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Derived subalgebras of centralisers and finite W-algebras

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Abstract

Let $\mathfrak{g} = \mathrm{Lie}(G)$ be the Lie algebra of a simple algebraic group G over an algebraically closed field of characteristic 0. Let e be a nilpotent element of $\mathfrak{g}$ and let $\mathfrak{g}_e = \mathrm{Lie}(G_e)$ where G_e stands for the stabiliser of e in G. For $\mathfrak{g}$ classical, we give an explicit combinatorial formula for the codimension of $[\mathfrak{g}_e, \mathfrak{g}_e]$ in $\mathfrak{g}_e$ and use it to determine those $e \in \mathfrak{g}$ for which the largest commutative quotient $U(\mathfrak{g}, e)^{\mathrm{ab}}$ of the finite W-algebra $U(\mathfrak{g}, e)$ is isomorphic to a polynomial algebra. It turns out that this happens if and only if e lies in a unique sheet of $\mathfrak{g}$. The nilpotent elements with this property are called *non-singular* in the paper. Confirming a recent conjecture of Izosimov, we prove that a nilpotent element $e \in \mathfrak{g}$ is non-singular if and only if the maximal dimension of the geometric quotients $\mathcal{S}/G$, where $\mathcal{S}$ is a sheet of $\mathfrak{g}$ containing e, coincides with the codimension of $[\mathfrak{g}_e, \mathfrak{g}_e]$ in $\mathfrak{g}_e$ and describe all non-singular nilpotent elements in terms of partitions. We also show that for any nilpotent element e in a classical Lie algebra $\mathfrak{g}$ the closed subset of $\mathrm{Specm}\, U(\mathfrak{g}, e)^{\mathrm{ab}}$ consisting of all points fixed by the natural action of the component group of G_e is isomorphic to an affine space. Analogues of these results for exceptional Lie algebras are also obtained and applications to the theory of primitive ideals are given.

1. Introduction and preliminaries

1.1 Associated varieties and associated cycles

Let $\Bbbk$ be an algebraically closed field of characteristic 0 and let G be a simple algebraic group of adjoint type over $\Bbbk$. Given an element x in the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, we write G_x for the (adjoint) stabiliser of x in G and denote by $\mathfrak{g}_x$ the Lie algebra of G_x. It is well known that $\mathfrak{g}_x$ coincides with the centraliser of x in $\mathfrak{g}$.

Let $U(\mathfrak{g})$ for the universal enveloping algebra of $\mathfrak{g}$ and denote by $\mathcal{X}$ the set of all primitive ideals of $U(\mathfrak{g})$. By the PBW theorem, the graded algebra associated with the canonical filtration of $U(\mathfrak{g})$ is isomorphic to the symmetric algebra $S(\mathfrak{g})$ which we identify with $S(\mathfrak{g}^*)$ by using the Killing form on $\mathfrak{g}$. Using commutative algebra, we then attach to $I \in \mathcal{X}$ two important invariants: the associated variety $\mathrm{VA}(I)$ and the associated cycle $\mathrm{AC}(I)$. The variety $\mathrm{VA}(I)$ is the zero locus in $\mathfrak{g}$ of the G-stable ideal $\mathrm{gr}(I)$ of $S(\mathfrak{g}^*)$, and $\mathrm{AC}(I)$ is a formal linear combination $\sum_{i=1}^{l} m_i[\mathfrak{p}_i]$ where $\mathfrak{p}_1, \ldots, \mathfrak{p}_l$ are the minimal primes of $S(\mathfrak{g}^*)$ over $\mathrm{Ann}_{S(\mathfrak{g}^*)}\,\mathrm{gr}(U(\mathfrak{g})/I)$ and $m_1, \ldots, m_l$ are their multiplicities; see [Jan04, § 9], where notation is slightly different. Since the variety $\mathrm{VA}(I)$ is irreducible by Joseph's theorem [Jos85a] and hence coincides with the Zariski closure of a nilpotent orbit $\mathcal{O} \subset \mathfrak{g}$, we have that $\mathrm{AC}(I) = m_I[J]$ where $m_I \in \mathbb{N}$ and $J = \sqrt{\mathrm{gr}(I)}$, a prime

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ideal of $S(\mathfrak{g}^*)$. The positive integer m_I is sometimes referred to as the *multiplicity* of $\mathcal{O}$ in the primitive quotient $U(\mathfrak{g})/I$ and abbreviated as $\mathrm{mult}_{\mathcal{O}}(U(\mathfrak{g})/I)$. It is well known that if $\mathcal{O} = \{0\}$ then I coincides with the annihilator in $U(\mathfrak{g})$ of a finite-dimensional irreducible $\mathfrak{g}$-module V, the radical $J = \sqrt{\mathrm{gr}(I)}$ identifies with the ideal $\bigoplus_{i>0} S^i(\mathfrak{g}^*)$ and $m_I = (\dim V)^2$.

1.2 Primitive ideals and finite *W*-algebras

From now on we let e be a non-zero nilpotent element of $\mathfrak{g}$ and include it in an $\mathfrak{sl}_2$-triple $\{e, h, f\} \subset \mathfrak{g}$. Let $U(\mathfrak{g}, e)$ be the finite W-algebra associated with the pair $(\mathfrak{g}, e)$, a non-commutative filtered deformation of the coordinate algebra $\Bbbk[e + \mathfrak{g}_f]$ on the Slodowy slice $e + \mathfrak{g}_f$ regarded with its Slodowy grading. Recall that $U(\mathfrak{g}, e) = \big(\mathrm{End}_{\mathfrak{g}}\, Q_e\big)^{\mathrm{op}}$ where Q_e stands for a generalised Gelfand–Graev $\mathfrak{g}$-module associated with e; see [Pre02, GG02] for more detail. By a result of Skryabin, proved in the appendix to [Pre02], the right $U(\mathfrak{g}, e)$-module Q_e is free and for any irreducible $U(\mathfrak{g}, e)$-module V the $\mathfrak{g}$-module $Q_e \otimes_{U(\mathfrak{g},e)} V$ is irreducible. As a consequence, the annihilator $I_V := \mathrm{Ann}_{U(\mathfrak{g})}\big(Q_e \otimes_{U(\mathfrak{g},e)} V\big)$ is a primitive ideal of $U(\mathfrak{g})$.

Let $\mathcal{O}$ be the adjoint G-orbit of e and define $\mathcal{X}_{\mathcal{O}} := \{I \in \mathcal{X} \mid \mathrm{VA}(I) = \overline{\mathcal{O}}\}$. By [Pre07], $I_V \in \mathcal{X}_{\mathcal{O}}$ for any finite-dimensional irreducible $U(\mathfrak{g}, e)$-module V, whilst [Gin09, Los10a, Pre10] show that any primitive ideal $I \in \mathcal{X}_{\mathcal{O}}$ has the form I_W for some finite-dimensional irreducible $U(\mathfrak{g}, e)$-module W. As explained in [Pre10], there is a natural action of the component group $\Gamma = G_e/G_e^\circ$ on the set $\mathrm{Irr}\, U(\mathfrak{g}, e)$ of all isoclasses of finite-dimensional irreducible $U(\mathfrak{g}, e)$-modules. It is straightforward to check that the primitive ideal I_W depends only on the isoclass of W, and so one can speak of a primitive ideal $I_{[W]}$ where $[W]$ is the isoclass of W in $\mathrm{Irr}\, U(\mathfrak{g}, e)$; see [Pre10, Corollary 4.1], for instance. In [Los11b] Losev showed that

$$\mathrm{mult}_{\mathcal{O}}(U(\mathfrak{g})/I_W) = [\Gamma : \Gamma_W] \cdot (\dim W)^2 \tag{1}$$

where Γ_W denotes the stabiliser of the isoclass $[W]$ in Γ. Furthermore, confirming a conjecture of the first-named author, he proved that the equality $I_{[W]} = I_{[W']}$ holds for $[W], [W'] \in \mathrm{Irr}\, U(\mathfrak{g}, e)$ if and only if $[W'] = {}^{\gamma}[W]$ for some $\gamma \in \Gamma$. In particular, this means that $\dim W$ is an intrinsic invariant of the primitive ideal $I = I_W \in \mathcal{X}_{\mathcal{O}}$.

By Goldie's theory, for any $I \in \mathcal{X}$ the prime Noetherian ring $U(\mathfrak{g})/I$ embeds into a full ring of fractions. The latter ring is prime Artinian and hence isomorphic to the matrix algebra $\mathrm{Mat}_n(\mathcal{D}_I)$ over a skew-field $\mathcal{D}_I$ called the *Goldie field* of $U(\mathfrak{g})/I$. The positive integer $n = n_I$ coincides with the Goldie rank of $U(\mathfrak{g})/I$ which is often abbreviated as $\mathrm{rk}(U(\mathfrak{g})/I)$.

Recall that a primitive ideal I is called *completely prime* if $U(\mathfrak{g})/I$ is a domain. It is well known that this happens if and only if $\mathrm{rk}(U(\mathfrak{g})/I) = 1$. Classifying the completely prime primitive ideals of $U(\mathfrak{g})$ is an long-standing classical (and much studied) problem of Lie theory. In general, it remains open outside type A although many important partial results can be found in [BJ01, Bor76, Bry03, BV85, Jos76, Jos85b, Los10b, Los11a, McG94, Mœg87, Mœg88] and references therein. If $I = I_V \in \mathcal{X}_{\mathcal{O}}$, where $[V] \in \mathrm{Irr}\, U(\mathfrak{g}, e)$, then the main result of [Pre11] states that the number

$$q_I := \frac{\dim V}{\mathrm{rk}(U(\mathfrak{g})/I)}$$

is an integer, and it is also proved in the same paper that $q_I = 1$ if the Goldie field $\mathcal{D}_I$ is isomorphic to the skew-field of fractions of a Weyl algebra. The integrality of q_I implies that I_V is completely prime whenever $\dim V = 1$ (this fact also follows from results of Mœglin [Mœg88] and Losev [Los10a]).

Obviously, $I = I_V$ is completely prime if and only if $q_I = \dim V$. If $\Gamma = \{1\}$ then, combining (1) with Joseph's results on Goldie-rank polynomials [Jos80] (as exposed in [Jan83, 12.7]), it is straightforward to see that the scale factor q_I takes the same value on coherent families of primitive ideals in $\mathcal{X}_{\mathcal{O}}$; see [Los12, 5.3] for more detail. It seems likely that this holds without any assumption on Γ and the entire set $\{q_I : \ I \in \mathcal{X}\}$ is finite. (By a coherent family of primitive ideals we mean any subset $\{I(w\cdot\mu) \colon \ \mu \in \Lambda^+\}$ of $\mathcal{X}_{\mathcal{O}}$ with μ and w satisfying the assumptions of [Jan83, 12.7].) We mention for completeness that outside type A there are examples of completely prime primitive ideals $I \in \mathcal{X}_{\mathcal{O}}$ for which $q_I > |\Gamma|$ (see [Pre11, Remark 4.3]), but it is proved in [Los12] for $\mathfrak{g}$ classical (and conjectured for $\mathfrak{g}$ exceptional) that $q_I = 1$ whenever the central character of I is integral.

