FREE RIGHT TYPE A SEMIGROUPS

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(Received 10 October, 1989)

Introduction. The relation $\mathscr{L}^*$ is defined on a semigroup S by the rule that $a\mathscr{L}^*b$ if and only if the elements a, b of S are related by Green's relation $\mathscr{L}$ in some oversemigroup of S. A semigroup S is an *E-semigroup* if its set $E(S)$ of idempotents is a subsemilattice of S. A *right adequate semigroup* is an E-semigroup in which every $\mathscr{L}^*$-class contains an idempotent. It is easy to see that, in fact, each $\mathscr{L}^*$-class of a right adequate semigroup contains a unique idempotent [**8**]. We denote the idempotent in the $\mathscr{L}^*$-class of a by a^*. Then we may regard a right adequate semigroup as an algebra with a binary operation of multiplication and a unary operation *. We will refer to such algebras as *-semigroups. In [**10**], it is observed that viewed in this way the class of right adequate semigroups is a quasi-variety.

In this paper, which is the promised sequel to [**10**], we are concerned with right type A semigroups. These are semigroups which are right adequate and in which $ea = a(ea)^*$ for each $a \in S$ and $e \in E(S)$; they form a sub-quasi-variety of the quasi-variety of all right adequate semigroups. Thus, from general results in universal algebra, we know that free right type A semigroups exist. It is the purpose of this paper to give an explicit description of these free objects and to discuss some of their properties.

Our approach is to make use of the construction of free right h-adequate semigroups in [**10**], a right adequate semigroup being right h-adequate when the mapping $\alpha_a : E(S)^1 \to E(S)^1$ defined by $x\alpha_a = (xa)^*$ is a homomorphism for each element a of S. By a *-congruence on a right adequate semigroup S, we mean a congruence on S regarded as a *-semigroup, that is, a semigroup congruence ρ on S which also satisfies $a\rho b$ implies $a^*\rho b^*$. A *-congruence ρ on a right adequate semigroup S is called a *right type A congruence* if S/ρ is a right type A semigroup, where the semigroup S/ρ is made into a *-semigroup by defining $(a\rho)^*$ to be $a^*\rho$. On any right adequate semigroup, there is a minimum right type A congruence and if γ is this congruence on P_X, the free right h-adequate semigroup on X, then P_X/γ is the free right type A semigroup on X. For any non-empty set X, we construct a semigroup A_X isomorphic to P_X/γ which is analogous to Scheiblich's construction [**20**] of the free inverse semigroup on X.

This construction allows us to obtain several results which are analogues of theorems on inverse semigroups. For example, the fact that free inverse semigroups are E-unitary gives rise to one proof that every inverse semigroup has an E-unitary cover [**18**, Theorem VIII.1.10]. The corresponding result for right type A semigroups is that every right type A semigroup has a proper cover [**8**]. On a right type A semigroup, the minimum left cancellative congruence is denoted by σ and the semigroup is proper if $\sigma \cap \mathscr{L}^* = \iota$. It is easily seen that A_X is proper and we give a new proof of the covering result modelled on that in the inverse case.

After recalling the basic properties of right type A semigroups in Section 1, we devote Section 2 to the construction outlined above. In Section 3, we show that A_X enjoys properties similar to those of P_X. Among other things we have that Green's relations on A_X are trivial, the word problem for A_X is solvable and A_X is residually finite.

We examine sets of free generators in right type A semigroups in Section 4. The

Glasgow Math. J. **33** (1991) 135–148.

results are similar to those of Section 4 in **[10]** and both sets of results are inspired by those of Reilly **[19]** in the inverse case.

In the final section we obtain a simple description of the free objects in the quasi-variety of adequate semigroups with central idempotents. This parallels the description of free Clifford semigroups in Chapter VIII of **[18]**.

1. Preliminaries. We begin by giving some elementary facts about right adequate and right type A semigroups. Alternative characterisations of the relation $\mathscr{L}^*$ are given by the following lemma from **[14]** and **[17]**.

LEMMA 1.1. *Let S be a semigroup and let a, b be elements of S. Then the following conditions are equivalent*:

(1) $a\mathscr{L}^*b$,

(2) *for all* $x, y \in S^1$, $ax = ay$ *if and only if* $bx = by$,

(3) *there is an* S^1-*isomorphism* $\phi: aS^1 \to bS^1$ *with* $a\phi = b$.

As an easy consequence we have the following corollary.

COROLLARY 1.2. *If e is an idempotent of a semigroup S then the following are equivalent for an element a of S*:

(1) $e\mathscr{L}^*a$,

(2) $ae = a$ *and, for all* x, y *in* S^1, $ax = ay$ *implies* $ex = ey$.

From the definition and Lemma 1.1, it follows that $\mathscr{L}^*$ is a right congruence and that $\mathscr{L} \subseteq \mathscr{L}^*$. It is well known and easy to see that, for regular elements a, b of S, we have $a\mathscr{L}^*b$ if and only if $a\mathscr{L}b$. In particular, if S is a regular semigroup then $\mathscr{L} = \mathscr{L}^*$.

We record next some elementary properties of right adequate semigroups which we use repeatedly.

PROPOSITION 1.3 **[9**, Proposition 1.6]. *Let S be a right adequate semigroup and a, b elements of S. Then*

(1) $a\mathscr{L}^*b$ *if and only if* $a^* = b^*$,

(2) $(ab)^* = (a^*b)^*$,

(3) $(ab)^* \leq b^*$, *where* $\leq$ *is the natural ordering on the semilattice* $E(S)$.

As noted in **[9**, Lemma 2.1], if S is a right type A semigroup then $(efa)^* = (ea)^*(fa)^*$ for all $e, f \in E(S)$, $a \in S$. Thus, in our present terminology, a right type A semigroup is right h-adequate.

From universal algebra, we have the notions of **-subsemigroup*, **-homomorphism* and **-congruence*. It is clear that if S is right adequate, right h-adequate or right type A then so is any *-subsemigroup of S. A *left *-ideal* of S is a left ideal of S which is also a *-subsemigroup of S. By a **-ideal* of S, we mean an ideal of S which is also a left *-ideal. In the case of a right adequate semigroup S, a left ideal I is a left *-ideal if and only if it is a union of $\mathscr{L}^*$-classes.

