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ALGEBRAS OF CANCELLATIVE SEMIGROUPS

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The Jacobson radical $J(K[S])$ of the semigroup ring $K[S]$ of a cancellative semigroup S over a field K is studied. We show that, if $J(K[S]) \neq 0$, then either S is a reversive semigroup or $K[S]$ has many nilpotents and $J(K[P]) \neq 0$ for a reversive subsemigroup P of S. This is used to prove that $J(K[S]) = 0$ for every unique product semigroup S.

Let $K[S]$ be the semigroup ring of a cancellative semigroup S over a field K. Our aim is to show that the semiprimitivity problem for $K[S]$ can often be reduced to the case where S has a group of fractions. This allows to prove that $J(K[S]) = 0$ whenever S is a unique product semigroup, which answers the question asked in [4, Problem 23]. Here, $J(K[S])$ denotes the Jacobson radical of $K[S]$. We refer to [1, 3, 4] for the basic facts on semigroups, semigroup rings and graded rings used in this note.

If S is not a monoid, then let S^1 be the monoid obtained by adjoining a unity element to S. Otherwise, let $S^1 = S$. Recall that S is left reversive if it satisfies the right Ore condition: $sS \cap tS \neq \emptyset$ for every $s, t \in S$. This is equivalent to the fact that S has a group of classical right fractions, see [1]. The left reversive congruence ρ_S on S^1 is defined for $s, t \in S^1$ by the rule $(s, t) \in \rho$ if $sxS \cap txS \neq \emptyset$ for every $x \in S$, [5]. The restriction of ρ_S to S will also be denoted by ρ_S, or by ρ if unambiguous. It is known that ρ is left cancellative. A subset Z of S is said to be left group-like (or left unitary) if $s \in Z$ whenever $z \in Z$, $s \in S$ and $zs \in Z$.

Our approach is based on the following observation, which allows us to cover S with a collection of its nice subsemigroups.

LEMMA 1. *For every $t \in S^1$ the set $S_t = \{s \in S \mid (t^r s, t^n) \in \rho$ for some $r, n \geqslant 1\}$ is a left group-like subsemigroup of S.*

PROOF: Let $s, u \in S_t$. Then $(t^r u, t^n) \in \rho$ and $(t^i s, t^j) \in \rho$ for some $r, n, i, j \geqslant 1$. The latter implies that $(t^{i+r} su, t^{j+r} u) \in \rho$. But $(t^{j+r} u, t^{j+n}) \in \rho$. Hence $(t^{i+r} su, t^{j+n}) \in \rho$, and so $su \in S_t$. Thus, S_t is a subsemigroup of S.

Assume also that $sx \in S_t$ for some $x \in S$. Then there exist $k, m \geqslant 1$ such that $(t^k sx, t^m) \in \rho$. Now $(t^{i+k} sx, t^{i+m}) \in \rho$ and also $(t^{i+k} sx, t^{j+k} x) \in \rho$. This implies that $(t^{j+k} x, t^{i+m}) \in \rho$. Hence $x \in S_t$, as desired. ∎

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Denote by ϕ the natural homomorphism $S \to S/\rho$. Let $U = \{s \in S \mid (sz, 1) \in \rho$ for some $z \in S\}$. Assume that $U \neq \emptyset$. If $(sz, 1) \in \rho$, then $(szs, s) \in \rho$, so that the left cancellativity of ρ implies that $(zs, 1) \in \rho$. Therefore $U = \phi^{-1}(H)$, the inverse image in S of the group H of units of S/ρ. In particular, U is a filter of S. Since $(sz, 1) \in \rho$ implies that $szxS \cap xS \neq \emptyset$ for every $x \in S$, it follows that $S = U$ if and only if S is left reversive.

The advantage of dealing with the rings $K[S_t]$, in place of $K[S]$, is that each $K[S_t]$ admits a very simple gradation. This will not be used explicitly, but it recovers the general flavour of our approach.