1.3 Commutative quotients of finite W-algebras and sheets

In this paper we begin a systematic investigation of those $I \in \mathcal{X}_{\mathcal{O}}$ for which $\mathrm{mult}_{\mathcal{O}}(U(\mathfrak{g})/I) = 1$; we call such primitive ideals *multiplicity-free*. For $\mathfrak{g}$ classical we impose no assumptions on e, but for $\mathfrak{g}$ exceptional we shall assume that the orbit $\mathcal{O}$ is induced in the sense of Lusztig and Spaltenstein from a nilpotent orbit in a proper Levi subalgebra of $\mathfrak{g}$. The remaining case of rigid (i.e. non-induced) orbits in exceptional Lie algebras is dealt with in [Pre14]. As we explained earlier, any multiplicity-free primitive ideal is completely prime, but the converse may not always be true outside type A.

Let $\mathcal{S}_1, \dots, \mathcal{S}_t$ be all sheets of $\mathfrak{g}$ containing $\mathcal{O}$. For $1 \leqslant i \leqslant t$, set $r_i = \dim \mathcal{S}_i - \dim \mathcal{O}$, the rank of $\mathcal{S}_i$, and define

$$r(e) := \max_{1 \leqslant i \leqslant t} r_i.$$

Let $\mathfrak{c}_e = \mathfrak{g}_e/[\mathfrak{g}_e, \mathfrak{g}_e]$. Since any one-dimensional torus of G_e and any unipotent element $u = \exp(\operatorname{ad} n)$ with $n \in \mathfrak{g}_e$ act trivially on $\mathfrak{c}_e$, it is straightforward to see that the adjoint action of G_e on $\mathfrak{g}_e$ induces the trivial action of the connected group G_e° on $\mathfrak{c}_e$ and hence gives rise to a natural action of Γ. We denote by $\mathfrak{c}_e^\Gamma$ the corresponding fixed point space, i.e. the set of all $x \in \mathfrak{c}_e$ such that $\gamma(x) = x$ for all $\gamma \in \Gamma$. We define

$$c(e) := \dim(\mathfrak{c}_e), \quad c_\Gamma(e) := \dim(\mathfrak{c}_e^\Gamma).$$

Let $U(\mathfrak{g}, e)^{\mathrm{ab}} = U(\mathfrak{g}, e)/I_c$ where I_c is the two-sided ideal of $U(\mathfrak{g}, e)$ generated by all commutators $u \cdot v - v \cdot u$ with $u, v \in U(\mathfrak{g}, e)$. Our assumption on $\mathcal{O}$ in conjunction with [Bry03, GRU10, Los10a] guarantees that I_c is a proper ideal of $U(\mathfrak{g}, e)$; see [Pre10] for more detail. We denote by $\mathcal{E}$ the maximal spectrum of the finitely generated commutative $\Bbbk$-algebra $U(\mathfrak{g}, e)^{\mathrm{ab}}$. This affine variety parameterises the one-dimensional representations of $U(\mathfrak{g}, e)$ and is acted upon by the component group Γ (it is known that Γ acts on $U(\mathfrak{g}, e)^{\mathrm{ab}}$ by algebra automorphisms). We denote by $\mathcal{E}^\Gamma$ the corresponding fixed point set which consists of all $\eta \in \mathcal{E}$ such that $\gamma(\eta) = \eta$ for all $\gamma \in \Gamma$. Let I_Γ be the ideal of $U(\mathfrak{g}, e)^{\mathrm{ab}}$ generated by all $\phi - \phi^\gamma$ with $\phi \in U(\mathfrak{g}, e)^{\mathrm{ab}}$ and $\gamma \in \Gamma$. It is straightforward to see that $\mathcal{E}^\Gamma$ coincides with the zero locus of I_Γ in $\mathcal{E}$. We define $U(\mathfrak{g}, e)^{\mathrm{ab}}_\Gamma := U(\mathfrak{g}, e)^{\mathrm{ab}}/I_\Gamma$.

It follows from (1) that $I = I_V$ is multiplicity-free if and only if $\dim V = 1$ and $\Gamma_V = \Gamma$. Thus, in order to classify the multiplicity-free primitive ideals in $\mathcal{X}_{\mathcal{O}}$ we need to determine the variety $\mathcal{E}^\Gamma$. This problem is important as solving it could eventually lead us to a complete description of *all* quantisations of nilpotent orbits; see [Mœg88] and [Los10b, Theorem 1.1] for precise statements.

Thanks to [Pre10, Theorem 1.2] we know that $\dim \mathcal{E} = r(e)$ and the number of irreducible components of $\mathcal{E}$ is greater than or equal to t. Thus, the variety $\mathcal{E}$ is irreducible only if e lies in a unique sheet of $\mathfrak{g}$. For $\mathfrak{g} = \mathfrak{sl}_n$, this condition is satisfied for any nilpotent element e, and [Pre10, Corollary 3.2] states that $U(\mathfrak{sl}_n, e)^{\mathrm{ab}}$ is a polynomial algebra in $r(e)$ variables. Our first main result is a generalisation of that to all Lie algebras of classical types. We call an element $a \in \mathfrak{g}$ *non-singular* if it lies in a unique sheet of $\mathfrak{g}$. If $\dim \mathfrak{g}_a = m$ and $\mathfrak{g}^{(m)} = \{x \in \mathfrak{g}\colon\ \dim \mathfrak{g}_x = m\}$, a locally closed subset of $\mathfrak{g}$, then it follows from the smoothness of sheets of classical Lie algebras (proved by Im Hof in [ImH05]) that a is non-singular if and only if a is a smooth point of the quasi-affine variety $\mathfrak{g}^{(m)}$ (hence the name).

Theorem 1. *If e is a nilpotent element in a classical Lie algebra $\mathfrak{g}$, then the following are equivalent:*

(i) *e is non-singular;*
(ii) *$c(e) = r(e)$;*
(iii) *$U(\mathfrak{g}, e)^{\mathrm{ab}}$ is isomorphic to a polynomial algebra in $r(e)$ variables.*

The equivalence of the first two statements of Theorem 1 was conjectured by Izosimov [Izo12] for all elements in a classical Lie algebra $\mathfrak{g}$. In Remark 5, we use the Jordan–Chevalley decomposition in $\mathfrak{g}$ to show that his conjecture is an immediate consequence of Theorem 1.

Although the polynomiality of $U(\mathfrak{g}, e)^{\mathrm{ab}}$ occurs rather infrequently outside type A, the algebras $U(\mathfrak{g}, e)^{\mathrm{ab}}_{\Gamma}$ exhibit a much more uniform behaviour.

Theorem 2. *If e is any nilpotent element in a classical Lie algebra $\mathfrak{g}$, then $U(\mathfrak{g}, e)^{\mathrm{ab}}_{\Gamma}$ is isomorphic to a polynomial algebra in $c_\Gamma(e)$ variables. In particular, $\mathcal{E}^\Gamma$ is a single point if and only if $c_\Gamma(e) = 0$.*

As an obvious corollary of Theorem 2 we deduce that the variety $\mathcal{E}^\Gamma$ is isomorphic to an affine space for any nilpotent element in a classical Lie algebra and hence is irreducible.

1.4 Derived subalgebras of centralisers in classical Lie algebras

In order to prove Theorems 1 and 2 we have to look very closely at the centralisers of nilpotent elements in classical Lie algebras. A link between completely prime primitive ideals and centralisers of nilpotent elements originates in the fact that for any nilpotent element $e \in \mathfrak{g}$ the finite W-algebra $U(\mathfrak{g}, e)$ is a filtered deformation of the universal enveloping algebra $U(\mathfrak{g}_e)$; see [BGK08, Pre07].

Suppose that $\mathfrak{g}$ is one of $\mathfrak{so}_N$ or $\mathfrak{sp}_N$. It is well known that to any nilpotent element $e \in \mathfrak{g}$ one can attach a partition $\lambda \in \mathcal{P}_\epsilon(N)$ where $\epsilon = 1$ if $\mathfrak{g} = \mathfrak{so}_N$ and $\epsilon = -1$ if $\mathfrak{g} = \mathfrak{sp}_N$. Recall that a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ of N with $\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 1$ is in $\mathcal{P}_\epsilon(N)$ if there is an involution $i \mapsto i'$ on the set of indices $\{1, \dots, n\}$ satisfying $i' \in \{i-1, i, i+1\}$ such that $\lambda_{i'} = \lambda_i$ and $i' = i$ if and only if $\epsilon(-1)^{\lambda_i} = -1$ for all i. We call a pair of indices $(i, i+1)$ with $1 \leqslant i < n$ a 2-*step* of λ if $i' = i$, $(i+1)' = i+1$ and $\lambda_{i-1} \neq \lambda_i \geqslant \lambda_{i+1} \neq \lambda_{i+2}$ where our convention is that $\lambda_i = 0$ for $i \in \{0, n+1\}$. We denote by $\Delta(\lambda)$ the set of all 2-steps of λ and set

$$s(\lambda) := \sum_{i=1}^{n} \lfloor (\lambda_i - \lambda_{i+1})/2 \rfloor.$$

We call λ *exceptional* if $\mathfrak{g}$ has type D and there exists a $k < n$ such that the parts λ_k, λ_{k+1} are odd and the parts λ_i with $i \notin \{k, k+1\}$ are all even.

It should be mentioned that for any $(i, i+1) \in \Delta(\lambda)$ the integers λ_i and λ_{i+1} have the same parity. If $(i, i+1) \in \Delta(\lambda)$ and $i > 1$ (respectively, $i = 1$), then we call λ_{i-1} and λ_{i+2} (respectively, λ_3) the *boundary* of $(i, i+1)$. We say that a 2-step $(i, i+1)$ is *good* if its boundary and λ_i have the opposite parity.

Theorem 3. *Let $\mathfrak{g}$ be one of $\mathfrak{so}_N$ or $\mathfrak{sp}_N$, where $N \geqslant 2$, and let e be a nilpotent element of $\mathfrak{g}$ associated with a partition $\lambda \in \mathcal{P}_\epsilon(N)$. Then the following hold:*

(i) $c(e) = s(\lambda) + |\Delta(\lambda)|$;

(ii) $c_\Gamma(e) = s(\lambda)$ *unless* $\mathfrak{g} = \mathfrak{so}_N$ *and* $\lambda \in \mathcal{P}_1(N)$ *is exceptional, in which case* $c_\Gamma(e) = s(\lambda) + 1$;

(iii) *e is non-singular if and only if all 2-steps of λ are good.*

If $\lambda \in \mathcal{P}_1(N)$ is exceptional, then it is immediate from the definitions that $|\Delta(\lambda)| = 1$ and the only 2-step of λ is good. Therefore, any nilpotent element $e \in \mathfrak{g}$ associated with λ is non-singular. It is also straightforward to see that any such e is a Richardson element of $\mathfrak{g}$.

For a nilpotent element e associated with a partition $\lambda \in \mathcal{P}_\epsilon(N)$, we give an explicit combinatorial formula for the number $r(e)$; see Corollary 9. It involves the notion of a *good* 2-*cluster* of λ introduced in § 3.3.

1.5 The case of exceptional Lie algebras

Now suppose that $\mathfrak{g}$ is an exceptional Lie algebra. In this case our results are less complete because we have to exclude the seven induced orbits in Table 0. Using [Car85, pp. 440–445] one observes that all orbits listed in Table 0 are non-special. (All tables have been gathered at the end of the article, beginning on page 1542.)