These ideas are connected by the following result from **[10]**.

PROPOSITION 1.4. *Let S be a right adequate semigroup and let I be a *-ideal of S. Then the Rees quotient semigroup S/I is right adequate and the natural map* $\nu: S \to S/I$ *is a *-homomorphism. Furthermore, if S is right h-adequate or right type A then so is S/I.*

Let ρ be a *-congruence on a right type A semigroup S. We conclude this section by considering the smallest *-congruence on S which induces the same partition of $E(S)$ as ρ. It is a relatively straightforward adaptation of the corresponding result for inverse semigroups [**18**]. We define the relation $\rho_{\min}$ on S by

$$a\rho_{\min}b \text{ if and only if } ae=be \text{ for some } e\in E(S) \text{ with } e\rho a^*\rho b^*.$$

PROPOSITION 1.5. *Let ρ be a *-congruence on a right type A semigroup S. Then $\rho_{\min}$ is a *-congruence on S, $\rho_{\min}\mid E(S)=\rho\mid E(S)$ and $\rho_{\min}\subseteq\tau$ for any *-congruence τ on S with $\tau\mid E(S)=\rho\mid E(S)$. Furthermore, $S/\rho_{\min}$ is right type A and*

$$E(S/\rho_{\min})=\{e\rho_{\min}:e\in E(S)\}.$$

Proof. Clearly $\rho_{\min}$ is reflexive and symmetric. If $a\rho_{\min}b$, $b\rho_{\min}c$ then there are idempotents e,f with $e\rho a^*\rho b^*\rho c^*\rho f$, $ae=be$ and $bf=cf$. Hence $e^2\rho ef$; so that $ef\rho a^*\rho c^*$ and $aef=bef=cef$, whence $a\rho_{\min}c$.

Now let $a\rho_{\min}b$ and $c\in S$. Then $ae=be$ for some idempotent e with $e\rho a^*\rho b^*$. Certainly $(ca)e(ca)^*=(cb)e(ca)^*$ and, from Proposition 1.3,

$$(ca)^*=a^*(ca)^*\rho e(ca)^*,$$

$$(cb)^*=b^*(cb)^*=b^*(cb)^*b^*\rho e(cb)^*e.$$

Now $cae=cbe$ so that $(ca)^*e=(cae)^*=(cbe)^*=(cb)^*e$ and hence

$$(cb)^*\rho e(ca)^*e=e(ca)^*.$$

Thus $ca\rho_{\min}cb$.

Since S is right type A, we have $ec=c(ec)^*$; so that

$$(ac)(ec)^*=aec=bec=(c)(ec)^*.$$

As $a^*\rho e$, we have $a^*c\rho ec$ and so $(a^*c)^*\rho(ec)^*$ since ρ is a *-congruence. But $(a^*c)^*=(ac)^*$; so that $(ac)^*\rho(ec)^*$. Similarly $(bc)^*\rho(ec)^*$, and hence $ac\rho_{\min}bc$.

Also, from $ae=be$, we obtain

$$a^*e=(ae)^*=(be)^*=b^*e$$

and, as $a^*\rho b^*\rho e$, we conclude that $a^*\rho_{\min}b^*$.

Thus $\rho_{\min}$ is a *-congruence on S.

If $e,f\in E(S)$ and $e\rho f$ then $e\rho ef$ and $e(ef)=f(ef)$; so that $e\rho_{\min}f$. Conversely, if $e\rho_{\min}f$ then it is immediate that $e\rho f$ and thus $\rho\mid E(S)=\rho_{\min}\mid E(S)$.

Now let τ be a *-congruence on S with $\rho\mid E(S)=\tau\mid E(S)$. If $a\rho_{\min}b$ then $ae=be$ for some idempotent e with $e\rho a^*\rho b^*$. Thus $e\tau a^*\tau b^*$ and so $aa^*\tau ae$, $bb^*\tau be$, whence $aa^*\tau bb^*$, that is, $a\tau b$. Thus $\rho_{\min}\subseteq\tau$.

To show that the *-semigroup $S/\rho_{\min}$ is right type A, it is sufficient to show that it is right adequate. To see this, let $a\in S$, $x,y\in S^1$ and suppose that $ax\rho_{\min}ay$. Then $axe=aye$ for some $e\in E(S)$ with $(ax)^*\rho(ay)^*\rho e$. Since $a^*\mathscr{L}^*a$, we thus have $a^*xe=a^*ye$ and, by Proposition 1.3, $(a^*x)^*\rho(a^*y)^*\rho e$; so that $a^*x\rho_{\min}a^*y$ as required and $S/\rho_{\min}$ is right type A.

If $a^2\rho_{\min}a$ then, from above, $a^*a\rho_{\min}a^*$ and hence $(a^*a)^*\rho_{\min}a^*$. Thus $e\rho_{\min}a^*$, where $e=a^*(a^*a)^*$. Now $e=(a(a^*a)^*)^*$ and $\rho_{\min}\subseteq\rho$; so that, from $ae=a(a^*a)^*e$, we conclude $a\rho_{\min}a(a^*a)^*$. But $a^*a=a(a^*a)^*$ since S is right type A and so $a\rho_{\min}a^*$. It follows that the set of idempotents of $S/\rho_{\min}$ is $\{e\rho_{\min}:e\in E(S)\}$.

By considering the case when $\rho = \omega$ is the universal congruence on S, it is straightforward to obtain the following corollary which is Lemma 1.3 of **[8]** (see also **[12**, Proposition 1.7]). We denote $\omega_{\min}$ by σ and note that $a\sigma b$ if and only if $ae = be$ for some idempotent e of S.

COROLLARY 1.6. *Let S be a right type A semigroup. Then σ is a *-congruence on S, S/σ is a left cancellative monoid and σ is the minimum left cancellative congruence on S.*

Following **[8]**, we define a right type A semigroup to be *proper* if $\sigma \cap \mathscr{L}^* = \iota$. Every proper right type A semigroup S is E-unitary (i.e. $a \in S$, ae, $e \in E(S)$ implies $a \in E(S)$) but the converse is not true as shown by Example 3 of **[8]**.