PROPOSITION. *Let $t \in S$. Then the image G_t of S_t under the natural homomorphism $\phi\colon S \to S/\rho$ is a cyclic group, a cyclic semigroup or a cyclic monoid generated by $\phi(t)$. Consequently, the ring $R = K[S_t]$ has a natural G_t-gradation given by $R_g = \phi^{-1}(Kg)$ for $g \in G_t$. Moreover, if $t \notin U$, then the set $I_t = \{s \in S \mid (s, t^n) \in \rho$ for some $n \geqslant 1\}$ is an ideal of S_t, $\phi(I_t)$ is an infinite cyclic semigroup and $K[I_t]$ has an induced $\phi(I_t)$-gradation.*

PROOF: If $\phi(s) \in G_t$, then there exist $r, n \geqslant 1$ such that $(t^r s, t^n) \in \rho$. Therefore, either $(s, t^k) \in \rho$ for some $k \geqslant 0$ or $(t^k s, 1) \in \rho$ for some $k \geqslant 1$. If for some $s \in S$ the latter holds, $\phi(t)$ lies in the group H of units of S/ρ, so that $G_t = \phi(S_t)$ is the cyclic subgroup of H generated by $\phi(t)$. Otherwise, G_t is the cyclic semigroup (or the cyclic monoid, if $S_1 \neq \emptyset$) generated by $\phi(t)$. Clearly, this gives the desired G_t-gradation on the ring $R = K[S_t]$. The remaining assertions follow easily. ∎

We refer to [**3**, Chapter 4], for a variety of results on rings graded by groups. In particular, for those concerning the homogenity of the Jacobson radical and the prime radical.

Every non-zero $c \in K[S]$ can be uniquely written in the form $c = c_1 + \ldots + c_n$, where each $\operatorname{supp}(c_i)$ lies in a different ρ-class of S. The elements $c_1, \ldots, c_n$ are called the ρ-components of c. We say that c is ρ-separated if $\operatorname{supp}(c_i)S \cap \operatorname{supp}(c_j)S = \emptyset$ for $i \neq j$. For convenience, the zero of $K[S]$ will also be called ρ-separated.

LEMMA 2. *Let V be the set of ρ-separated elements of $K[S]$ and let $W = V \cap J(K[S \setminus U])$. Then*

(1) *V is a subsemigroup of the multiplicative semigroup of $K[S]$, in particular $VS, SV \subseteq V$;*
(2) *if $b \in W$, then the ρ-components of b generate a finite nilpotent semigroup, in particular W is a nil semigroup;*
(3) *for every $a \in K[S]$ there exists $s \in S$ such that $as \in V$.*

PROOF: For $a, b \in V$, $a, b \neq 0$, let $a = a_1 + \ldots + a_r$, $b = b_1 + \ldots + b_n$ be the decompositions of a, b into ρ-components. Choose $t_i, s_j \in S^1$ such that $(t_i, \operatorname{supp}(a_i)) \in \rho$

and $(s_j, \operatorname{supp}(b_j)) \in \rho$. Suppose that $t_i s_j c = t_k s_m d$ for some $c, d \in S$ and some i, j, k, m. Since $a \in V$, it follows that $i = k$ because otherwise $t_i S \cap t_k S = \emptyset$. Hence $s_j c = s_m d$. Similarly, $j = m$ because $b \in V$, so that (1) follows.

Assume further that $b \in W$. Let $c \in K[S \setminus U]$ be a right quasi inverse of b, that is, $b + c = bc$. Let $c_1, \ldots, c_m$ be the ρ-components of c. By induction on k we show that each non-zero $e = b_{i_1} b_{i_2} \ldots b_{i_k}$, $i_j \in \{1, \ldots, n\}$, lies in the set $C = \{-c_1, \ldots, -c_m\}$. By (1) we know that each non-zero $b_i c_j$ lies in a different ρ-class of S. If $b_i \notin C$, then $(\operatorname{supp}(b_i), \operatorname{supp}(b_p c_q)) \in \rho$ for some p, q. Then $i = p$ since $b \in V$. Therefore $(\operatorname{supp}(c_q), 1) \in \rho$, which contradicts the fact that $c_q \in K[S \setminus U]$. Hence $b_i \in C$. Assume now that $k > 1$. Since, by the induction hypothesis, $-b_{i_2} \ldots b_{i_k}$ is a ρ-component of c, $-e$ must be a ρ-component of bc. As before, from the left cancellativity of ρ it follows that $(\operatorname{supp}(e), \operatorname{supp}(b)) \notin \rho$ because $\operatorname{supp}(b) \cap U = \emptyset$ and $b \in V$. Hence, the equality $b + c = bc$ implies that $e \in C$, as claimed. Now, the semigroup B generated by $b_1, \ldots, b_n$ is finite. Moreover, each $e \in B$ is nilpotent because $e^p = e^q \neq 0$ for $p > q$ would again contradict the fact that $\operatorname{supp}(e) \cap U = \emptyset$. Therefore B is nilpotent, so that b is a nilpotent element. This proves that (2) holds.

