrm{Alt}(a)$ and a is odd, according to the definition in [2, Section 2], H is product indecomposable. From [2, Proposition 5.9 (5)], we deduce that $\mathcal{F}(H)$ is isomorphic to the dual of $\mathcal{O}_H(J) \setminus \{H\}$, where $J := \mathbf{N}_H(L)$ is the normaliser of a component L of H. Since $\mathbf{F}^*(H) = (\mathrm{Alt}(a))^b$, we have

$$J = \mathrm{Sym}(a) \times (\mathrm{Sym}(a)\mathrm{wr}\, Y) = \mathrm{Sym}(a) \times (\mathrm{Sym}(a)^{b-1} \rtimes Y),$$

with Y the stabiliser of a point in the imprimitive action of X of degree b. In particular, $\mathcal{O}_H(J) \setminus \{H\} \cong \mathcal{O}_X(Y) \setminus \{X\}$.

The proper subgroups of X containing the point stabiliser Y are in one-to-one correspondence with the regular partitions Σ of $\{1, \dots, b\}$ normalised by X and with at least two blocks. Notice that for any $i \in \{1, \dots, \ell\}$, there is an embedding of X in $\mathrm{Sym}(c_i)\mathrm{wr}\ \mathrm{Sym}(d_i)$, and therefore X normalises a regular (c_i, d_i)-partition, which we call $\Sigma_{\ell+1-i}$. An iterated application of Lemma 9.1 implies that $\Sigma_1 < \cdots < \Sigma_\ell$ are the unique nontrivial regular partitions normalised by X. □

Theorem 9.3. *If $H \le K \le \mathrm{Sym}(n)$, then $K = M_I$ for some subset I of $\{1, \dots, \ell\}$.*

Proof. Clearly, without loss of generality we can suppose that $H < K < \mathrm{Sym}(n)$. We apply [29, Proposition 7.1] to the inclusion (H, K). Since H has primitive components isomorphic to $\mathrm{Alt}(a)$, with a odd, only cases (ii,a) and (ii,b) can occur.

Assume that (H, K) is an inclusion of type (ii,a). In this case we have $H < K \le \mathrm{Sym}(a)\mathrm{wr}\ \mathrm{Sym}(b)$. Since $\mathrm{Sym}(a)^b \le H \le K$, we deduce that $K = \mathrm{Sym}(a)\mathrm{wr}\, Y$, with $X \le Y \le \mathrm{Sym}(b)$. So it suffices to notice that the only subgroups of $\mathrm{Sym}(b)$ containing X are those of the kind

$$\mathrm{Sym}(b_1 \cdots b_{t_1})\mathrm{wr}\ \mathrm{Sym}(b_{t_1+1} \cdots b_{t_2})\mathrm{wr} \cdots \mathrm{wr}\ \mathrm{Sym}(b_{t_s+1} \cdots b_\ell\},$$

for some subset $\{t_1, \dots, t_s\}$ of $\{1, \dots, \ell\}$. Indeed, this fact follows from Section 8.

Assume that (H, K) is an inclusion of type (ii,b). (In what follows, the precise meaning of the term “blow-up” can be found in [29, Section 2]; we refer the reader to that article for details. Here we do not give a full account because we are interested only in a particular consequence.) In this case, following the terminology in [29, Sections 2 and 7], $n = a^b = \alpha^{\gamma\delta}$, H is a blow-up of a subgroup Z of $\mathrm{Sym}(\alpha^\gamma)$ and (H, K) is a blow-up of a natural inclusion (Z, L), where $\mathrm{Alt}(\alpha^\gamma) \le L \le \mathrm{Sym}(\alpha^\gamma)$. From this we immediately deduce that H normalises a regular (α^γ, δ)-product structure $\mathcal{F}$. By Lemma 9.2, we have $\mathcal{F} \in \{\mathcal{F}_1, \dots, \mathcal{F}_\ell\}$. In particular, $\alpha^\gamma = a^{c_i}$, $\delta = d_i$ and $Z = \mathrm{Sym}(a)\mathrm{wr}\ \mathrm{Sym}(b_1)\mathrm{wr}\ \mathrm{Sym}(b_2)\mathrm{wr} \cdots \mathrm{wr}\ \mathrm{Sym}(b_i)$. Since a is odd, $Z \not\le \mathrm{Alt}(a^{c_i})$, so $L = \mathrm{Sym}(a^{c_i})$ and $(\mathrm{Sym}(a^{c_i}))^{d_i} \le K \le \mathrm{Sym}(a^{c_i})\mathrm{wr}\ \mathrm{Sym}(d_i)$. If H is maximal in K, then $i = 1$ and $K = \mathrm{Sym}(a^{b_1})\mathrm{wr}\ \mathrm{Sym}(b_2)\mathrm{wr} \cdots \mathrm{wr}\ \mathrm{Sym}(b_\ell) = M_{\{1,\dots,\ell-1\}}$; otherwise, we can proceed by induction on ℓ. □