PROPOSITION 1.7. *If ρ is a *-congruence on a proper right type A semigroup S then $S/\rho_{\min}$ is proper.*

Proof. Write τ for $\rho_{\min}$ and suppose that $a\tau$, $b\tau$ are $(\sigma \cap \mathscr{L}^*)$-related in S/τ. Then $(a\tau)^* = (b\tau)^*$ so that $a^*\tau b^*$ and hence $a^*f = b^*f$ for some $f \in E(S)$ with $f\rho a^*\rho b^*$.

Now $(a\tau)(e\tau) = (b\tau)(e\tau)$ for some $e \in E(S)$ since $a\tau$, $b\tau$ are σ-related in S/τ. Thus $ae\tau be$ and so $aeh = beh$ for some idempotent h. This gives $a\sigma b$ and so $af\sigma bf$. But $(af)^* = a^*f = b^*f = (bf)^*$ so that $af\mathscr{L}^*bf$ and, as S is proper, we get $af = bf$. But $f\rho a^*\rho b^*$ and so $a\tau b$ as required and S/τ is proper.

2. The semigroup A_X. In this section, we construct the free right type A semigroup A_X on a set X. We show that A_X is free by showing that it is isomorphic to P_X/γ, where P_X is the free right h-adequate semigroup on X and γ is the minimum right type A congruence on P_X. We begin by recalling the construction of P_X in **[10]**.

Let F_X be the free semigroup on X and partially order F_X by putting $u \leqslant v$ if and only if u is a final segment of v. For any subset A of F_X, we write

$$\max A = \{a \in A : a \text{ is maximal in } A \text{ under } \leqslant\}.$$

Now let

$$E_X = \{A : A \subseteq F_X,\ A \text{ is finite and non-empty},\ A = \max A\}.$$

Thus E_X is the set of all finite suffix codes over X. For A, $B \in E_X$, let $AB = \max(A \cup B)$. Then E_X is a semilattice; in fact, if we consider F_X as partially ordered by the dual of the above ordering then E_X is the free semilattice on this partially ordered set **[11]**. We note that the following statements are equivalent for members A, B of E_X where we use $\leqslant$ for the order relation in E_X as well as that in F_X:

$A \leqslant B$; $AB = A$; $\max(A \cup B) = A$; for each b in B, there is an a in A such that $b \leqslant a$; each element in B is a final segment of some element in A.

For $w \in F_X$, $A \in E_X$, we put $A\,.\,w = \{aw : a \in A\}$. Clearly $A\,.\,w \in E_X$ and we have an action of F_X on E_X. Furthermore, if $w \in F_X$, A, $B \in E_X$ then it is routine to verify that

$$(AB)\,.\,w = (A\,.\,w)(B\,.\,w)$$

and consequently the action is order-preserving.

For each element w of F_X, we define w^* to be the singleton $\{w\} \in E_X$. We note that if $A = \{w_1, \ldots, w_k\} \in E_X$ then $A = w_1^* \ldots w_k^*$; so that E_X is generated by the set $\{w^* : w \in F_X\}$. We also observe that, for any $A \in E_X$, $w \in F_X$, we have $A\,.\,w \leqslant \{w\} = w^*$.

Consider the free product $F_X * E_X$ in the category of semigroups. Its elements can be written uniquely as words $a = s_1 \ldots s_n$, where $s_i \in F_X \cup E_X$ and, for $i = 1, \ldots, n-1$, the elements s_i, s_{i+1} are not both in the same factor F_X or E_X.

We extend * from F_X to $F_X * E_X$ as follows: for $A \in E_X$, we put $A^* = A$ and if $a = s_1 \ldots s_n$ (as above), $b = s_1 \ldots s_{n-1}$ and $b^* \in E_X$ has been defined then

$$a^* = \begin{cases} b^* s_n & \text{if } s_n \in E_X, \\ b^* \,.\, s_n & \text{if } s_n \in F_X. \end{cases}$$

Let $\sim$ be the congruence on $F_X * E_X$ generated by the relation $\{(aa^*, a) : a \in F_X * E_X\}$ and put $P_X = (F_X * E_X)/\sim$. It is shown in **[10]** that P_X is the free right h-adequate semigroup on X. Further, every element of P_X can be represented as $w_0 A_1 \ldots A_n w_n$, where each A_i is in E_X, each w_i is in $F_X \cup \{1\}$ with $w_i \neq 1$ when $i \neq 0, n$, and, for each i, $A_i < (w_0 A_1 \ldots w_{i-1})^*$. An element of P_X written in such a way is said to be in *normal form*.

From **[10]**, we know that if $a \in P_X$ has normal form $w_0 A_1 \ldots A_n w_n$ then $a^* = A_n \,.\, w_n$ and thus we have the following lemma which is part of Proposition 3.3 of **[10]**.

Lemma 2.1. *Let a, b be elements of P_X with normal forms $w_0 A_1 \ldots A_n w_n$, $v_0 B_1 \ldots B_m v_m$ respectively. Then*

$$a \mathscr{L}^* b \text{ if and only if } A_n \,.\, w_n = B_m \,.\, w_m.$$

On any right adequate semigroup there is a minimum left cancellative congruence σ. A description of σ on right type A semigroups was given in Section 1. Now we want a characterisation of σ on P_X. For any element $a = w_0 A_1 \ldots A_k w_k$ of $F_X * E_X$, we define $c(a) = w_0 \ldots w_k$. If $a, b \in F_X * E_X$ and $a \sim b$ then $c(a) = c(b)$, so that we may regard c as defined on P_X. Then c is a *-homomorphism from P_X onto F_X^1 and we have the following result.