(3) was established in [5]. ∎

LEMMA 3. *Let $t \in S \setminus U$. Assume that $a + b = ab$ for some $a, b \in K[S_t]$. Then $b \in K[A]$ for the subsemigroup A generated in S by $\operatorname{supp}(a)$. Consequently, $J(K[P]) \cap K[T] \subseteq J(K[T])$ for any subsemigroups T, P of S_t.*

PROOF: Assume that $a \neq 0$. Substituting $b = ab - a$ we come to

$$b = ab - a = a^2 b - a^2 - a = \ldots = a^n b - a^n - a^{n-1} - \ldots - a$$

for every $n \geqslant 1$. Suppose that there exists $s \in \operatorname{supp}(b) \setminus A$. Then $s \in \operatorname{supp}(a^n b)$ for every $n \geqslant 1$, hence there exist $t_n \in \operatorname{supp}(b)$ and $s_{n,i_j} \in \operatorname{supp}(a)$, $j = 1, \ldots, n$, such that $s = s_{n,i_1} \ldots s_{n,i_n} t_n$. Therefore, there are infinitely many equal elements of the form $s_{n,i_1} \ldots s_{n,i_n} t_n$. Since $t \notin U$ and ρ is left cancellative, there exists $N \geqslant 1$ such that each s_{n,i_j} is ρ-related to some t^r, $1 \leqslant r \leqslant N$. It follows that $(t^p, t^Q) \in \rho$ for some $p < q$. This contradicts the fact that $t \notin U$. Therefore $\operatorname{supp}(b) \subseteq A$. The assertion follows. ∎

We show that, if $J(K[S]) \neq 0$ for a non-left reversive semigroup S, then the semigroup ring $K[T]$ of a left reversive subsemigroup T of S is not semiprimitive and contains many nilpotents.

THEOREM. *Let S be a cancellative semigroup that is not left reversive. Assume that $0 \neq c \in J(K[S])$. Then there exists $s \in S$ such that*

(i) $S^1 cs S^1 \subseteq W \setminus \{0\}$.

(ii) *If c_1 is a ρ-component of cs and $t \in \operatorname{supp}(c_1 s)$, then $c_1 \in J(K[S_t])$ and $S^1 c_1 S^1$ consists of nilpotents.*

(iii) *There exists a left reversive subsemigroup T of S and an element $u \in S$ such that the natural K-linear projection f of csu onto $K[T]$ is a non-zero element of $J(K[T])$ for which $T^1 f T^1$ consists of nilpotents.*

PROOF: Since S is not left reversive, $S \neq U$. By Lemma 2 there exists $z \in S$ such that $cz \in V$. Then $czq \in W$ for any $q \in S \setminus U$. Hence, (i) follows with $s = zq$.

Let $c_1, \ldots, c_m$ be the ρ-components of cs. Note that for $w \in S_t$ we have $wS \cap tS \neq \emptyset$. Hence

$$(*) \qquad yx \notin S_t \text{ for every } x \in S^1 \text{ and every } y \in \operatorname{supp}(c_j), \quad j \neq 1$$

Let $\pi \colon K[S] \to K[S_t]$ be the natural K-linear projection. Let $a \in K[S_t]$. Since $csa \in J(K[S])$, there exists $d \in K[S]$ such that $csa + d = csad$. Then (*) shows that $\pi(csa) = \pi(c_1 a) = c_1 a$ and $\pi(csad) = \pi(c_1 ad)$. Since S_t is a left group-like subsemigroup of S, from [4, Lemma 4.14], it follows that $\pi(c_1 ad) = c_1 a \pi(d)$. This shows that $c_1 a$ is quasi invertible in $K[S_t]$, so that $c_1 \in J(K[S_t])$. For every $x, y \in S^1$, xc_1y is a ρ-component of $xcsy$. Hence, the remaining assertion of (ii) follows from Lemma 2.