Theorem 4. *Let $\mathfrak{g}$ be an exceptional Lie algebra and suppose that e is an induced nilpotent element of $\mathfrak{g}$. Then the following hold:*

(i) $\mathcal{E}^\Gamma \neq \emptyset$;

(ii) *if e is not listed in the first six columns of Table 0 and lies in a single sheet of $\mathfrak{g}$, then $U(\mathfrak{g}, e)^{\mathrm{ab}}$ is isomorphic to a polynomial algebra in $c(e)$ variables;*

(iii) *if e is not listed in Table 0, then $U(\mathfrak{g}, e)^{\mathrm{ab}}_\Gamma$ is isomorphic to a polynomial algebra in $c_\Gamma(e)$ variables.*

The numbers $c(e)$ and $c_\Gamma(e)$ are listed in the last two columns of Tables 1–6.

Curiously, there are instances where for an induced element e the variety $\mathcal{E}^\Gamma$ is a single point. For $\mathfrak{g}$ exceptional there are four such cases (two in type E_7 and two in type E_8) and for $\mathfrak{g}$ classical this occurs when e is associated with a partition $\lambda \in \mathcal{P}_\epsilon(N)$ for which $\lambda_i - \lambda_{i+1} \in \{0, 1\}$ for all i (we call such partitions *almost rigid*). The nilpotent elements from the four orbits in types E_7 and E_8 have already appeared in the literature under three different names: *p-compact*, *compact* and *reachable*; see [BB92, EG93, deG13, Pan04, Yak10]. It is worth mentioning that almost rigid and exceptional partitions in $\mathcal{P}_\epsilon(N)$ also played a special role in Namikawa's work [Nam09] on $\mathbb{Q}$-factorial terminalisations of nilpotent orbit closures in classical Lie algebras.

In proving Theorem 4 we rely heavily on results of de Graaf [deG13] and Lawther and Testerman [LT07] obtained by computational methods. It seems plausible that the algebra $U(\mathfrak{g}, e)^{\mathrm{ab}}_\Gamma$ is reduced and the variety $\mathcal{E}^\Gamma$ is equidimensional in all cases, but to prove this for the orbits listed in Table 0 one would have to use different methods (a computational approach in the spirit of [GRU10] would certainly do the trick).

1.6 Multiplicity-free primitive ideals associated with induced orbits

The traditional way to classify the completely prime ideals $I \in \mathcal{X}_\mathcal{O}$ parallels Borho's classification of the sheets of $\mathfrak{g}$; see [Bor81]. Here one aims to show that if the orbit $\mathcal{O}$ is induced from a rigid orbit $\mathcal{O}_0$ in a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$, then the majority of I as above can be obtained as the annihilators in $U(\mathfrak{g})$ of (not necessarily irreducible) induced $\mathfrak{g}$-modules

$$\mathrm{Ind}_\mathfrak{p}^\mathfrak{g}(E) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} E,$$

where $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ is a parabolic subalgebra of $\mathfrak{g}$ with nilradical $\mathfrak{n}$ and E is an irreducible $\mathfrak{p}$-module with $\mathfrak{n} \cdot E = 0$ such that the annihilator $I_0 := \mathrm{Ann}_{U(\mathfrak{l})} E$ is a completely prime primitive ideal of $U(\mathfrak{l})$ with $\mathrm{VA}(I_0) = \overline{\mathcal{O}}_0$. The ideals

$$I(\mathfrak{p}, E) := \mathrm{Ann}_{U(\mathfrak{g})}(\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(E))$$

are referred to as *induced*. It should be mentioned that $I(\mathfrak{p}, E)$ does not have to be primitive and completely prime, in general, but this holds under the additional assumption that I_0 is completely prime thanks to Conze's theorem [Con74] and the Dixmier–Mœglin equivalence [Dix74, 8.5.7]. It is well known that $I(\mathfrak{p}, E)$ coincides with the largest two-sided ideal of $U(\mathfrak{g})$ contained in the left ideal $U(\mathfrak{g})(\mathfrak{n} + I_0)$ and hence depends only on $\mathfrak{p}$ and I_0; see [BGR73, 10.4]. We shall sometimes use a more flexible notation $\mathfrak{I}_{\mathfrak{p}}^{\mathfrak{g}}(I_0)$ when referring to $I(\mathfrak{p}, E)$.

Motivated by the natural desire to keep things simple, one wants *all* completely prime primitive ideals in $\mathcal{X}_{\mathcal{O}}$ to be induced, but since this fails outside type A one must find a way to determine the non-induced ones. This is, of course, the hardest part of the problem and the main reason why the classification remains open outside type A; see [BJ01] for more detail.

Fortunately, this issue does not arise for the multiplicity-free primitive ideals. The following is the main result of this paper.

Theorem 5. *Let $I \in \mathcal{X}_{\mathcal{O}}$ be a multiplicity-free primitive ideal associated with an induced nilpotent orbit $\mathcal{O} \subset \mathfrak{g}$. If $\mathfrak{g}$ is exceptional, assume further that $\mathcal{O}$ is not listed in Table 0. Then there exists a proper parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ with a Levi subalgebra $\mathfrak{l}$ and a rigid nilpotent orbit $\mathcal{O}_0$ in $\mathfrak{l}$ such that $\mathcal{O}$ is induced from $\mathcal{O}_0$ and $I = I(\mathfrak{p}, E)$, where E is an irreducible $U(\mathfrak{p})$-module with the trivial action of the nilradical of $\mathfrak{p}$. Moreover, the primitive ideal $I_0 = \mathrm{Ann}_{U(\mathfrak{l})} E$ is completely prime and $\mathrm{VA}(I_0) = \overline{\mathcal{O}}_0$.*

Theorem 5 can be regarded as a generalisation of Mœglin's theorem [Mœg87] on completely prime primitive ideals of $U(\mathfrak{sl}_n)$. From the main body of the paper one can obtain more information on the parabolic subalgebra $\mathfrak{p}$ and the $\mathfrak{p}$-module E. It is quite possible that Theorem 5 holds for all induced orbits in $\mathfrak{g}$ and this would follow (by the same argument) if the variety $\mathcal{E}^{\Gamma}$ turned out to be irreducible for all orbits listed in Table 0.

2. The derived subalgebra of a centraliser

2.1 A basis for centralisers in classical Lie algebras

Let $\Bbbk$ be an algebraically closed field of any characteristic except 2. Fix $N \geqslant 2$ and denote by V an N-dimensional vector space over $\Bbbk$. In this section we denote by G the algebraic group $GL(V)$ with Lie algebra $\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{gl}(V)$ and let $\Psi = (\,\cdot\,,\,\cdot\,)$ be a symmetric or skew-symmetric non-degenerate bilinear form on V with values in $\Bbbk$, so that $(u, v) = \epsilon(v, u)$ for all $u, v \in V$ where $\epsilon = \pm 1$. Choose a basis for V to identify $\mathfrak{gl}(V)$ with $\mathfrak{gl}_N$ and let J be the matrix associated with Ψ with respect to that basis. If X is an endomorphism of V then $X^{\top}$ denotes the transpose of X. There is a Lie algebra automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ of order 2 taking $X \in \mathfrak{g}$ to $-J^{-1}X^{\top}J$ which is independent of our choice of basis. Then σ induces a $\mathbb{Z}_2$-grading $\mathfrak{g} = \mathfrak{g}_0 \bigoplus \mathfrak{g}_1$. Make the notation $\mathfrak{k} = \mathfrak{g}_0$. If $\epsilon = 1$ then $\mathfrak{k}$ is an orthogonal algebra, and if $\epsilon = -1$ then $\mathfrak{k}$ is a symplectic algebra. In either case $\mathfrak{g}_1$ is a $\mathfrak{k}$-module. Let K denote the connected component of the associated orthogonal or symplectic group.

The conjugacy classes of nilpotent elements in $\mathfrak{g}$ are in one-to-one correspondence with ordered partitions of N: we associate with a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of N with $\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 1$ the G-orbit of the nilpotent element in Jordan normal form with Jordan block sizes $\lambda_1, \ldots, \lambda_n$.

Let $e \in \mathfrak{k}$ be a nilpotent element with Jordan block sizes $\lambda_1 \geqslant \cdots \geqslant \lambda_n$. Since $\mathfrak{k}$ acts naturally on V we may decompose V uniquely into minimal e-stable subspaces $V = \bigoplus_{i=1}^n V[i]$, and shall call these $V[i]$ the Jordan block spaces of e in V. Since e restricts to a regular nilpotent endomorphism on each $V[i]$, there exist vectors $\{w_i\}$ such that $\{e^s w_i : 1 \leqslant i \leqslant n, 0 \leqslant s < \lambda_i\}$ forms a basis for V. When dealing with partitions λ as above we always assume that $\lambda_0 = 0$ and $\lambda_i = 0$ for all $i > n$.

The following condition on the Jordan block sizes can be found in [Jan04, Theorem 1.4], for example. The final statement follows from [CM93, Theorem 5.1.6].

LEMMA 1. *The $w_i \in V$ can be chosen so that there exists an involution $i \mapsto i'$ on the set $\{1, \ldots, n\}$ such that:*

(1) $\lambda_i = \lambda_{i'}$ *for all* $i = 1, \ldots, n$;
(2) $(V[i], V[j]) = 0$ *if* $i \neq j'$;
(3) $i = i'$ *if and only if* $\epsilon(-1)^{\lambda_i} = -1$.

The lemma states that for a nilpotent element in a symplectic Lie algebra each Jordan block of odd dimension can be paired with a different Jordan block of the same dimension; in an orthogonal algebra each Jordan block of even dimension can be paired with a different Jordan block of the same dimension; and that this pairing is involutory. Renumbering the vectors w_i if necessary, we may (and will) assume from now on that

$$i' \in \{i-1, i, i+1\} \quad \text{for all } 1 \leqslant i \leqslant n.$$

As an immediate consequence of this convention we have that $j' > i'$ whenever $1 \leqslant i < j \leqslant n$ and $j \neq i'$. Following [CM93], we denote by $\mathcal{P}_\epsilon(N)$ the set of partitions of N which are associated with nilpotent elements of $\mathfrak{k}$ (i.e. fulfilling the parity conditions of Lemma 1).

If $\mathcal{L}$ is a Lie algebra and $x \in \mathcal{L}$ then we write $\mathcal{L}_x$ for the centraliser of x in $\mathcal{L}$. Since $\sigma(e) = e$, the centraliser of e in $\mathfrak{g}$ is σ-stable, inducing a decomposition $\mathfrak{g}_e = \mathfrak{k}_e \oplus (\mathfrak{g}_e)_1$ where $(\mathfrak{g}_e)_1 = (\mathfrak{g}_1)_e$ is a $\mathfrak{k}_e$-module. Thanks to [Jan04, Theorems 2.5 and 2.6] we may identify $\mathfrak{k}_e$ with $\mathrm{Lie}(K_e)$. We shall normalise the basis for V. Let $\{w_i\}$ be chosen in accordance with the above and fix $1 \leqslant i \leqslant n$, $0 < s$. We have $(e^{\lambda_i - 1} w_i, e^s w_{i'}) = (-1)^s (e^{\lambda_i - 1 + s} w_i, w_{i'})$ and $e^{\lambda_i - 1 + s} w_i = 0$ so $e^{\lambda_i - 1} w_i$ is orthogonal to all $e^s w_{i'}$ with $s > 0$. There is a (unique up to scalar) vector $v \in V[i]$ which is orthogonal to all $e^s w_{i'}$ for $s < \lambda_i - 1$. This v does not lie in $\mathrm{Im}(e)$ for otherwise it would be orthogonal to all of $V[i] + V[i']$. This is not possible since the restriction of Ψ to $V[i] + V[i']$ is non-degenerate. It does no harm to replace w_i by v and normalise according to the rule

$$(w_i, e^{\lambda_i - 1} w_{i'}) = 1 \quad \text{whenever } i \leqslant i'.$$

With respect to this basis the matrix of the restriction of Ψ to $V[i] + V[i']$ is antidiagonal with entries ± 1.