Proposition 2.2 [**10**, Proposition 3.7]. *On P_X,*
(1) *$(a, b) \in \sigma$ if and only if $c(a) = c(b)$ and*
(2) *$P_X/\sigma \cong F_X^1$.*

The class $\mathscr{A}$ of right type A semigroups is contained in the class of right h-adequate semigroups and so, in the terminology of **[15]**, P_X is free *for* $\mathscr{A}$ over X, that is, P_X is generated by X and, for every $S \in \mathscr{A}$ and for every mapping $\alpha : X \to S$, there is a *-homomorphism $\beta : P_X \to S$ which extends α. Since $\mathscr{A}$ is a quasi-variety, it follows from Lemma 4.112 of **[15]** that P_X/γ is free in $\mathscr{A}$ over $\bar{X} = \{x\gamma : x \in X\}$, where γ is the intersection of all the *-congruences ρ on P_X such that P_X/ρ is right type A. In other words, γ is the minimum right type A*-congruence on P_X and P_X/γ is the free right type A semigroup on $\bar{X}$. As $|\bar{X}| = |X|$, we may regard P_X/γ as the free right type A semigroup on X.

Our next task is to describe γ on P_X. To help do this, we introduce the semigroup A_X:

$$A_X = \{(w, A) \in F_X^1 \times E_X : w \leqslant a \text{ for some } a \in A\}$$

and the multiplication in A_X is given by

$$(w, A)(v, B) = (wv, A \,.\, v \wedge B).$$

https://doi.org/10.1017/S0017089500008168 Published online by Cambridge University Press

It is an easy matter to verify that A_X is a right type A semigroup with semilattice of idempotents $E(A_X)=\{(1,A):A\in E_X\}$ and with $(w,A)^*=(1,A)$.

THEOREM 2.3. *On* P_X, $\gamma=\sigma\cap\mathscr{L}^*$ *and* $P_X/\gamma\cong A_X$.

Proof. Define a mapping $\theta:P_X\to A_X$ by putting $a\theta=(c(a),a^*)$ for each a in P_X. Then $a^*\theta=(c(a^*),a^*)=(1,a^*)$ since c maps all idempotents of P_X to 1. Thus $a^*\theta=(a\theta)^*$.

Next we note that if $C_1,\dots,C_k\in E_X$, $u_1,\dots,u_{k-1}\in F_X$ and $u_k\in F_X^1$ then, using the definition of a^*, an easy induction argument yields

$$(C_1u_1\dots C_ku_k)^*=(C_1\,.\,(u_1\dots u_k))(C_2\,.\,(u_2\dots u_k))\dots(C_k\,.\,u_k).$$

Thus if $a,b\in P_X$ have normal forms $w_0A_1w_1\dots A_nw_n$, $v_0B_1v_1\dots B_mv_m$ respectively then

$$\begin{aligned}(a^*b)^*&=((A_n\,.\,w_n)v_0B_1v_1\dots B_mv_m)^*\\&=((A_n\,.\,w_n)\,.\,(v_0\dots v_m))(B_1\,.\,(v_1\dots v_m))\dots(B_m\,.\,v_m)\\&=((A_n\,.\,w_n)\,.\,c(b))(B_m\,.\,v_m)\\&=a^*\,.\,c(b)\wedge b^*.\end{aligned}$$

Hence

$$\begin{aligned}(ab)\theta=(c(ab),(ab)^*)&=(c(a)c(b),(a^*b)^*)\\&=(c(a)c(b),a^*\,.\,c(b)\wedge b^*)\\&=(c(a),a^*)(c(b),b^*)\\&=(a\theta)(b\theta).\end{aligned}$$

Thus θ is a *-homomorphism. Further, θ is surjective because if $(w,A)\in A_X$ then wA is an element of P_X. Also $A\leqslant\{w\}$ since $w\leqslant a$ for some $a\in A$ and so $(wA)^*=A$. Hence $(w,A)=(wA)\theta$.

Thus $P_X/\ker\theta\cong A_X$, so that $\ker\theta$ is a right type A congruence and $\gamma\subseteq\ker\theta$. Now $(a,b)\in\ker\theta$ if and only if $c(a)=c(b)$ and $a^*=b^*$; so that, by Proposition 2.2, $\ker\theta=\sigma\cap\mathscr{L}^*$.

On the other hand, for all elements c and all idempotents e in P_X, we have $ec\gamma c(ec)^*$. Repeated application of this to A_1w_1, A_2w_2, $(A_1w_1)^*w_2$ etc., where $w_0A_1w_1\dots A_nw_n$ is the normal form of an element a of P_X, leads to

$$a\gamma v_0(A_1v_1)^*\dots(A_nv_n)^*,$$

where $v_i=w_i\dots w_n$. Now $A_{i+1}<A_i\,.\,w_i$ gives

$$(A_nw_n)^*=A_n\,.\,w_n<A_i\,.\,v_i=(A_iv_i)^*$$

for $i=1,\dots,n-1$ and hence $a\gamma c(a)a^*$. Thus if a,b are elements of P_X with $c(a)=c(b)$ and $a^*=b^*$ then $a\gamma b$. In view of Proposition 2.2, we thus have $\sigma\cap\mathscr{L}^*\subseteq\gamma$ and so $\gamma=\sigma\cap\mathscr{L}^*$.

COROLLARY 2.4. A_X *is the free right type A semigroup on X.*

By comparing A_X with the free objects in the category of SL2 δ-semigroups as described in [**1**], we see that A_X is the free SL2 δ-semigroup on X. Thus, as noted in [**3**], we have the following corollary.

COROLLARY 2.5. *In the class of *-semigroups, the variety generated by the quasi-variety of right type A semigroups is the variety of* SL2 δ*-semigroups.*

Part (1) of the following result is an easy consequence of the definition of σ on a right type A semigroup; part (2) follows from $(v, A)^* = (1, A)$.

PROPOSITION 2.6. *Let* (v, A), (w, B) *be elements of* A_X; *then*
(1) $(v, A)\sigma(w, B)$ *if and only if* $v = w$,
(2) $(v, A)\mathscr{L}^*(w, B)$ *if and only if* $A = B$.