10. Application to Brown’s problem

In this section we will prove Theorem 1.3 (where part 4 is a direct application of Theorem 1.2), which in these cases proves Conjecture 1.1 and provides a positive answer to the relative Brown’s problem.

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10.1. Some general lemmas

In this subsection we will prove some lemmas for every finite group. Let G be a finite group and H a subgroup such that the overgroup lattice $\mathcal{O}_G(H)$ is Boolean of rank ℓ, and let $M_1, \ldots, M_\ell$ be its coatoms. For any K in $\mathcal{O}_G(H)$, let us denote with $K^\complement$ its *lattice-complement*, that is, $K \wedge K^\complement = H$ and $K \vee K^\complement = G$.

Lemma 10.1. *If $\mathcal{O}_G(H)$ is Boolean of rank 2 and if H is normal in M_i ($i = 1, 2$), then $|M_1 : H| \neq |M_2 : H|$.*

Proof. As an immediate consequence of the assumption, H is normal in $M_1 \vee M_2 = G$, but then G/H is a group and $\mathcal{L}(G/H)$ is Boolean, so distributive, and G/H is cyclic by Ore's theorem; thus $|M_1/H| \neq |M_2/H|$. □

Lemma 10.2. *If $\mathcal{O}_G(H)$ is Boolean of rank 2, then $(|M_1 : H|, |M_2 : H|) \neq (2, 2)$.*

Proof. If $(|M_1 : H|, |M_2 : H|) = (2, 2)$, then H is normal in M_i ($i = 1, 2$), which is a contradiction by Lemma 10.1. □

Lemma 10.3. *If $\mathcal{O}_G(H)$ is Boolean of rank $\ell \leq 2$, then $\hat{\varphi}(H, G) \geq 2^{\ell-1}$.*

Proof. If $\ell = 1$, then

$$\hat{\varphi}(H, G) = |G : H| - |G : G| \geq 2 - 1 = 2^{\ell-1}.$$

If $\ell = 2$, by Lemma 10.2 there is an i with $|M_i : H| \geq 3$. Then

$$\begin{aligned} \hat{\varphi}(H, G) &= |G : H| - |G : M_1| - |G : M_2| + |G : G| \\ &= |G : H|(1 - |M_1 : H|^{-1} - |M_2 : H|^{-1}) + 1 \\ &\geq 6(1 - 1/3 - 1/2) + 1 = 2^{\ell-1}. \end{aligned}$$ □

Remark 10.4 (Product formula)**.** *Let A be a finite group and B, C two subgroups; then $|B| \cdot |C| = |BC| \cdot |B \cap C|$, so*

$$|B| \cdot |C| \leq |B \vee C| \cdot |B \wedge C| \ \text{ and } \ |B : B \wedge C| \leq |B \vee C : C|.$$

Lemma 10.5. *Let A be a finite group and B, C two subgroups. If $|A : C| = 2$ and $B \not\subseteq C$, then $|B : B \wedge C| = 2$.*

Proof. By the product formula, $2 \leq |B : B \wedge C| \leq |A : C| = 2$, because $A = B \vee C$. □

Lemma 10.6. *Let A be an atom of $\mathcal{O}_G(H)$. If $K_1, K_2 \in \mathcal{O}_{A^\complement}(H)$ with $K_1 < K_2$, then*

$$|K_1 \vee A : K_1| \leq |K_2 \vee A : K_2|.$$

Equivalently, if $K_1, K_2 \in \mathcal{O}_G(A)$ with $K_1 < K_2$, then

$$|K_1 : K_1 \wedge A^\complement| \leq |K_2 : K_2 \wedge A^\complement|.$$

Moreover, if $|G : A^\complement| = 2$, then $|K \vee A : K| = 2$ for all K in $\mathcal{O}_{A^\complement}(H)$.