Let $\xi \in \mathfrak{g}_e$. Then $\xi(e^s w_i) = e^s(\xi w_i)$, showing that ξ is determined by its action on the w_i. If we define

$$\xi_i^{j,s} w_k = \begin{cases} e^s w_j & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}$$

and extend the action to $\{e^s w_i\}$ by the requirement that $\xi_i^{j,s}$ is linear and centralises e then

$$\{\xi_i^{j,\lambda_j - 1 - s} : 1 \leqslant i, j \leqslant n, 0 \leqslant s < \min(\lambda_i, \lambda_j)\} \tag{2}$$

forms a basis for $\mathfrak{g}_e$; see [Yak06], for example. Our next aim is to describe a basis for $\mathfrak{k}_e$. The following approach is implicit in [Yak06], but we shall recover the details for the reader's

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convenience. Since $\sigma : \mathfrak{g}_e \to \mathfrak{g}_e$ is an involution the maps $\xi + \sigma(\xi)$, with $\xi \in \mathfrak{g}_e$, span $\mathfrak{k}_e$. Thanks to (2) we may define $\zeta_i^{j,s} = \xi_i^{j,\lambda_j-1-s} + \sigma(\xi_i^{j,\lambda_j-1-s})$ and conclude that $\{\zeta_i^{j,s} : 1 \leqslant i,j \leqslant n, 0 \leqslant s < \min(\lambda_i,\lambda_j)\}$ is the required spanning set for $\mathfrak{k}_e$. This leaves us with two immediate tasks: evaluate $\sigma(\xi_i^{j,\lambda_j-1-s})$ and determine the linear relations between the $\zeta_i^{j,s}$. Using the fact that $\zeta_i^{j,s}$ is skew self-adjoint with respect to Ψ, we deduce that

$$\sigma(\xi_i^{j,\lambda_j-1-s}) = \varepsilon_{i,j,s}\xi_{j'}^{i',\lambda_i-1-s} \tag{3}$$

where $\varepsilon_{i,j,s}$ is defined by the relationship $(e^{\lambda_j-1-s}w_j, e^s w_{j'}) = -\varepsilon_{i,j,s}(w_i, e^{\lambda_i-1}w_{i'})$. This requires a little calculation. We now have the notation

$$\zeta_i^{j,s} = \xi_i^{j,\lambda_j-1-s} + \varepsilon_{i,j,s}\xi_{j'}^{i',\lambda_i-1-s}.$$

We further write

$$\varpi_{i\leqslant j} = \begin{cases} 1 & \text{if } i \leqslant j \\ -1 & \text{if } i > j \end{cases}$$

and, comparing with Lemma 1, we see that $\varpi_{i\leqslant i'}\varpi_{i'\leqslant i} = \epsilon(-1)^{\lambda_i-1}$, which will prove useful in some later calculations. The next lemma settles the question of which linear relations exist between the maps $\zeta_i^{j,s}$. The proof may be found in [Top14].

Lemma 2. *The following are true:*

(1) $\varepsilon_{i,j,s} = (-1)^{\lambda_j-s}\varpi_{i\leqslant i'}\varpi_{j\leqslant j'}$;

(2) $\varepsilon_{i,j,s} = \varepsilon_{j',i',s}$;

(3) *the only linear relations amongst the* $\zeta_i^{j,s}$ *are those of the form* $\zeta_i^{j,s} = \varepsilon_{i,j,s}\zeta_{j'}^{i',s}$.

Thanks to the above lemma we may refine a basis from the spanning set of vectors $\{\zeta_i^{j,s}\}$ by removing any zero elements and excluding precisely one of the pair $(\zeta_i^{j,s}, \zeta_{j'}^{i',s})$ when these vectors are non-zero. With this in mind, define

$$\begin{aligned} H &:= \{\zeta_i^{i,s} : i < i', 0 \leqslant s < \lambda_i\} \cup \{\zeta_i^{i,s} : i = i', 0 \leqslant s < \lambda_i, \lambda_i - s \text{ even}\}, \\ N_0 &:= \{\zeta_i^{i',s} : i \neq i', 0 \leqslant s < \lambda_i, \lambda_i - s \text{ odd}\}, \\ N_1 &:= \{\zeta_i^{j,s} : i < j \neq i', 0 \leqslant s < \lambda_j\}. \end{aligned}$$

Define also

$$\begin{aligned} \mathfrak{H} &:= \operatorname{span}(H), \\ \mathfrak{N}_0 &:= \operatorname{span}(N_0), \\ \mathfrak{N}_1 &:= \operatorname{span}(N_1). \end{aligned}$$

If U_0 and U_1 are subspaces of V then $\operatorname{End}(U_0, U_1)$ shall denote the space of all linear maps $U_0 \to U_1$. We consider $\operatorname{End}(U_0, U_1)$ to be a subspace of $\operatorname{End}(V)$ under the natural embedding induced by the inclusions of U_0 and U_1 into V.

Lemma 3. *The set* $H \sqcup N_0 \sqcup N_1$ *forms a basis for* $\mathfrak{k}_e$. *Furthermore, we have the following characterisation of the three spaces:*

(1) $\mathfrak{H}$ *is precisely the subspace of* $\mathfrak{k}_e$ *which preserves each Jordan block space* $V[i]$,

$$\mathfrak{H} = \mathfrak{k}_e \cap \Big(\bigoplus_i \operatorname{End}(V[i])\Big);$$

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(2) $\mathfrak{N}_0$ *is precisely the subspace of* $\mathfrak{k}_e$ *which 'interchanges'* $V[i]$ *and* $V[i']$ *for* $i \neq i'$ *and annihilates* $V[i]$ *for* $i = i'$,

$$\mathfrak{N}_0 = \mathfrak{k}_e \cap \Big(\bigoplus_{i\neq i'} \mathrm{End}(V[i], V[i'])\Big);$$

(3) $\mathfrak{N}_1$ *is the subspace of* $\mathfrak{k}_e$ *which does neither of the above,*

$$\mathfrak{N}_1 = \mathfrak{k}_e \cap \Big(\bigoplus_{i}\Big(\bigoplus_{j\notin\{i,i'\}} \mathrm{End}(V[i], V[j])\Big)\Big).$$

Proof. First we show that all elements of $H \sqcup N_0 \sqcup N_1$ are non-zero. Clearly $\zeta_i^{j,s} = 0$ if and only if $\xi_i^{j,\lambda_j-1-s} = -\varepsilon_{i,j,s}\xi_{j'}^{i',\lambda_i-1-s}$. For this we require that $i = j'$ and $\varepsilon_{i,j,s} = -1$. For $i = j'$ we must have $i = i' = j$ or $i \neq i' = j$. In the first case, $\varepsilon_{i,j,s} = (-1)^{\lambda_j - s}$ which equals -1 only if $\lambda_i - s$ is odd. But the maps $\zeta_i^{i,s}$ do not occur in H when $i = i'$ and $\lambda_i - s$ is odd. In the second case, $\varepsilon_{i,j,s} = (-1)^{\lambda_i-1-s}$ which equals -1 only if $\lambda_i - s$ is even. However, the maps $\zeta_i^{i',s}$ do not occur in N_0 when $i \neq i'$ and $\lambda_i - s$ is even.

Next observe that when $\zeta_i^{j,s} \neq 0$ exactly one of the two maps $\zeta_i^{j,s}$ and $\zeta_{j'}^{i',s}$ occurs in $H \sqcup N_0 \sqcup N_1$, thus showing this set to be a basis by part (3) of Lemma 2. The three characterisations are clear upon inspection of the definitions of the sets H, N_0 and N_1. □

2.2 Decomposing $\mathfrak{k}_e$

It is our intention to decompose $[\mathfrak{k}_e, \mathfrak{k}_e]$ into subspaces. In order to do so we must first decompose $\mathfrak{H}$ and $\mathfrak{N}_1$. Let

$$\mathfrak{H}_0 := \mathrm{span}\{\zeta_i^{i,s} \in \mathfrak{H} : \lambda_i - s \text{ even}\},$$
$$\mathfrak{H}_1 := \mathrm{span}\{\zeta_i^{i,s} \in \mathfrak{H} : \lambda_i - s \text{ odd}\},$$

so that $\mathfrak{H} = \mathfrak{H}_0 \bigoplus \mathfrak{H}_1$. The space $\mathfrak{H}_0$ can be further decomposed as $\bigoplus_{m=1}^{\lfloor \lambda_1/2\rfloor} \mathfrak{H}_0^m$ where

$$\mathfrak{H}_0^m := \mathrm{span}\{\zeta_i^{i,\lambda_i-2m} \in \mathfrak{H} : 1 \leqslant i \leqslant n\}.$$

Next we must decompose each $\mathfrak{H}_0^m$ into subspaces $\mathfrak{H}_{0,j}^m$ for $j \geqslant 1$.

Fix $0 < m \leqslant \lfloor \lambda_1/2 \rfloor$, put $a_{1,m} := 1$ and let $1 = a_{1,m} < a_{2,m} < \cdots < a_{t(m),m} \leqslant n+1$ be the set of all integers such that

$$\lambda_{a_{j,m}-1} - \lambda_{a_{j,m}} \geqslant 2m, \quad 2 \leqslant j \leqslant t(m).$$

For $1 \leqslant j < t(m)$ we define

$$\mathfrak{H}_{0,j}^m := \mathrm{span}\{\zeta_i^{i,\lambda_i-2m} \in \mathfrak{H} : a_{j,m} \leqslant i < a_{j+1,m}\}$$

and set

$$\mathfrak{H}_{0,t(m)}^m := \mathrm{span}\{\zeta_i^{i,\lambda_i-2m} \in \mathfrak{H} : a_{t(m),m} \leqslant i < n+1\}.$$

Lemma 4. *The following are true:*

(1) *if* $\lambda_{a_{t(m),m}} < 2m$ *then* $\mathfrak{H}_{0,t(m)}^m = \{0\}$;

(2) $\mathfrak{H}_0^m = \bigoplus_{j=1}^{t(m)} \mathfrak{H}_{0,j}^m$.