COROLLARY 2.7. *The free right type A semigroup* A_X *is proper.*

We can now give an alternative proof of Theorem 3.3 of [**8**]. A homomorphism $\theta: S \to T$ of semigroups is called an $\mathscr{L}^*$*-homomorphism* in [**8**] if $a\theta = b\theta$ implies $a\mathscr{L}^*b$. When S, T are right adequate, it is easy to see that if θ is an idempotent-separating *-homomorphism then θ is an $\mathscr{L}^*$-homomorphism. Thus the formulation in the next theorem does give Theorem 3.3 of [**8**].

THEOREM 2.8. *Let S be a right type A semigroup. Then S is the image of a proper right type A semigroup under an idempotent-separating *-homomorphism.*

Proof. Every right type A semigroup is a *-homomorphic image of some free right type A semigroup. Thus $S \cong A_X/\rho$ for some X and some *-congruence ρ on A_X. Now $\rho_{\min} \subseteq \rho$; so $A_X/\rho \cong (A_X/\rho_{\min})/(\rho/\rho_{\min})$. Since A_X is proper, it follows from Proposition 1.7 that $A_X/\rho_{\min}$ is proper. Since $\rho \mid E(A_X) = \rho_{\min} \mid E(A_X)$, the *-congruence $\rho/\rho_{\min}$ is idempotent-separating and the result follows.

We remark that it is evident from the definition of $\rho_{\min}$ that $\rho_{\min} \subseteq \sigma$; so that, in the terminology of [**1**] and [**2**], $A_X/\rho_{\min}$ is *quasi-free*. A result similar to Theorem 2.8 is stated in [**1**] and [**2**], where it is asserted that every SL2 γ-semigroup is the image of a quasi-free SL2 γ-semigroup under an idempotent-separating *-homomorphism.

We also remark that Theorem 2.8 was proved in [**16**] by analysing *-congruences on M-semigroups and using the characterisation of A_X as an M-semigroup.

3. Properties of A_X. The relation $\mathscr{R}^*$ on a semigroup is the dual of $\mathscr{L}^*$; $\mathscr{D}^*$ is the join of $\mathscr{L}^*$ and $\mathscr{R}^*$. We describe the relation $\mathscr{R}^*$ on A_X and show that A_X is a single $\mathscr{D}^*$-class. By contrast, all Green's relations on A_X are trivial. We then show that A_X satisfies certain maximal conditions, has solvable word problem and is residually finite.

For any element (v, A) of A_X, we define the subset $\mathrm{IS}(v, A)$ of F_X^1 by

$$\mathrm{IS}(v, A) = \{u \in F_X^1 : uv \in A\}.$$

Since (v, A) is a member of A_X, v is a final segment of some element of A and so $\mathrm{IS}(v, A)$ is not empty. Furthermore, since $A = \max A$, it is clear that $\mathrm{IS}(v, A) = \{1\}$ or $\mathrm{IS}(v, A) = \max \mathrm{IS}(v, A)$. In the latter case, $\mathrm{IS}(v, A)$ is a member of E_X.

PROPOSITION 3.1. *Let* (v, A), (w, B) *be elements of* A_X; *then* $(v, A)\mathscr{R}^*(w, B)$ *if and only if* $\mathrm{IS}(v, A) = \mathrm{IS}(w, B)$.

Proof. Put $H = \mathrm{IS}(v, A)$. Then $A = H.v \cup (A \setminus H.v)$. If $t \in A \setminus H.v$ then t and uv are incomparable for any element u of F_X since $A = \max A$. Hence we see that $A \setminus H.v$ is

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a subset of $C.v \wedge A$ for any member C of E_X. Thus, for $C, D \in E_X$, we have $C.v \wedge A = D.v \wedge A$ if and only if $C.v \wedge H.v = D.v \wedge H.v$. If $H = \{1\}$, this latter condition is equivalent to $C.v = D.v$ and it is not difficult to see that (v, A) is right cancellable. If $H \neq \{1\}$ then, since F_X acts on E_X by semilattice homomorphisms, we have $C.v \wedge A = D.v \wedge A$ if and only if $(C \wedge H).v = (D \wedge H).v$, that is, if and only if $C \wedge H = D \wedge H$. It is now easy to see that $(v, A)\mathcal{R}^*(1, H)$ and the result follows.

Corollary 3.2. *On A_X, the relation $\mathcal{D}^*$ is the universal relation, that is, A_X is $\mathcal{D}^*$-simple.*

Proof. Let $A \in E_X$ and let $v \in A$. Then (v, A) is a member of A_X and $\mathrm{IS}(v, A) = \{1\}$; so that (v, A) is right cancellable. Since $(1, A)\mathcal{L}^*(v, A)$, it follows that every idempotent of A_X is $\mathcal{L}^*$-related to a right cancellable element of A_X and hence all idempotents of A_X are in the same $\mathcal{D}^*$-class. The corollary now follows.

We now turn to Green's relations.

Proposition 3.3. *On A_X, $\mathcal{J} = \iota$.*

Proof. Suppose that (v, A), (w, B) are $\mathcal{J}$-related. It follows that v, w are subwords of each other and hence that $v = w$. Thus

$$(v, A) = (1, C)(v, B)(1, D)$$

for some $C, D \in E_X$; whence $A \leq B$. Similarly $B \leq A$ and so $(v, A) = (w, B)$.

As pointed out in **[10]**, it is easy to see that the semilattice E_X satisfies the ascending chain condition. We use this fact in the proof of the next result.

Proposition 3.4. *A_X satisfies the maximal condition for principal left ideals, for principal right ideals and for principal ideals.*

Proof. Since $\mathcal{J}$ is trivial, it suffices to prove the condition for principal ideals. If $a_1, a_2, \ldots$ are the generators of an increasing sequence of principal ideals in A_X then $a_1\sigma, a_2\sigma, \ldots$ are the generators of a similar sequence in F_X. Hence $a_k\sigma = a_{k+1}\sigma = \ldots$ for some k. Write $v = a_k\sigma$ and, for $j \geq k$, let $a_j = (v, A_j)$. It is easy to see that $A_k \leq A_{k+1} \leq \ldots$; so that, for some positive integer n, $A_n = A_{n+1} = \ldots$. The result follows.