Proof. By the product formula,

$$|K_1 \vee A| \cdot |K_2| \leq |(K_1 \vee A) \vee K_2| \cdot |(K_1 \vee A) \wedge K_2|,$$

but $K_1 \wedge K_2 = K_1$, $K_1 \vee K_2 = K_2$ and $A \wedge K_2 = H$, so by distributivity,

$$|K_1 \vee A| \cdot |K_2| \leq |K_2 \vee A| \cdot |K_1|.$$

Finally, $A^{\complement} \vee A = G$, so if $H \le K \le A^{\complement}$ and $|G : A^{\complement}| = 2$, then

$$2 \le |K \vee A : K| \le |A^{\complement} \vee A : A^{\complement}| = 2.$$

It follows that $|K \vee A : K| = 2$. □

Lemma 10.7. *If $\mathcal{O}_G(H)$ is Boolean of rank 2, then $|M_1 : H| = 2$ if and only if $|G : M_2| = 2$.*

Proof. If $|G : M_2| = 2$, then $|M_1 : H| = 2$ by Lemma 10.5. Now if $|M_1 : H| = 2$, then $H \lhd M_1$ and $M_1 = H \sqcup H\tau$ with $\tau H = H\tau$ and $(H\tau)^2 = H$, so $H\tau^2 = H$ and $\tau^2 \in H$. Now $M_2 \in (H, G)$, and thus $\tau M_2 \tau^{-1} \in (\tau H \tau^{-1}, \tau G \tau^{-1}) = (H, G)$, so by assumption, $\tau M_2 \tau^{-1} \in \{M_1, M_2\}$. If $\tau M_2 \tau^{-1} = M_1$, then $M_2 = \tau^{-1} M_1 \tau = M_1$, which is a contradiction. So $\tau M_2 \tau^{-1} = M_2$. Now $\tau^2 \in H < M_2$, so $M_2 \tau^2 = M_2$. It follows that $G = \langle M_2, \tau \rangle = M_2 \sqcup M_2 \tau$, and $|G : M_2| = 2$. □

Lemma 10.8. *If there are $K, L \in \mathcal{O}_G(H)$ such that $K < L$ and $|L : K| = 2$, then there is an atom A such that $L = K \vee A$ and $|G : A^{\complement}| = 2$.*

Proof. By the Boolean structure and because K must be a maximal subgroup of L, there is an atom A of $\mathcal{O}_G(H)$ such that $L = K \vee A$. Let

$$K = K_1 < K_2 < \cdots < K_r = A^{\complement}$$

be a maximal chain from K to $A^{\complement}$. Let $L_i = K_i \vee A$; then the overgroup lattice $\mathcal{O}_{L_{i+1}}(K_i)$ is Boolean of rank 2, and $|L_1 : K_1| = 2$, so by Lemma 10.7,

$$2 = |L_1 : K_1| = |L_2 : K_2| = \cdots = |L_r : K_r| = |G : A^{\complement}|.$$ □

Note that for a B that is an index 2 subgroup of A, if $|B|$ is odd then $A = B \rtimes C_2$, but this is not true in general if $|B|$ is even.

Lemma 10.9. *If there is i such that for all K in $\mathcal{O}_{M_i}(H)$, $|K \vee M_i^{\complement} : K| = |M_i^{\complement} : H|$, then*

$$\hat{\varphi}(H, G) = (|M_i^{\complement} : H| - 1)\hat{\varphi}(H, M_i).$$

Proof. By assumption we deduce that $\hat{\varphi}(H, M_i) = \hat{\varphi}(M_i^{\complement}, G)$, but by definition, $\hat{\varphi}(H, G) = |M_i^{\complement} : H|\hat{\varphi}(H, M_i) - \hat{\varphi}(H, M_i)$. The result follows. □

Lemma 10.10. *If there is i such that $|M_i^{\complement} : H| = 2$, then $\hat{\varphi}(H, G) = \hat{\varphi}(H, M_i)$.*

Proof. By assumption and Lemma 10.8, $|G : M_i| = 2$, so by Lemma 10.6, if $H \le K \le M_i$ then $|K \vee M_i^{\complement} : K| = 2$. Thus, by Lemma 10.9, $\hat{\varphi}(H, G) = (2 - 1)\hat{\varphi}(H, M_i)$. □