Proof. If $a_{t(m),m} = n+1$ then certainly $\mathfrak{H}^m_{0,t(m)} = 0$, so assume not. If $\lambda_{a_{t(m),m}} < 2m$ then the ordering $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ implies that $\lambda_i - 2m < 0$ for all $i \geqslant a_{t(m),m}$. Then $\zeta_i^{i,\lambda_i-2m} = 0$ for all $\zeta_i^{i,\lambda_i-2m} \in \mathfrak{H}^m_{0,t(m)}$, proving part (1). The choice of m (and the fact that $a_{1,m} = 1$) ensures that $\bigoplus_{l=1}^{t(m)} \mathfrak{H}^m_{0,j} = \operatorname{span}\{\zeta_i^{i,\lambda_i-2m} \in \mathfrak{H} : 1 \leqslant i \leqslant n\} = \mathfrak{H}^m_0$, hence part (2). □

It should be noted that if $i \neq i'$ then $\varepsilon_{i,i,\lambda_i-2m} = 1$ by Lemma 2. In this case $\zeta_i^{i,\lambda_i-2m} = \zeta_{i'}^{i',\lambda_{i'}-2m}$ by the same lemma. In order to overcome this notational problem and concisely refer to a basis for $\mathfrak{H}^m_{0,j}$ it will be convenient to use an indexing set slightly different from $\{1,\ldots,n\}$. Extend the involution $i \mapsto i'$ to all of $\mathbb{Z}$ by the rule $i = i'$ for $i > n$ or $i < 1$. We adopt the convention $\lambda_i = 0$ for all $i > n$ or $i < 1$, which immediately implies $\zeta_i^{i,s} = 0$ for any such i. We shall index our maps and partitions by the set $\mathbb{Z}/\!\sim$ where $i \sim j$ if $i = j'$. We denote by $[i]$ the class of i in $\mathbb{Z}/\!\sim$. We have $\lambda_i = \lambda_{i'}$ for all i, so we may introduce the notation $\lambda_{[i]}$. As was observed a moment ago, $\zeta_i^{i,\lambda_i-2m} = \zeta_{i'}^{i',\lambda_{i'}-2m}$. Hence we may also use the notation $\zeta_{[i]}^{[i],\lambda_{[i]}-2m}$. Furthermore, since $i' \in \{i-1, i, i+1\}$ we have a well-defined order on $\mathbb{Z}/\!\sim$ inherited from $\mathbb{Z}$: let $[i] \leqslant [j]$ if $i \leqslant j$. As a result there exists a unique isomorphism of totally ordered sets $\psi : (\mathbb{Z}/\!\sim) \to \mathbb{Z}$ with $\psi([1]) = 1$. Using this isomorphism, we define analogues of addition and subtraction $+, -\colon (\mathbb{Z}/\!\sim) \times \mathbb{Z} \to (\mathbb{Z}/\!\sim)$ by the rules

$$[i] + j := \psi^{-1}(\psi(i) + j),$$
$$[i] - j := \psi^{-1}(\psi(i) - j).$$

To clarify, $[i]+1$ is the class in $(\mathbb{Z}/\!\sim)$ succeeding $[i]$ and $[i]-1$ is the class preceding $[i]$ in the ordering.

For $1 \leqslant j < t(m)$, Lemma 2(3) yields that the set

$$\left\{\zeta_{[i]}^{[i],\lambda_{[i]}-2m} \in \mathfrak{H} :\ [a_{j,m}] \leqslant [i] < [a_{j+1,m}]\right\}$$

is a basis for $\mathfrak{H}^m_{0,j}$. Using this basis we may describe an important hyperplane $\mathfrak{H}^{m,+}_{0,j}$ of $\mathfrak{H}^m_{0,j}$. First we define the augmentation map $\mathfrak{H}^m_{0,j} \twoheadrightarrow \Bbbk$ by sending $\zeta_{[i]}^{[i],\lambda_{[i]}-2m}$ to 1 for all $[a_{j,m}] \leqslant [i] < [a_{j+1,m}]$ and extending to $\mathfrak{H}^m_{0,j}$ by $\Bbbk$-linearity. Let $\mathfrak{H}^{m,+}_{0,j}$ denote the kernel of this map. It was noted in Lemma 4 that $\mathfrak{H}^m_{0,t(m)}$ might be zero. If this is not the case then a basis for $\mathfrak{H}^m_{0,t(m)}$ is the span of those $\zeta_{[i]}^{[i],\lambda_{[i]}-2m}$ which are non-zero with $[a_{t(m),m}] \leqslant [i] \leqslant [n]$. Using this basis, we can define the augmentation map $\mathfrak{H}^m_{0,t(m)} \twoheadrightarrow \Bbbk$ and hyperplane $\mathfrak{H}^{m,+}_{0,t(m)}$ of $\mathfrak{H}^m_{0,t(m)}$ in a similar fashion. Make the notation

$$\mathfrak{H}^+_0 := \sum_{m=1}^{\lfloor \lambda_1/2 \rfloor} \left(\bigoplus_{j=1}^{t(m)-1} \mathfrak{H}^{m,+}_{0,j} + \mathfrak{H}^m_{0,t(m)} \right) \subseteq \mathfrak{H}_0.$$

Before we continue we must decompose $\mathfrak{N}_1$ into a direct sum of two subspaces. We shall need the following definition, first stated in the introduction.

Definition 1. Given $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathcal{P}_\epsilon(N)$, we denote by $\Delta(\lambda)$ the set of all pairs $(i, i+1)$ with $1 \leqslant i < n$ such that $i' = i$, $(i+1)' = i+1$ and $\lambda_{i-1} \neq \lambda_i \geqslant \lambda_{i+1} \neq \lambda_{i+2}$. If $(i, i+1) \in \Delta(\lambda)$ then the pair will be called a 2-step of λ. If $i > 1$ and $(i, i+1)$ is a 2-step of λ then λ_{i-1} and λ_{i+2} are referred to as the boundary of $(i, i+1)$. If $(1,2) \in \Delta(\lambda)$ then λ_3 is referred to as the boundary of $(1,2)$ (if $n = 2$ then $\lambda_3 = 0$ by convention).

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Here and throughout we adopt the convention that $\lambda_0 = \lambda_{n+1} = 0$. Take note that if $(n-1, n) \in \Delta(\lambda)$ then λ_{n-2} and $\lambda_{n+1} = 0$ form the boundary of $(n-1,n)$. We define $\mathfrak{N}_1^-$ to be the span of the basis vectors $\zeta_i^{i+1,\lambda_{i+1}-1} \in N_1$ such that $(i,i+1) \in \Delta(\lambda)$ and we let $\mathfrak{N}_1^+$ be the complement to $\mathfrak{N}_1^-$ in $\mathfrak{N}_1$ which is spanned by the remaining basis vectors $\zeta_i^{j,s} \in N_1$.

2.3 Decomposing $[\mathfrak{k}_e, \mathfrak{k}_e]$

It is the intention of this section to decompose $[\mathfrak{k}_e, \mathfrak{k}_e]$ into a finite collection of those subspaces of $\mathfrak{k}_e$ defined in the previous section. Our calculations will be quite explicit and depend principally upon the following.

Lemma 5. *For all indices i, j, s and k, l, r,*

$$[\zeta_i^{j,s}, \zeta_k^{l,r}] = \delta_{il}\zeta_k^{j,r+s-(\lambda_i-1)} - \delta_{jk}\zeta_i^{l,r+s-(\lambda_j-1)} + \varepsilon_{k,l,r}\big(\delta_{k,i'}\zeta_{l'}^{j,r+s-(\lambda_i-1)} - \delta_{j,l'}\zeta_i^{k',r+s-(\lambda_j-1)}\big).$$

The proof is a short calculation which we leave to the reader. The following proposition will be central in the process of decomposing $[\mathfrak{k}_e, \mathfrak{k}_e]$.

Proposition 1. *The following inclusions hold:*

$$\begin{aligned} [\mathfrak{H},\mathfrak{H}] = \{0\}, \quad & [\mathfrak{H},\mathfrak{N}_0] \subseteq \mathfrak{N}_0, \quad [\mathfrak{H},\mathfrak{N}_1] \subseteq \mathfrak{N}_1, \\ & [\mathfrak{N}_0,\mathfrak{N}_0] \subseteq \mathfrak{H}, \quad [\mathfrak{N}_0,\mathfrak{N}_1] \subseteq \mathfrak{N}_1. \end{aligned}$$

Furthermore, for any two elements $\zeta_i^{j,s}, \zeta_k^{l,r} \in N_1$ the commutator $[\zeta_i^{j,s}, \zeta_k^{l,r}]$ lies in either $\mathfrak{H}$, $\mathfrak{N}_0$ or $\mathfrak{N}_1^+$. More precisely,

$$[\zeta_i^{j,s}, \zeta_k^{l,r}] \in \begin{cases} \mathfrak{N}_1^+ & \text{if } i = l \text{ or } j = k \\ \mathfrak{N}_0 \text{ or } \mathfrak{N}_1^+ & \text{if } k = i' \text{ or } j = l' \text{ but not both} \\ \mathfrak{H} & \text{if } k = i' \text{ and } j = l' \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We shall call on the characterisations of $\mathfrak{H}, \mathfrak{N}_0$ and $\mathfrak{N}_1$ given in Lemma 3. Thanks to [Yak06, Theorem 1] we have $\mathfrak{H} = \mathfrak{k} \cap (\mathfrak{g}_e)_\alpha$ where α is a certain regular element of $\mathfrak{g}_e^*$. By [Dix74, 1.11.7] the stabiliser $(\mathfrak{g}_e)_\alpha$ is abelian, hence $[\mathfrak{H},\mathfrak{H}] = 0$. The elements of $\mathfrak{N}_0$ are characterised by the fact that they exchange the spaces $V[i]$ and $V[i']$ with $i \neq i'$. Therefore the elements of $[\mathfrak{H},\mathfrak{N}_0]$ must exchange them also, implying $[\mathfrak{H},\mathfrak{N}_0] \subseteq \mathfrak{N}_0$. Each $\zeta_i^{j,s} \in \mathfrak{N}_1$ transports $V[i]$ to $V[j]$ and $V[j']$ to $V[i']$. Therefore $[\mathfrak{H}, \zeta_i^{j,s}]$ does likewise and $[\mathfrak{H},\mathfrak{N}_1] \subseteq \mathfrak{N}_1$. Since each element of $\mathfrak{N}_0$ exchanges the spaces $V[i]$ and $V[i']$ with $i \neq i'$ and annihilates all $V[i]$ with $i = i'$, the commutator space $[\mathfrak{N}_0,\mathfrak{N}_0]$ must stabilise all $V[i]$, hence be contained in $\mathfrak{H}$. The inclusion $[\mathfrak{N}_0,\mathfrak{N}_1] \subseteq \mathfrak{N}_1$ is best checked using Lemma 5. Let $i \neq i'$ and $l > k \neq l'$. Then $[\zeta_i^{i',s}\zeta_k^{l,r}]$ is non-zero only if $i = l$ or $i' = k$. Our restrictions on i, l and k ensure that these two possibilities are mutually exclusive. In the first case,

$$[\zeta_i^{i',s}, \zeta_k^{l,r}] = \zeta_k^{l',r+s-(\lambda_i-1)} - \varepsilon_{k,l,r}\zeta_l^{k',r+s-(\lambda_i-1)}$$

which lies in $\mathfrak{N}_1$. The second case is very similar.

We now consider the final claim. Suppose that $j > i \neq j'$ and $l > k \neq l'$. By Lemma 5 the bracket $[\zeta_i^{j,s}, \zeta_k^{l,r}]$ is only non-zero when one or more of the following equalities hold: $i = l, j = k$, $i' = k$, $j' = l$. We consider these four possibilities one by one.

Since the bracket is anticommutative, the reasoning in the case $i = l$ is identical to the case $j = k$ and so we need to consider only the first of these two possibilities. If $i = l$ then the

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relations $i' \neq j > i$ and $l > k \neq l'$ ensure that $j \neq k, i' \neq k$ and $j' \neq l$. Therefore $[\zeta_i^{j,s}, \zeta_k^{l,r}] = \zeta_k^{j,r+s-(\lambda_i-1)} \in \mathfrak{N}_1$. In order for this map to lie in $\mathfrak{N}_1^-$ we would require $j = k+1$; however, we have $j > i = l > k$ which makes this impossible. Thus $[\zeta_i^{j,s}, \zeta_k^{l,r}] \in \mathfrak{N}_1^+$.