In **[10]**, it is shown that an element of the free *-semigroup on X can be effectively reduced to a normal form in P_X. Given an element a of P_X in normal form, it is clear that there is a finite procedure for finding $c(a)$ and a^*. We can determine in an effective way whether two elements of F_X^1 (resp. E_X) are equal and so, by virtue of Lemma 2.1, Proposition 2.2 and Theorem 2.3, we can decide when two elements in normal form in P_X are related by the congruence γ. The following result is an immediate consequence.

Proposition 3.5. *The word problem for A_X is solvable.*

We conclude this section by considering the residual finiteness of A_X.

Proposition 3.6. *A_X is residually finite in the class of right type A semigroups.*

Proof. Let (v, A), (w, B) be distinct elements of A_X and let Y be the set of letters (elements of X) occurring in words in $A \cup B$. Let Q be the set of elements C of E_X for which an element of $X\setminus Y$ occurs in a word belonging to C. It is clear that Q is an ideal of E_X. It follows that

$$J=\{(u,C)\in A_X : X\in Q\}$$

is an ideal of A_X and it is clear that it is in fact a *-ideal.

Next we let $k-1$ be the length of the longest word in $A\cup B$ and let J_k be the ideal of E_X consisting of all members of E_X which contain a word of length at least k. Then

$$K=\{(u,C)\in A_X : C\in J_k\}$$

is a *-ideal of A_X. Now let $I=J\cup K$. Then I is a *-ideal of A_X; so that, by Proposition 1.4, A_X/I is a right type A semigroup and the natural homomorphism ν from A_X onto A_X/I is a *-homomorphism. Since neither (v,A) nor (w,B) is a member of I, we have $(v,A)\nu\neq(w,B)\nu$. However, it is easy to see that A_X/I is finite.

We recall that an algebra is *hopfian* if all its surjective endomorphisms are automorphisms. It is pointed out in **[10]** that the general result of Evans **[6]** that a finitely generated residually finite algebra in a variety of algebras is hopfian applies equally well to a quasi-variety of algebras. The following corollary is therefore immediate from Proposition 3.6.

COROLLARY 3.7. *For a finite set X, A_X is hopfian.*

4. Sets of free generators. For a subset Y of a right adequate semigroup S, we denote by $\langle Y\rangle^*$ the *-subsemigroup generated by Y. As we observed in Section 1, if S is right type A then so is $\langle Y\rangle^*$. If the inclusion map $Y\to\langle Y\rangle^*$ extends to an isomorphism of A_Y onto $\langle Y\rangle^*$ then we say that Y is a set of free generators for $\langle Y\rangle^*$.

The results of this section are inspired by the corresponding ones of Reilly **[19]** as were those of Section 4 in **[10]**. We start observing that A_X has only one set of free generators.

PROPOSITION 4.1. *Let X be a non-empty set. Then the subset $\bar{X}=\{(x,\{x\}):x\in X\}$ is the only set of free generators for A_X.*

Proof. For any subset Y of A_X, let $Y_r=\{y^*:y\in Y\}$. Then $\bar{X}_r=\{(1,\{x\}):x\in X\}$ is the set of maximal elements of $E(A_X)$. We can thus characterize $\bar{X}$ as the unique subset of A_X such that $\bar{X}\cap\bar{X}_r=\varnothing$ and $\bar{X}_r$ is the set of maximal elements of $E(A_X)$.

For any right type A semigroup S, we define the relation $\leqslant$ by

$$a\leqslant b \text{ if and only if } a=ba^*.$$

As noted in **[1]** and **[5]**, "$a=ba^*$" is equivalent to "$a=be$ for some idempotent e in S" and the relation $\leqslant$ is a compatible partial order on S which extends the natural order of the semilattice of idempotents of S. It will always be clear from the context when $\leqslant$ is being used for the relation just described and when it is the partial order on F_X used throughout the paper.

We now give criteria for a subset of a right type A semigroup to be a set of free generators for the *-subsemigroup which it generates.

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PROPOSITION 4.2. *Let Y be subset of a right type A semigroup S. Then Y is a set of free generators for $\langle Y\rangle^*$ if and only if*

(1) *no pair of elements of $\langle Y\rangle$ have a lower bound in $\langle Y\rangle^*$, and*

(2) *if $(y_t \dots y_1)^* \geq \prod_{j=1}^{m} (y_{jp(j)} \dots y_{j1})^*$, where the y_i and y_{jk} are elements of Y, then there is a $j \in \{1, \dots, m\}$ such that $y_i = y_{ji}$ for $i = 1, \dots, t$.*

Proof. Let X be a set in one–one correspondence with Y and let $\theta: X \to Y$ be a bijection. Then there is a unique *-homomorphism ψ from A_X onto $\langle Y\rangle^*$ such that $(x, \{x\})\psi = x\theta$ for each $x \in X$. Thus Y is a set of free generators for $\langle Y\rangle^*$ if and only if ψ is injective.

Suppose first that ψ is injective. We identify $\bar{X}$ with Y and A_X with $\langle Y\rangle^*$. A typical element of $\langle \bar{X}\rangle$ is $(v, \{v\})$, where $v \in F_X$, and it is easily seen that condition (1) holds. If the hypothesis of condition (2) holds then we have

$$(1, \{x_t \dots x_1\}) \geq \prod_{j=1}^{m} (1, \{x_{jp(j)} \dots x_{j1}\});$$

so that $x_t \dots x_1$ is a final segment of $x_{jp(j)} \dots x_{j1}$ for some j. Since F_X is the free semigroup on X, we have $x_i = x_{ji}$ for $i = 1, \dots, t$ and the conclusion of condition (2) now follows.