Let $n \geqslant 1$ be the minimal integer satisfying the following condition:

There exists a subsemigroup Q of S_t and an element $u \in S$ such that $csu = f + f_0$, where $f \in J(K[Q])$, $f_0 \in K[S]$, $\operatorname{supp}(f_0)Q \cap \operatorname{supp}(f)Q = \emptyset$, $|\operatorname{supp}(f)| = n$ and $Q^1 f Q^1$ consists of nilpotents.

In view of (ii), n is well-defined. Let $T \subseteq Q$ be the semigroup generated by $\operatorname{supp}(f)$. Lemma 3 implies that $f \in J(K[T])$. Suppose that T is not left reversive. From [5, Lemma 2], it follows that $\operatorname{supp}(f)$ does not lie in a single ρ_T-class of T. Proceeding as at the beginning of the proof, we can find an element $w \in T$ such that fw is ρ_T-separated, so that $fw = f_1 + \ldots + f_z$, $z \geqslant 2$, with $\operatorname{supp}(f_i)T \cap \operatorname{supp}(f_j)T = \emptyset$ for $i \neq j$ and each $\operatorname{supp}(f_i)$ lying in a different ρ_T-class of T. Moveover, $f_1 \in J(K[T_v])$ for $v \in \operatorname{supp}(f_1)$ and $T^1 f_1 T^1$ consists of nilpotents. The choice of f implies that $|\operatorname{supp}(f_1)| = |\operatorname{supp}(f)|$, so that $z = 1$, a contradiction. Hence T is a left reversive semigroup. This completes the proof of the theorem. ∎

An induction, as that in the proof of (iii) above, can also be carried out with respect to the congruence ρ', that is right-left dual to ρ, see [5]. Applying both procedures alternately a number of times, one derives the following consequence.

COROLLARY 1. *If $J(K[S]) \neq 0$ for a cancellative semigroup S, then there exists a (left and right) reversive subsemigroup P of S such that $J(K[P]) \neq 0$.*

If K is not algebraic over its prime subfield K_0 and $J(K[S]) \neq 0$, then $K_0[S]$ has a non-zero nil ideal, see [6, Chapter 7]. Our techniques allow us to find a reversive subsemigroup P of S such that $K_0[P]$ has a non-zero nil ideal.

The above theorem often reduces the semiprimitivity problem for algebras $K[S]$ to the case where S is reversive, so S has a group of fractions G. When studying $K[S]$, one can then apply a variety of group ring techniques and results. For example, it is known that $K[G]$ is a domain for a wide class of groups G, and it is conjectured that this is always the case if G is a torsion-free group, see [7, Chapter 9].

Recall that a semigroup S is a u.p. (unique product) semigroup if for any nonempty finite subsets A, B of S with $|A| + |B| > 2$, there exists an element $s \in AB$ with a unique presentation in the form $s = ab$, where $a \in A$, $b \in B$, see [4, Chapter 10]. In this case $K[S]$ is a domain and in particular S is cancellative. Similarly, S is called a t.u.p. (two unique product) semigroup if there are at least two elements with unique presentation in each AB. Then $K[S^1]$ has no nontrivial units, so that $J(K[S]) = 0$. Note that there exist u.p. semigroups that do not have the t.u.p. property, [4, Chapter 10].

COROLLARY 2. *Let S be a u.p. semigroup. Then $J(K[S]) = 0$.*

PROOF: The theorem allows us to assume that S is left reversive. It is known that every u.p. semigroup that is left reversive must be a t.u.p. semigroup, [8], see [4, Theorem 10.6]. As noted above, this implies that $J(K[S]) = 0$. ∎

Let A be a domain that is nontrivially graded (that is, $A \neq A_1$) by a cancellative semigroup S. Assume that $J(A) \neq 0$. If S is not left reversive, then, as in the proof of assertion (ii) of the theorem, one shows that $J(R) \neq 0$ for a subring R of A that is graded by an infinite cyclic semigroup. It is known that R contains nontrivial nilpotents, [2], see [3, Theorem 32.5], a contradiction. Hence S is left reversive. Therefore, if S is a u.p. semigroup, then it is a t.u.p. semigroup. This again contradicts [2]. Hence, the assertion of Corollary 2 can be extended to any domain A that is nontrivially graded by a u.p. semigroup S.

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