By Lemma 2 we have $\zeta_i^{j,s} = \pm\zeta_{j'}^{i',s}$ and $\zeta_k^{l,r} = \pm\zeta_{l'}^{k',r}$, so the reasoning in the case $i = k'$ is identical to the case $j' = l$. Therefore we need to consider only the first of these two possibilities. Suppose that $i = k'$. Then certainly $i \neq l$ and $j \neq k$. If $j' = l$ then

$$[\zeta_i^{j,s}, \zeta_k^{l,r}] = \varepsilon_{k,l,r}(\zeta_j^{j,r+s-(\lambda_i-1)} - \zeta_i^{i,r+s-(\lambda_j-1)}) \in \mathfrak{H},$$

so assume henceforth that $j' \neq l$. Then

$$[\zeta_i^{j,s}, \zeta_k^{l,r}] = \varepsilon_{k,l,r}\zeta_{l'}^{j,r+s-(\lambda_i-1)}.$$

If $j = l$ then the product lies in $\mathfrak{N}_0$. Assume that $j \neq l$. Thanks to the relation $\zeta_{l'}^{j,r+s-(\lambda_i-1)} = \pm\zeta_{j'}^{l,r+s-(\lambda_i-1)}$ from Lemma 2 we may assume that $j > l'$, and from here it is easily seen that the product lies in $\mathfrak{N}_1$. In order for the product to lie in $\mathfrak{N}_1^-$ we require $\lambda_{l'-1} \neq \lambda_{l'}$, which implies $\lambda_l < \lambda_i$ since $i = k' < l$. From the bounds $0 \leqslant r < \lambda_l$ and $0 \leqslant s < \lambda_j$ we deduce that $r + s - (\lambda_i - 1) < \lambda_j - 1$, which confirms that the term $\zeta_{l'}^{j,r+s-(\lambda_i-1)}$ does not lie in $\mathfrak{N}_1^-$. □

Proposition 2. *The following are true:*

1. $\mathfrak{N}_0 \subset [\mathfrak{k}_e, \mathfrak{k}_e]$;
2. $\mathfrak{N}_1 \cap [\mathfrak{k}_e, \mathfrak{k}_e] = \mathfrak{N}_1^+$.

Proof. Assume that $i \neq i'$ and $\lambda_i - s$ is odd. We have $\varepsilon_{i,i,s} = (-1)^{\lambda_i - s}$, so

$$[\zeta_i^{i',s}, \zeta_i^{i,\lambda_i-1}] = \zeta_i^{i',s} - \varepsilon_{i,i,\lambda_i-1}\zeta_i^{i',s} = 2\zeta_i^{i',s} \in [\mathfrak{k}_e, \mathfrak{k}_e].$$

Since $\mathrm{char}(\mathbb{k}) \neq 2$ we get $\mathfrak{N}_0 = [\mathfrak{H}, \mathfrak{N}_0] \subseteq [\mathfrak{k}_e, \mathfrak{k}_e]$. This proves part (1).

For the sake of clarity we shall divide the proof of part (2) of the current proposition into subsections (i)–(ix). In parts (i)–(v) we demonstrate that $\mathfrak{N}_1^+ \subseteq [\mathfrak{k}_e, \mathfrak{k}_e]$ by showing that if $\zeta_i^{j,s} \in N_1$ is amongst the basis vectors spanning $\mathfrak{N}_1^+$ then some multiple of $\zeta_i^{j,s}$ may be found as a product of two basis elements in $\mathfrak{k}_e$. Recall that these vectors are defined to be those for which $(i, i+1) \notin \Delta(\lambda)$, or for which $j \neq i+1$, or for which $s < \lambda_j - 1$. In parts (vi)–(viii) we show that the reverse inclusion holds by noting that $\mathfrak{N}_1 \cap [\mathfrak{k}_e, \mathfrak{k}_e]$ is the sum of those products $[\zeta_i^{j,s}, \zeta_k^{l,r}]$ which lie in $\mathfrak{N}_1$, and showing that all such products actually lie in $\mathfrak{N}_1^+$. For the remainder of the proof we shall fix $l > k \neq l'$ so that $\min(\lambda_k, \lambda_l) = \lambda_l$.

(i) *If $l \neq l'$ or $k \neq k'$, then $\zeta_k^{l,r} \in [\mathfrak{k}_e, \mathfrak{k}_e]$ for $r = 0, 1, \dots, \lambda_l - 1$.* Suppose first that $l \neq l'$. We have

$$[\zeta_l^{l,\lambda_l-1}, \zeta_k^{l,r}] = \zeta_k^{l,r} \in [\mathfrak{k}_e, \mathfrak{k}_e],$$

whence we obtain $\zeta_k^{l,r} \in [\mathfrak{k}_e, \mathfrak{k}_e]$ for $r = 0, 1, \dots, \lambda_l - 1$. Now suppose that $k \neq k'$. Then

$$[\zeta_k^{k,\lambda_k-1}, \zeta_k^{l,r}] = -\zeta_k^{l,r} \in [\mathfrak{k}_e, \mathfrak{k}_e],$$

so that $\zeta_k^{l,r} \in [\mathfrak{k}_e, \mathfrak{k}_e]$ for all $r = 0, 1, \dots, \lambda_l - 1$.

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(ii) *If $l' = l$ and $k = k'$ then $\zeta_k^{l,r} \in [\mathfrak{k}_e, \mathfrak{k}_e]$ for $r = 0, 1, \ldots, \lambda_l - 2$.* With l and k as above,

$$[\zeta_l^{l,\lambda_l-2}, \zeta_k^{l,r}] = \zeta_k^{l,r-1} - \varepsilon_{k,l,r}\zeta_{l'}^{k',r-1} \in [\mathfrak{k}_e, \mathfrak{k}_e].$$

By part (3) of Lemma 2 this final expression is $(1 - \varepsilon_{k,l,r}\varepsilon_{k,l,r-1})\zeta_k^{l,r-1}$. Since $\varepsilon_{k,l,r} = (-1)^{\lambda_l - r}$ this expression actually equals $2\zeta_k^{l,r-1}$. Allowing r to run from 0 to $\lambda_l - 1$, we obtain the desired result.

(iii) *If $l' = l$, $k = k'$ and $k \neq j - 1$ then $\zeta_k^{l,r} \in [\mathfrak{k}_e, \mathfrak{k}_e]$ for $r = 0, 1, \ldots, \lambda_l - 1$.* We may assume there exists j fulfilling $l > j > k$. Then $k \neq l$ and $k' \neq j \neq l'$, so that

$$[\zeta_j^{l,r}, \zeta_k^{j,\lambda_j-1}] = \zeta_k^{l,r} \in [\mathfrak{k}_e\mathfrak{k}_e].$$

(iv) *If $l = l'$, $k = k'$ and either $\lambda_k = \lambda_{k-1}$ or $\lambda_l = \lambda_{l+1}$, then $\zeta_k^{l,r} \in [\mathfrak{k}_e, \mathfrak{k}_e]$ for $r = 0, 1, \ldots, \lambda_l - 1$.* First suppose that $\lambda_k = \lambda_{k-1}$. Since $k = k'$ we have $k - 1 = (k-1)'$, so that

$$[\zeta_{k-1}^{l,r}, \zeta_{k-1}^{k,\lambda_k-1}] = \varepsilon_{k-1,k,\lambda_k-1}\zeta_k^{l,r} \in [\mathfrak{k}_e, \mathfrak{k}_e]$$

for $r = 0, 1, \ldots, \lambda_l - 1$. Next suppose that $\lambda_l = \lambda_{l+1}$. Since $l = l'$ we have $l + 1 = (l+1)'$, and so

$$[\zeta_k^{l+1,\lambda_{l+1}-1}\zeta_l^{l+1,r}] = -\varepsilon_{l,l+1,r}\zeta_k^{l,r} \in [\mathfrak{k}_e, \mathfrak{k}_e]$$

for $r = 0, 1, \ldots, \lambda_l - 1$.

(v) $\mathfrak{N}_1^+ \subseteq [\mathfrak{k}_e, \mathfrak{k}_e]$. This follows by combining the deductions of parts (i)–(iv).

(vi) $[\mathfrak{H}, \mathfrak{N}_1] \subseteq \mathfrak{N}_1^+$. We continue to fix $l > k \neq l'$. The bracket $[\zeta_i^{i,s}, \zeta_k^{l,r}]$ is non-zero only if $i = k$ or $i = l$. Assume that $i = l$ (the case $i = k$ is similar). Then $[\zeta_i^{i,s}, \zeta_k^{l,r}] = \zeta_k^{l,r+s-(\lambda_i-1)}$, which lies in either $\mathfrak{N}_1^-$ or $\mathfrak{N}_1^+$. In order for $\zeta_k^{l,r+s-(\lambda_i-1)}$ to lie in $\mathfrak{N}^-$ we must have $l = l'$. But in that case $i = i'$, and so $\lambda_i - s$ must be even by the definition of $\mathfrak{H}$. In particular, $s \leqslant \lambda_i - 2$ and $r + s - (\lambda_i - 1) \leqslant r - 1 < \lambda_l - 1$, forcing $[\zeta_i^{i,s}, \zeta_k^{l,r}] \in \mathfrak{N}_1^+$.

(vii) $[\mathfrak{N}_0, \mathfrak{N}_1] \subseteq \mathfrak{N}_1^+$. The product $[\zeta_i^{i',s}, \zeta_k^{l,r}]$ with $i \neq i'$ is non-zero only if $i = l$ or $i' = k$. Our restrictions on i, l and k ensure that these two possibilities are mutually exclusive. In the case $i = l$,

$$[\zeta_i^{i',s}, \zeta_k^{l,r}] = \zeta_k^{l',r+s-(\lambda_i-1)} - \varepsilon_{k,l,r}\zeta_l^{k',r+s-(\lambda_i-1)} = (1 - \varepsilon_{k,l,r}\varepsilon_{k,l',r+s-(\lambda_i-1)})\zeta_k^{l',r+s-(\lambda_i-1)}.$$

If $\zeta_k^{l',r+s-(\lambda_i-1)} \in \mathfrak{N}_1^-$ then $l = l'$ by the definition of $\mathfrak{N}_1^-$. But then $i = l = l'$ yields $i = i'$ contrary to our assumptions. We deduce that $\zeta_k^{l',r+s-(\lambda_i-1)} \in \mathfrak{N}_1^+$. Now consider the case $i' = k$. A calculation similar to the above gives

$$[\zeta_i^{i',s}, \zeta_k^{l,r}] = (\varepsilon_{k,l,r}\varepsilon_{k,l,r+s-(\lambda_i-1)} - 1)\zeta_i^{l,r+s-(\lambda_i-1)}.$$

Since $i \neq i'$ we see, as before, that the right-hand side lies in $\mathfrak{N}_1^+$, hence (vii).

(viii) $\mathfrak{N}_1 \cap [\mathfrak{N}_1, \mathfrak{N}_1] \subseteq \mathfrak{N}_1^+$. This follows immediately from the last statement of Proposition 1.