Lemma 10.11. *Let G be a finite group and H a subgroup such that the overgroup lattice $\mathcal{O}_G(H)$ is Boolean of rank ℓ, and let $A_1, \dots, A_\ell$ be its atoms. If $|A_i : H| \ge 2^i$, then $\hat{\varphi}(H, G) \ge 2^{\ell - 1}$.*

Proof. Let I be a subset of $\{1, \dots, \ell\}$ and let A_I be $\bigvee_{i \in I} A_i$. Then $\mathcal{O}_G(H) = \{A_I \mid I \subseteq \{1, \dots, \ell\}\}$ and

$$\hat{\varphi}(H, G) = \sum_{I \subseteq \{1, \dots, \ell\}} (-1)^{|I|} |G : A_I|.$$

By assumption and Lemma 10.6, if $j \notin I$ then $|G : A_I| \ge 2^j |G : A_I \vee A_j|$. It follows that

$$|G : A_J| \le \frac{1}{|J|} \sum_{j \in J} 2^{-j} |G : A_{J \setminus \{j\}}|,$$

from which we get

$$\begin{aligned}
\hat{\varphi}(H,G) &\geq \sum_{|I| \text{ even}} |G:A_I| - \sum_{|I| \text{ odd}} \frac{1}{|I|} \sum_{i\in I} 2^{-i}|G:A_{I\setminus\{i\}}| \\
&= \sum_{|I| \text{ even}} |G:A_I|(1 - \frac{\sum_{i\notin I} 2^{-i}}{|I|+1}) \\
&= \sum_{|I| \text{ even}} |G:A_I| \frac{|I| + 2^{-\ell} + \sum_{i\in I} 2^{-i}}{|I|+1} \\
&\geq |G:A_\emptyset| 2^{-\ell} = 2^{-\ell}|G:H| \\
&\geq 2^{-\ell} \prod_{i=1}^{\ell} 2^i = 2^{\ell(\ell-1)/2} \geq 2^{\ell-1}. \qquad \square
\end{aligned}$$

Lemma 10.12. *Let G be a finite group and H a subgroup such that the overgroup lattice $\mathcal{O}_G(H)$ is Boolean of rank ℓ, and let $A_1, \ldots, A_\ell$ be its atoms. If $|A_i : H| \geq a_i > 0$, then $\hat{\varphi}(H,G) \geq (1 - \sum_i a_i^{-1}) \prod_i a_i$.*

Proof. This proof works exactly like the proof of Lemma 10.11. $\square$

10.2. Proof of Theorem 1.3 part 1

Proof. The case where $n \leq 2$ is precisely Lemma 10.3. It remains to consider the case where $n = 3$.

If there is i such that $|M_i^\complement : H| = 2$, then by Lemma 10.2 and the Boolean structure, for all $j \neq i$, $|M_j^\complement : H| \geq 3$, and by Lemma 10.10, $\hat{\varphi}(H,G) = \hat{\varphi}(H,M_i)$. But as for the proof of Lemma 10.3, we have

$$\hat{\varphi}(H,M_i) \geq 9(1 - 1/3 - 1/3) + 1 = 2^{n-1}.$$

Otherwise, for all i we have $|M_i^\complement : H| \geq 3$. Then (using Lemma 10.6),

$$\begin{aligned}
\hat{\varphi}(H,G) &= |G:H| - \sum_i |G:M_i^\complement| + \sum_i |G:M_i| - |G:G| \\
&\geq |G:H|(1 - \sum_i |M_i^\complement : H|^{-1}) + \sum_i |M_i^\complement : H| - 1 \\
&\geq 27(1 - \sum_i 1/3) + \sum_i (3) - 1 = 8 > 2^{n-1}. \qquad \square
\end{aligned}$$

10.3. Proof of Theorem 1.3 parts 2 and 3

Let $M_1, \ldots, M_\ell$ be the coatoms of $\mathcal{O}_G(H)$. The Boolean lattice $\mathcal{O}_G(H)$ is called *group-complemented* if $KK^\complement = K^\complement K$ for every $K \in \mathcal{O}_G(H)$.