Now suppose that conditions (1) and (2) hold and let $A, B \in E_X$ be such that $(1, A)\psi = (1, B)\psi$. Let $A = \{v_1, \dots, v_m\}$ and $B = \{w_1, \dots, w_n\}$. Then, since $A = \max A$, $B = \max B$, no two of $v_1, \dots, v_m$ are comparable and no two of $w_1, \dots, w_n$ are comparable. Further,

$$(1, A) = (1, \{v_1\}) \dots (1, \{v_m\}) = (v_1, \{v_1\})^* \dots (v_m, \{v_m\})^*,$$
$$(1, B) = (1, \{w_1\}) \dots (1, \{w_n\}) = (w_1, \{w_1\})^* \dots (w_n, \{w_n\})^*.$$

Thus $(1, A)\psi = (1, B)\psi \leq (1, \{w_1\})\psi = ((w_1, \{w_1\})\psi)^*$. For each $j = 1, \dots, m$, let $v_j = x_{jp(j)} \dots x_{j1}$, where $x_{jk} \in X$ for $1 \leq k \leq p(j)$, and let $w_1 = x_t \dots x_1$, where $x_1, \dots, x_t \in X$. For each x_{jk} and each x_i, put $x_{jk}\theta = y_{jk}$ and $x_i\theta = y_i$. Then

$$(y_t \dots y_1)^* = ((w_1, \{w_1\})\psi)^* = (1, \{w_1\})\psi$$

and

$$\prod_{j=1}^{m} (y_{jp(j)} \dots y_{j1})^* = \prod_{j=1}^{m} ((v_j, \{v_j\})\psi)^* = (1, A)\psi.$$

Hence

$$(y_t \dots y_1)^* \geq \prod_{j=1}^{m} (y_{jp(j)} \dots y_{j1})^*$$

and so, by condition (2), there is some j in $\{1, \dots, m\}$ such that $y_i = y_{ji}$ for $i = 1, \dots, t$. It follows that $x_i = x_{ji}$ for $i = 1, \dots, t$ and so w_1 is a final segment of some v_j. A similar argument shows that v_j is a final segment of some w_k. Now we have that w_1 and w_k are comparable so that $k = 1$ and $w_1 = v_j$. Replacing w_1 by w_i shows that each w_i is some v_j; similarly each v_j is some w_i. Thus $A = B$ and ψ is idempotent-separating.

Now let (v, A), (w, B) be elements of A_X with $(v, A)\psi = (w, B)\psi$. Since ψ is a *-homomorphism, $(1, A)\psi = (1, B)\psi$ and so $A = B$. Hence

$$(v, \{v\})\psi(1, A)\psi = (v, A)\psi = (w, A)\psi$$
$$= (w, \{w\})\psi(1, A)\psi;$$

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so that $(v, \{v\})\psi$, $(w, \{w\})\psi$ have a lower bound in $\langle Y\rangle^*$. It follows from condition (1) that $(v, \{v\})\psi = (w, \{w\})\psi$ and, since ψ is a *-homomorphism, we obtain $(1, \{v\})\psi = (1, \{w\})\psi$. As ψ is idempotent separating, we conclude that $v = w$ and hence ψ is injective.

Let $\bar{F}_X = \{(w, \{w\}) : w \in F_X\}$. We now consider which subsets of $\bar{F}_X$ are sets of free generators for the *-subsemigroups of A_X which they generate. It is easy to see that any subset of $\bar{F}_X$ satisfies condition (1) of Proposition 4.2. Thus for a subset Y of $\bar{F}_X$, condition (2) is necessary and sufficient for Y to be a set of free generators for $\langle Y\rangle^*$. We recall that a subset C of F_X is a *suffix code over* X if $F_X C \cap C = \varnothing$. We refer the reader to Chapter 5 of **[13]** for the essential facts about suffix codes.

COROLLARY 4.3. *Let Y be a subset of $\bar{F}_X$ and let $C = \{w \in F_X : (w, \{w\}) \in Y\}$. Then the *-subsemigroup $\langle Y\rangle^*$ of A_X is freely generated by Y if and only if C is a suffix code over X.*

Proof. Suppose that $\langle Y\rangle^*$ is freely generated by Y. If $w \in F_X C \cap C$ then w has a proper final segment, say u, in C. Let $y = (w, \{w\})$, $y' = (u, \{u\})$; then $(y')^* > y^*$, contradicting condition (2). Hence $F_X C \cap C = \varnothing$ and C is a suffix code over X.

Now let C be a suffix code over X. As remarked above, we need only prove that condition (2) holds. Suppose then that

$$(y_t \dots y_1)^* \geqslant \prod_{j=1}^{m} (y_{jp(j)} \dots y_{j1})^*.$$

Let $y_i = (w_i, \{w_i\})$ for $i = 1, \dots, t$ and $y_{jk} = (w_{jk}, \{w_{jk}\})$ for $j = 1, \dots, m$ and $k = 1, \dots, p(j)$; so that w_i and w_{jk} are in F_X. Then our assumption is

$$\{w_t \dots w_1\} \geqslant \max\{w_{1p(1)} \dots w_{11}, \dots, w_{mp(m)} \dots w_{m1}\}$$

and so $w_t \dots w_1$ is a final segment of $w_{jp(j)} \dots w_{j1}$ for some j. Thus either $w_1 = w_{j1}$ or one of w_1, w_{j1} is a proper final segment of the other. The latter is impossible since $w_1, w_{j1} \in C$ and C is a suffix code. Hence $w_1 = w_{j1}$ and so $w_t \dots w_2$ is a final segment of $w_{jp(j)} \dots w_{j2}$. We now get $w_2 = w_{j2}$ and similarly we obtain $w_i = w_{ji}$ for $i = 1, \dots, t$. Thus $y_i = y_{ji}$ for $i = 1, \dots, t$ and condition (2) holds as required.

Over sets with at least two elements there are infinite suffix codes and so we have the following immediate consequence of Corollary 4.3.

COROLLARY 4.4. *If $2 \leqslant |X|$ then there is a countably infinite subset Y of $\bar{F}_X$ such that Y is a set of free generators for the *-subsemigroup $\langle Y\rangle^*$ of A_X.*

Since a right type A semigroup S is right h-adequate we know by Lemma 4.5 of **[10]** that if $S = \langle a\rangle^*$ for some a in S then $E(S) = \{(a^k)^* : k \in \mathbb{Z}, k \geqslant 1\}$. In particular, $E(S)$ is a chain, so that if s, t are any two elements of S then s, t cannot freely generate $\langle s, t\rangle^*$ because $E(\langle s, t\rangle^*)$ is not isomorphic to $E(A_{\{x,y\}})$. Thus if $|Z| = 1$ then A_Z does not contain copies of A_X for any set X with $2 \leqslant |X|$. We conclude this section by showing that a non-idempotent element a in any A_X freely generates $\langle a\rangle^*$.