(ix) $\mathfrak{N}_1 \cap [\mathfrak{k}_e, \mathfrak{k}_e] = \mathfrak{N}_1^+$. By (v) we know that $\mathfrak{N}_1^+ \subseteq \mathfrak{N}_1 \cap [\mathfrak{k}_e, \mathfrak{k}_e]$. By Proposition 1, $\mathfrak{N}_1 \cap [\mathfrak{k}_e, \mathfrak{k}_e]$ is equal to the span of those products $[\zeta_i^{j,s}, \zeta_k^{l,r}]$ which lie in $\mathfrak{N}_1$. By that same proposition and parts (vi)–(viii) we see that every product $[\zeta_i^{j,s}, \zeta_k^{l,r}]$ which lies in $\mathfrak{N}_1$ actually lies in $\mathfrak{N}_1^+$. The claim follows. □

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Proposition 3. *The following are true:*

(1) $\mathfrak{H}_1 \subset [\mathfrak{k}_e, \mathfrak{k}_e]$;
(2) $\mathfrak{H}_0 \cap [\mathfrak{k}_e, \mathfrak{k}_e] = \mathfrak{H}_0^+$.

Proof. $\mathfrak{H}_1$ has a basis consisting of vectors $\zeta_i^{i,s}$ with $i < i'$ and $\lambda_i - s$ odd. Fix such a choice of i and s, and choose r such that $\lambda_i - r$ is odd. By Lemma 5, we have that

$$[\zeta_{i'}^{i,s}, \zeta_i^{i',r}] = (1 + \varepsilon_{i,i',r})(\zeta_i^{i,s+r-(\lambda_i-1)} - \zeta_{i'}^{i',s+r-(\lambda_i-1)}).$$

Since $\varepsilon_{i,i,r+s-(\lambda_i-1)} = (-1)^{\lambda_i-(s+r-(\lambda_i-1))} = (-1)^{(\lambda_i-s)+(\lambda_i-r)+1} = -1$ it follows that $\zeta_{i'}^{i',s+r-(\lambda_i-1)} = -\zeta_i^{i,s+r-(\lambda_i-1)}$ by part (3) of Lemma 2. Also $\varepsilon_{i,i',r} = (-1)^{\lambda_i-r+1} = 1$. Therefore

$$[\zeta_{i'}^{i,s}, \zeta_i^{i',r}] = 4\zeta_i^{i,s+r-(\lambda_i-1)}$$

which is non-zero since $\mathrm{char}(\Bbbk) \neq 2$. We make the observation that the above expression lies in $\mathfrak{H}_1$ for any choice of r and s with $\lambda_i - r$ and $\lambda_i - s$ both odd. Taking $r = \lambda_i - 1$, we obtain $\zeta_i^{i,s} \in \mathfrak{H} \cap [\mathfrak{k}_e, \mathfrak{k}_e]$. Since $\mathfrak{H}_1$ is spanned by those $\zeta_i^{i,s}$ such that $i < i'$ and $\lambda_i - s$ is odd we have $\mathfrak{H}_1 \subseteq [\mathfrak{k}_e, \mathfrak{k}_e]$. This completes part (1).

For the sake of clarity we shall divide the proof of part (2) of the current proposition into subsections (i)–(vii). The approach is much the same as for part (2) of Proposition 2. In parts (i)–(iv) we show that a spanning set for $\mathfrak{H}_0^+$ may be found in $[\mathfrak{k}_e, \mathfrak{k}_e]$, and in the subsequent parts (v)–(vii) we demonstrate that any product $[\zeta_i^{j,s}, \zeta_k^{l,r}]$ which lies in $\mathfrak{H}_0$ actually lies in $\mathfrak{H}_0^+$.

(i) *The subspace $\mathfrak{H}_0 \cap [\mathfrak{N}_1, \mathfrak{N}_1]$ is spanned by all $\zeta_{[j]}^{[j],\lambda_{[j]}-2m} - \zeta_{[i]}^{[i],\lambda_{[i]}-2m}$ such that $[1] \leqslant [i] < [j] \leqslant [n]$ and $\lambda_i - \lambda_j < 2m < \lambda_j + \lambda_i$.* Indeed by Proposition 1 we see that $\mathfrak{H} \cap [\mathfrak{N}_1, \mathfrak{N}_1]$ is spanned by commutators $[\zeta_i^{j,s}, \zeta_{i'}^{j',r}]$ with $[j] > [i]$. In turn,

$$[\zeta_i^{j,s}, \zeta_{i'}^{j',r}] = \varepsilon_{i',j',r}[\zeta_i^{j,s}, \zeta_j^{i,r}] = \varepsilon_{i',j',r}(\zeta_j^{j,r+s-(\lambda_i-1)} - \zeta_i^{i,r+s-(\lambda_j-1)}).$$

The reader will notice that

$$[\zeta_i^{j,s}, \zeta_j^{i,r}] \in \begin{cases} \mathfrak{H}_1 & \text{if } \lambda_i + \lambda_j - (r+s) - 1 \text{ odd} \\ \mathfrak{H}_0 & \text{if } \lambda_i + \lambda_j - (r+s) - 1 \text{ even.} \end{cases}$$

As a consequence $\mathfrak{H}_0 \cap [\mathfrak{N}_1, \mathfrak{N}_1]$ is spanned by all $[\zeta_i^{j,s}, \zeta_j^{i,r}]$ with $[1] \leqslant [i] < [j] \leqslant [n]$ and $0 \leqslant s, r < \lambda_i, \lambda_i + \lambda_j - (r+s) - 1$ even. If we pick $[1] \leqslant [i] < [j] \leqslant [n]$ and $0 \leqslant s, r < \lambda_i$ such that $\lambda_i + \lambda_j - (r+s) - 1 = 2m$ then we have

$$[\zeta_i^{j,s}, \zeta_j^{i,r}] = \varepsilon_{i',j',r}\left(\zeta_{[j]}^{[j],\lambda_{[j]}-2m} - \zeta_{[i]}^{[i],\lambda_{[i]}-2m}\right).$$

The constraints placed on s and r are equivalent to $\lambda_i - \lambda_j < 2m < \lambda_i + \lambda_j$, and (i) follows.

(ii) $\mathfrak{H}_0 \cap [\mathfrak{k}_e, \mathfrak{k}_e] = \mathfrak{H}_0 \cap [\mathfrak{N}_1, \mathfrak{N}_1]$. By Proposition 1 we see that

$$\mathfrak{H} \cap [\mathfrak{k}_e, \mathfrak{k}_e] = [\mathfrak{N}_0, \mathfrak{N}_0] + (\mathfrak{H} \cap [\mathfrak{N}_1, \mathfrak{N}_1]),$$

whereas our observation in part (1) of the current proposition shows that $[\mathfrak{N}_0, \mathfrak{N}_0] \subseteq \mathfrak{H}_1$. Since $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$ and

$$\mathfrak{H} \cap [\mathfrak{N}_1, \mathfrak{N}_1] = (\mathfrak{H}_0 \cap [\mathfrak{N}_1, \mathfrak{N}_1]) \bigoplus (\mathfrak{H}_1 \cap [\mathfrak{N}_1, \mathfrak{N}_1])$$

by our discussion in (i) we obtain $\mathfrak{H}_0 \cap [\mathfrak{k}_e, \mathfrak{k}_e] = \mathfrak{H}_0 \cap [\mathfrak{N}_1, \mathfrak{N}_1]$.

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(iii) *Each spanning vector from (i) lies in a unique $\mathfrak{H}_{0,l}^m$, and in particular we have that $\mathfrak{H}_0 \cap [\mathfrak{N}_1, \mathfrak{N}_1] = \bigoplus_{l,m}(\mathfrak{H}_{0,l}^m \cap [\mathfrak{N}_1, \mathfrak{N}_1])$.* Fix m in the appropriate range and suppose that $1 \leqslant i < a_{t(m),m}$. We claim that if $[j] > [i]$ then each $\zeta_{[j]}^{[j],\lambda_{[j]}-2m} - \zeta_{[i]}^{[i],\lambda_{[i]}-2m} \in \mathfrak{H}_0 \cap [\mathfrak{N}_1, \mathfrak{N}_1]$ lies in $\mathfrak{H}_{0,l}^m$ where l is the unique integer fulfilling $[a_{l,m}] \leqslant [i] < [a_{l+1,m}]$. It will suffice to show that, given i, j, l and m as above, we have $[j] < [a_{l+1,m}]$. To see this, suppose that $[j] \geqslant [a_{l+1,m}]$. Then by our choice of $a_{l+1,m}$ we have $\lambda_{a_{l+1,m}-1} - \lambda_{a_{l+1,m}} \geqslant 2m$ which implies $\lambda_i - \lambda_j \geqslant 2m$ contrary to the restriction $\lambda_i - \lambda_j < 2m$ noted in the statement of (i). We conclude that $[a_{l,m}] \leqslant [i] < [j] < [a_{l+1,m}]$ and that the corresponding spanning vector lies in $\mathfrak{H}_{0,l}^m$. In the case $a_{t(m),m} \leqslant i$ we have $\zeta_{[j]}^{[j],\lambda_{[j]}-2m} - \zeta_{[i]}^{[i],\lambda_{[i]}-2m} \in \mathfrak{H}_{0,t(m)}^m$ by definition. Thus we have shown that the spanning vectors of $\mathfrak{H}_0 \cap [\mathfrak{N}_1, \mathfrak{N}_1]$ each lie in some $\mathfrak{H}_{0,l}^m$, as claimed.

(iv) *The inclusion $\mathfrak{H}_{0,l}^{m,+} \subseteq \mathfrak{H}_{0,l}^m \cap [\mathfrak{N}_1, \mathfrak{N}_1]$ holds for all l and m.* Suppose that $1 \leqslant i < a_{t(m),m}$. Since $\lambda_{a_{t(m),m}-1} - \lambda_{a_{t(m),m}} \geqslant 2m$ we know that $\lambda_{a_{t(m),m}-1} \geqslant 2m$ and so $\lambda_i \geqslant 2m$. It follows that $\zeta_{[i]}^{[i],\lambda_{[i]}-2m} \neq 0$ for all such i. Fix $[i]$ with $[a_{l,m}] < [i] < [a_{l+1,m}]$. By our choice of integers $\{a_{1,m}, \ldots, a_{t(m),m}\}$ we know that $\lambda_{[i]-1} - \lambda_{[i]} < 2m$ and since $\lambda_{[i]-1}, \lambda_{[i]} \geqslant \lambda_{a_{t(m),m}} \geqslant 2m$ we have $\lambda_{[i]-1} + \lambda_{[i]} > 2m$. By these remarks, using (i), it follows that $\zeta_{[i]-1}^{[i]-1,\lambda_{[i]-1}-2m} - \zeta_{[i]}^{[i],\lambda_{[i]}-2m}$ is a non-zero element of $\mathfrak{H}_{0,l}^m \cap [\mathfrak{N}_1, \mathfrak{N}_1]$. These vectors span all of $\mathfrak{H}_{0,l}^{m,+}$ so (iv) follows for $l < t(m)$.

The argument for $l = t(m)$ is similar. Let $k = \max\{i : \lambda_i \geqslant 2m\}$. Then $\zeta_i^{i,\lambda_i-2m} \neq 0$ if and only if $i \leqslant k$ so $\mathfrak{H}_{0,t(m)}^m = \operatorname{span}\left\{\zeta_{[i]}^{[i],\lambda_{[i]}-2m} : [a_{t(m),m}] \leqslant [i] \leqslant [k]\right\}$. Fix $[i]$ with $[a_{t(m),m}] < [i] \leqslant [k]$. By our choice of integers $\{a_{1,m}, \ldots, a_{t(m),m}\}$ we know that $\lambda_{[i]-1} - \lambda_{[i]} < 2m$, and by our choice of k we have $\lambda_{[i]-1} + \lambda_{[i]} > 2m$. The argument now concludes exactly as above.