Lemma 10.13. *If the Boolean lattice $\mathcal{O}_G(H)$ is group-complemented, then $\hat{\varphi}(H,G) = \prod_i (|G:M_i| - 1)$.*

Proof. By assumption, $KK^\complement = K^\complement K$, which means that $KK^\complement = K \vee K^\complement = G$, which also means (by the product formula) that $|G:K| = |K^\complement : H|$. Then by Lemma 10.6, for all i and for all K in $\mathcal{O}_G(M_i^\complement)$, $|K : K \wedge M_i| = |G:M_i|$. Now for all K in $\mathcal{O}_G(H)$ there is $I \subseteq \{1, \ldots, \ell\}$ such that $K = M_I = \bigwedge_{i\in I} M_i$;

it follows that $|G : K| = \prod_{i \in I} |G : M_i|$ and then

$$\hat{\varphi}(H, G) = (-1)^{\ell} \sum_{I \subseteq \{1,\ldots,\ell\}} (-1)^{|I|} |G : M_I|$$
$$= (-1)^{\ell} \sum_{I \subseteq \{1,\ldots,\ell\}} \prod_{i \in I} (-|G : M_i|) = \prod_i (|G : M_i| - 1). \qquad \square$$

Theorem 1.3 part 2 follows from Lemmas 10.2 and 10.13. Moreover, if G is solvable and $\mathcal{O}_G(H)$ is Boolean, then $\mathcal{O}_G(H)$ is also group-complemented by [19, Theorem 1.5] and the proof of Lemma 10.13. The proof of Theorem 1.3 part 3 follows.

10.4. Proof of Theorem 1.3 part 4

Proof. By Theorem 1.3 part 1, we are reduced to considering $\ell \geq 4$ on cases (1)–(6) of Theorem 1.2, from where we take the notation.

1. Take $G = Sym(\Omega)$. By Section 8, the rank ℓ Boolean lattice $\mathcal{O}_G(H)$ is made of

$$M_I \cong \mathrm{Sym}(n/n_{i_1}) \mathrm{wr}\, \mathrm{Sym}(n_{i_1}/n_{i_2}) \mathrm{wr} \cdots \mathrm{wr}\, \mathrm{Sym}(n_{i_{\kappa-1}}/n_{i_\kappa}) \mathrm{wr}\, \mathrm{Sym}(n_{i_\kappa}),$$

with $I = \{i_1, i_2, \ldots, i_\kappa\} \subseteq \{1, \ldots, \ell\}$, but

$$|M_I| = \left(\frac{n}{n_{i_1}}!\right)^{n_{i_1}} \left(\frac{n_{i_1}}{n_{i_2}}!\right)^{n_{i_2}} \cdots \left(\frac{n_{i_{\kappa-1}}}{n_{i_\kappa}}!\right)^{n_{i_\kappa}} n_{i_\kappa}!$$

In particular, with $n_0 = n$, $n_{\ell+1} = 1$, $H = M_{\{1,\ldots,\ell\}}$ and $A_i = M_i^{\complement}$, we have

$$|H| = \prod_{i=0}^{\ell} \left(\frac{n_i}{n_{i+1}}!\right)^{n_{i+1}}, \quad |A_j| = \left(\frac{n_{j-1}}{n_{j+1}}!\right)^{n_{j+1}} \prod_{i \neq j, j+1} \left(\frac{n_i}{n_{i+1}}!\right)^{n_{i+1}}.$$

It follows that

$$|A_j : H| = \frac{\left(\frac{n_{j-1}}{n_{j+1}}!\right)^{n_{j+1}}}{\left(\frac{n_{j-1}}{n_j}!\right)^{n_j} \left(\frac{n_j}{n_{j+1}}!\right)^{n_{j+1}}} = \left[\frac{\left(\frac{n_{j-1}}{n_{j+1}}!\right)}{\left(\frac{n_{j-1}}{n_j}!\right)^{\frac{n_j}{n_{j+1}}} \left(\frac{n_j}{n_{j+1}}!\right)}\right]^{n_{j+1}} \geq 3^{n_{j+1}}.$$

Take the atom $B_i := A_{\ell+1-i}$ and $m_i := n_{\ell+1-i}$; then

$$|B_i : H| \geq 3^{m_{i-1}} \geq 3^{2^{i-1}} > 2^i.$$

It follows by Lemma 10.11 that $\hat{\varphi}(H, G) \geq 2^{\ell-1}$.