PROPOSITION 4.5. *If $(v, A) \in A_X$ and $v \neq 1$ then (v, A) is a free generator of $\langle (v, A)\rangle^*$.*

Proof. If $(v, A)^t(1, B) = (v, A)^k(1, C)$ for any $B, C \in E_X$ and positive integers t, k then $v^t = v^k$; so that $t = k$ and condition (1) of Proposition 4.2 holds.

Let $a = (v, A)$ and suppose that

$$(a^t)^* \geq \prod_{j=1}^{m} (a^{p(j)})^*.$$

Then $t \leq \max\{p(1), \ldots, p(m)\}$ and hence condition (2) of Proposition 4.2 holds.

5. Free right type A semigroups with central idempotents. Let $\mathcal{B}$ be the class of right type A semigroups with central idempotents. From [7], we know that if $S \in \mathcal{B}$ then S is a strong semilattice of left cancellative monoids. In this section we describe the free objects in $\mathcal{B}$. On any right type A semigroup S there is a *-congruence ν which is minimum among those *-congruences ρ such that S/ρ has central idempotents. As $\mathcal{B}$ is a quasi-variety of *-semigroups and is contained in the quasi-variety of right type A semigroups, it follows (as in Section 2) from Lemma 4.112 of [15] that A_X/ν is the free object in $\mathcal{B}$ on X. To obtain a more explicit description, we begin by considering some properties of ν.

For a word w in F_X, we define $\text{alph}(w)$ to be the subset of X consisting of those elements which actually occur in w. For a subset A of F_X we define

$$\text{alph}(A) = \bigcup \{\text{alph}(w) : w \in A\}.$$

LEMMA 5.1. *For each* $w \in F_X$, $(1, \{w\})\nu(1, \text{alph}(w))$.

Proof. We use induction on the length of w. Assuming the result for words of length n, let $w = x_1 \ldots x_{n+1}$.

Then $(x_{n+1}, \text{alph}(w)) \in A_X$ and

$$\begin{aligned}(x_{n+1}, \text{alph}(w)) &= (x_{n+1}, \{x_{n+1}\})(1, \text{alph}(x_1 \ldots x_n)) \\ &\nu(1, \text{alph}(x_1 \ldots x_n))(x_{n+1}, \{x_{n+1}\}) \\ &\nu(1, \{x_1 \ldots x_n\})(x_{n+1}, \{x_{n+1}\}) \\ &= (x_{n+1}, \{x_1 \ldots x_{n+1}\}) \\ &= (x_{n+1}, \{w\}).\end{aligned}$$

As ν is a *-congruence, we thus have $(1, \text{alph}(w))\nu(1, \{w\})$ and the result follows.

COROLLARY 5.2. *If* $(1, A) \in A_X$ *then* $(1, A)\nu(1, \text{alph}(A))$.

Proof. Let $A = \{w_1, \ldots, w_m\}$. Then, by Lemma 5.1,

$$(1, A) = (1, \{w_1\}) \ldots (1, \{w_m\})\nu(1, \text{alph}(w_1)) \ldots (1, \text{alph}(w_m)) = (1, \text{alph}(A)).$$

COROLLARY 5.3. *If* $(w, A), (w, B) \in A_X$ *and* $\text{alph}(A) = \text{alph}(B)$ *then* $(w, A)\nu(w, B)$.

Proof. Since $(w, A) = (w, \{w\})(1, A)$ and $(w, B) = (w, \{w\})(1, B)$, we need only show that $(1, A)\nu(1, B)$. But this follows from Corollary 5.2 since $\text{alph}(A) = \text{alph}(B)$.

For a non-empty set X, we denote the free semilattice with identity on X by Y_X. The elements of Y_X are all finite subsets of X and the operation is set-theoretic union. We define a subsemigroup S_X of the direct product $F_X^1 \times (Y_X \setminus \{1\})$ as follows:

$$S_X = \{(w, T) \in F_X^1 \times (Y_X \setminus \{1\} : \text{alph}(w) \subseteq T\}$$

where $\text{alph}(1) = \varnothing$.

It is easy to verify that S_X is right type A with central idempotents.

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Proposition 5.4. *The mapping* $\psi : A_X \to S_X$ *defined by*

$$(w, A)\psi = (w, \text{alph}(A))$$

*is a surjective *-homomorphism and* $\ker \psi = \nu$.

Proof. Let $(w, A), (v, B) \in A_X$. Then v is a final segment of some member of B and so

$$\text{alph}(A \,.\, v \wedge V) = \text{alph}(A \cup B) = \text{alph}(A) \cup \text{alph}(B).$$

It is now easy to see that ψ is a *-homomorphism. Further if $(w, T) \in S_X$ then $(w, A) \in A_X$, where

$$A = \{w\} \cup (T \setminus \text{alph}(w))$$

and $(w, A)\psi = (w, T)$; so that ψ is surjective.

Thus $\ker \psi$ is a *-congruence and $A_X/\ker \psi$ is right type A with central idempotents; so that $\nu \subseteq \ker \psi$.

If $((w, A), (w, B)) \in \ker \psi$ then $w = v$ and $\text{alph}(A) = \text{alph}(B)$ so that, by Corollary 5.3, $\ker \psi \subseteq \nu$. Thus $\nu = \ker \psi$ as required.

Corollary 5.5. *The *-semigroup* S_X *is the free object in* $\mathcal{B}$ *on* X.

We conclude by noting that we can represent S_X as a strong semilattice of free monoids as follows:

$$S_X \cong \bigcup_{A \in E} F_A^1,$$

where $E = Y_X \setminus \{1\}$ and the linking homomorphisms $\phi_{A,B} : F_A^1 \to F_B^1$ are simply the inclusion mappings for $A \subseteq B$.

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