Next, if $B_i \subseteq \mathrm{Alt}(\Omega)$, then so is H, and obviously $|\mathrm{Alt}(\Omega) \cap B_i : \mathrm{Alt}(\Omega) \cap H| = |B_i : H|$, or else by Lemma 10.5 $|B_i : \mathrm{Alt}(\Omega) \cap B_i| = 2$; now $|H : \mathrm{Alt}(\Omega) \cap H| = 1$ or 2 whether $H \subseteq \mathrm{Alt}(\Omega)$ or not. In any case,

$$|\mathrm{Alt}(\Omega) \cap B_i : \mathrm{Alt}(\Omega) \cap H| \geq |B_i : H|/2 > 3^{2^{i-1}-1},$$

and we can also apply Lemma 10.11.

2. Let $A_\ell = G_\ell^{\complement}$; then $|A_\ell : H| = 2$. Next, we can, as earlier, order the remaining atoms $A_1, \ldots, A_{\ell-1}$ such that $|A_i : H| \geq 3^{2^{i-1}}$, because by assumption, $|A_\ell : \mathrm{Alt}(\Omega) \cap A_\ell| = 2$. The result follows from

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Lemma 10.12, because

$$1-\left(\frac{1}{2}+\sum_{i=1}^{\ell-1}3^{-2^{i-1}}\right)\geq\frac{1}{2}-\sum_{i=1}^{\ell-1}3^{-i}=\sum_{i=\ell}^{\infty}3^{-i}=\frac{3}{2}3^{-\ell}.$$

3. Following the notation of Section 9, for $I=\{r_1,r_2,\ldots,r_s\}$ we have that

$$|M_I|=(a^{b_1\cdots b_{r_1-1}}!)^{b_{r_1}\cdots b_\ell}\prod_{i=1}^{s}((b_{r_i}\cdots b_{r_{i+1}-1})!)^{b_{r_{i+1}}\cdots b_\ell}.$$

The atom $A_i=M_i^{\complement}$ is of the form $M_{\{i\}^{\complement}}$, whereas $H=M_{\{1,\ldots,\ell\}}$; then (with $b_0=1$)

$$|H|=(a!)^{b_1\cdots b_\ell}\prod_{i=1}^{\ell}(b_i!)^{b_{i+1}\cdots b_\ell}\quad\text{and}$$

$$A_j=(a^{b_1^{\delta_{1,j}}}!)^{b_1^{-\delta_{1,j}}\prod_i b_i}((b_{j-1}b_j)!)^{\delta_{1,j}b_{j+1}\cdots b_\ell}\prod_{i\neq j-1,j}(b_i!)^{b_{i+1}\cdots b_\ell}.$$

Let $j>1$. It follows that

$$|A_j:H|=\left[\frac{(b_{j-1}b_j)!}{((b_{j-1})!)^{b_j}b_j!}\right]^{b_{j+1}\cdots b_\ell}\quad\text{and}\quad|A_1:H|=\left[\frac{a^{b_1}!}{(a!)^{b_1}b_1!}\right]^{b_2\cdots b_\ell}.$$

The rest is similar to case 1.

4. Similar to case 2.
5. Here $n=a^b$ is a prime power p^d so that $a=p^{d'}$ with $bd'=d$, $b=b_1\cdots b_{\ell-1}$ and $G_\ell=AGL_d(p)$. We can deduce, by using [1, Theorem 13 (3)], that

$$AGL_d(p)\cap(\mathrm{Sym}(a^{b_1\cdots b_{r_1}})\,\mathrm{wr}\,\mathrm{Sym}(b_{r_1+1}\cdots b_{r_2})\,\mathrm{wr}\cdots\mathrm{wr}\,\mathrm{Sym}(b_{r_s+1}\cdots b_{\ell-1}))$$

$$=AGL_{d'b_1\cdots b_{r_1}}(p)\,\mathrm{wr}\,\mathrm{Sym}(b_{r_1+1}\cdots b_{r_2})\,\mathrm{wr}\cdots\mathrm{wr}\,\mathrm{Sym}(b_{r_s+1}\cdots b_{\ell-1}).$$

But $|AGL_k(p)|=p^k\prod_{i=0}^{k-1}(p^k-p^i)$. The rest is similar to case 3.

6. Similar to case 3. □

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