(v) *The equality $\mathfrak{H}_{0,l}^m \cap [\mathfrak{N}_1, \mathfrak{N}_1] = \mathfrak{H}_{0,l}^{m,+}$ holds for all $1 \leqslant l < t(m)$.* The discussion in (iii) confirms that $\mathfrak{H}_{0,l}^m \cap [\mathfrak{N}_1, \mathfrak{N}_1]$ is spanned by all $\zeta_{[j]}^{[j],\lambda_{[j]}-2m} - \zeta_{[i]}^{[i],\lambda_{[i]}-2m}$ with $[a_{l,m}] \leqslant [i] < [j] < [a_{l+1,m}], \lambda_i - \lambda_j < 2m < \lambda_j + \lambda_i$. This space is clearly contained in $\mathfrak{H}_{0,l}^{m,+}$. Now (v) follows from (iv).

(vi) $\mathfrak{H}_{0,t(m)}^m \cap [\mathfrak{N}_1, \mathfrak{N}_1] = \mathfrak{H}_{0,t(m)}^m$. First we note that $\mathfrak{H}_{0,t(m)}^{m,+} \subseteq \mathfrak{H}_{0,t(m)}^m \cap [\mathfrak{N}_1, \mathfrak{N}_1]$ by (iv). If $\lambda_{a_{t(m),m}} < 2m$ then $\mathfrak{H}_{0,t(m)}^m = 0$ by part (1) of Lemma 4 and the statement holds trivially. So assume that $\lambda_{a_{t(m),m}} \geqslant 2m$ and let $k = \max\{i : \lambda_i \geqslant 2m\}$. Then $\mathfrak{H}_{0,t(m)}^m$ is spanned by all $\zeta_{[i]}^{[i],\lambda_{[i]}-2m}$ with $[a_{t(m),m}] \leqslant [i] \leqslant [k]\}$. We claim that $[k] + 1 \leqslant [n]$. If not then $[k] = [n]$, which implies that $\lambda_k - \lambda_{k+1} = \lambda_k \geqslant 2m$, forcing $k + 1 \in \{a_{1,m}, \ldots, a_{t(m),m}\}$. However, $k + 1 > a_{t(m),m}$ and $a_{1,m} \leqslant \cdots \leqslant a_{t(m),m}$. This contradiction confirms the claim. By the very same reasoning we know that $\lambda_{[k]} - \lambda_{[k]+1} = \lambda_k - \lambda_{k+1} < 2m$ and the inequality $[k] + 1 \leqslant n$ gives us $\lambda_{[k]+1} > 0$, which in turn implies $\lambda_{[k]} + \lambda_{[k]+1} > 2m$. By (i) and (iii) we have $\zeta_{[k]+1}^{[k]+1,\lambda_{[k]+1}-2m} - \zeta_{[k]}^{[k],\lambda_{[k]}-2m} \in \mathfrak{H}_{0,t(m)}^m$. Since $\lambda_{k+1} < 2m$ we know that $\zeta_{[k]+1}^{[k]+1,\lambda_{[k]+1}-2m} = 0$. Since $\zeta_{[k]}^{[k],\lambda_{[k]}-2m} \notin \mathfrak{H}_{0,t(m)}^{m,+}$ and $\mathfrak{H}_{0,t(m)}^{m,+}$ has codimension 1 in $\mathfrak{H}_{0,t(m)}^m$, statement (vi) follows.

(vii) $\mathfrak{H}_0 \cap [\mathfrak{k}_e, \mathfrak{k}_e] = \mathfrak{H}_0^+$. By (ii) and (iii) we have

$$\mathfrak{H}_0 \cap [\mathfrak{k}_e, \mathfrak{k}_e] = \bigoplus_{l,m}(\mathfrak{H}_{0,l}^m \cap [\mathfrak{N}_1, \mathfrak{N}_1]).$$

The proposition now follows from (v) and (vi). □

Theorem 6. *The derived subalgebra* $[\mathfrak{k}_e, \mathfrak{k}_e]$ *coincides with* $\mathfrak{N}_0 \bigoplus \mathfrak{N}_1^+ \bigoplus \mathfrak{H}_0^+ \bigoplus \mathfrak{H}_1$.

Proof. The sum of the above subspaces is direct by construction. By Proposition 1 we know that $[\mathfrak{k}_e, \mathfrak{k}_e]$ is the sum of the three spaces

$$[\mathfrak{k}_e, \mathfrak{k}_e] = (\mathfrak{N}_0 \cap [\mathfrak{k}_e, \mathfrak{k}_e]) + (\mathfrak{N}_1 \cap [\mathfrak{k}_e, \mathfrak{k}_e]) + (\mathfrak{H} \cap [\mathfrak{k}_e, \mathfrak{k}_e]).$$

By Proposition 2 we have that $(\mathfrak{N}_0 \cap [\mathfrak{k}_e, \mathfrak{k}_e]) + (\mathfrak{N}_1 \cap [\mathfrak{k}_e, \mathfrak{k}_e]) = \mathfrak{N}_0 + \mathfrak{N}_1^+$. By Proposition 3, using the fact that $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$, we have $\mathfrak{H} \cap [\mathfrak{k}_e, \mathfrak{k}_e] = \mathfrak{H}_1 + \mathfrak{H}_0^+$. The theorem follows. □

2.4 A combinatorial formula for dim $\mathfrak{k}_e^{\mathrm{ab}}$

As a corollary to the previous theorem we obtain an expression for the dimension of the maximal abelian quotient $\mathfrak{k}_e^{\mathrm{ab}} := \mathfrak{k}_e/[\mathfrak{k}_e, \mathfrak{k}_e]$. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathcal{P}_\epsilon(N)$, we have defined $\Delta(\lambda)$ to be the set of pairs $(i, i+1)$ with $1 \leqslant i < n, i' = i, (i+1)' = i+1$ and $\lambda_{i-1} \neq \lambda_i \geqslant \lambda_{i+1} \neq \lambda_{i+2}$; see Definition 1. Recall that the elements of $\Delta(\lambda)$ are referred to as 2-steps. Now set

$$s(\lambda) := \sum_{i=1}^{n} \lfloor (\lambda_i - \lambda_{i+1})/2 \rfloor.$$

Note that if $(i, i+1) \in \Delta(\lambda)$ then $\epsilon(-1)^{\lambda_i} = \epsilon(-1)^{\lambda_{i+1}} = -1$ and recall our convention that $\lambda_0 = 0$ and $\lambda_i = 0$ for all $i > n$. We may now state and prove the formula for dim $\mathfrak{k}_e^{\mathrm{ab}}$.

Corollary 1. *Let* $\mathfrak{k}$ *be one of the classical Lie algebras* $\mathfrak{so}_N$ *or* $\mathfrak{sp}_N$ *where* $N \geqslant 2$ *and suppose that* char$(\Bbbk) \neq 2$. *Then* $\dim \mathfrak{k}_e^{\mathrm{ab}} = s(\lambda) + |\Delta(\lambda)|$ *for any nilpotent element* $e = e(\lambda) \in \mathfrak{k}$.

Proof. Recall that $\mathfrak{k}_e = \mathfrak{H} \bigoplus \mathfrak{N}_0 \bigoplus \mathfrak{N}_1$, that $\mathfrak{N}_1 = \mathfrak{N}_1^- \bigoplus \mathfrak{N}_1^+$, and that $\mathfrak{H} = \mathfrak{H}_0 \bigoplus \mathfrak{H}_1$ with $\mathfrak{H}_0^+ \subseteq \mathfrak{H}_0$. By Theorem 6 we have that $\mathfrak{k}_e^{\mathrm{ab}} \cong (\mathfrak{N}_1/\mathfrak{N}_1^+) \bigoplus (\mathfrak{H}_0/\mathfrak{H}_0^+)$ as vector spaces. We claim that $\dim(\mathfrak{N}_1/\mathfrak{N}_1^+) = |\Delta(\lambda)|$ and that $\dim(\mathfrak{H}_0/\mathfrak{H}_0^+) = s(\lambda)$, whence the theorem shall follow. First of all observe that $\dim(\mathfrak{N}_1/\mathfrak{N}_1^+) = \dim(\mathfrak{N}_1^-)$. By part (3) of Lemma 2 the maps $\zeta_{i-1}^{i,\lambda_i-1}$ spanning $\mathfrak{N}_1^-$ are all linearly independent. Out last remark in § 2.2 defines the set $\mathfrak{N}_1^-$ to be the space spanned by

$$N_1^- := \{\zeta_i^{i+1,\lambda_{i+1}-1} : (i, i+1) \in \Delta(\lambda)\}.$$

The map $(i, i+1) \mapsto \zeta_i^{i+1,\lambda_{i+1}-1}$ is clearly a bijection $\Delta(\lambda) \leftrightarrow N_1^-$. We conclude that $\dim(\mathfrak{N}_1/\mathfrak{N}_1^+) = \dim(\mathfrak{N}_1^-) = |\Delta(\lambda)|$.

We must now show that $\dim(\mathfrak{H}_0/\mathfrak{H}_0^+) = s(\lambda)$. Observe that $\mathfrak{H}_0 = \bigoplus_{l,m} \mathfrak{H}_{0,l}^m$ (part (2) of Lemma 4) and that each $\mathfrak{H}_{0,l}^{m,+}$ has codimension 1 in $\mathfrak{H}_{0,l}^m$. Furthermore, if $l < t(m)$ then $\mathfrak{H}_{0,l}^m \neq 0$. We conclude that $\dim(\mathfrak{H}_0/\mathfrak{H}_0^+) = |\mathcal{D}|$ where

$$\mathcal{D} = \{(l, m) : 1 \leqslant l \leqslant t(m) - 1,\ 1 \leqslant m \leqslant \lfloor \lambda_1/2 \rfloor\}.$$

On the other hand, $s(\lambda) = |\mathcal{D}'|$ where

$$\mathcal{D}' = \{(i, m) \in \{2, \ldots, n+1\} \times \{1, \ldots, \lfloor \lambda_1/2 \rfloor\} : \lambda_{i-1} - \lambda_i \geqslant 2m\}.$$

If we construct a bijection from $\mathcal{D}$ to $\mathcal{D}'$ then the result follows. Define a map from $\mathcal{D}$ to $\{2, \ldots, n+1\} \times \{1, \ldots, \lfloor \lambda_1/2 \rfloor\}$ by the rule

$$(i, m) \longmapsto (a_{i+1,m}, m).$$

By the definition of the integers $\{a_{1,m}, a_{2,m}, \ldots, a_{t(m),m}\}$ it is a well-defined injection into $\mathcal{D}'$. Fix $1 \leqslant m \leqslant \lfloor \lambda_1/2 \rfloor$. Since $\lambda_0 = 0$ and $\lambda_1 \geqslant \cdots \geqslant \lambda_n$, we have $a_{1,m} = 1$ and $\{a_{2,m}, \ldots, a_{t(m),m}\}$ is the set of all integers i with $2 \leqslant i \leqslant n+1$ and $\lambda_{i-1} - \lambda_i \geqslant 2m$. Thus the map is surjective and $\dim(\mathfrak{H}_0/\mathfrak{H}_0^+) = s(\lambda)$. □

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Remark 1. If $\mathfrak{g} = \mathfrak{sl}_N$ where $N \geqslant 2$ and e is a nilpotent element of $\mathfrak{g}$ corresponding to a partition $(\lambda_1, \ldots, \lambda_n)$ of N then $\dim \mathfrak{g}_e^{\mathrm{ab}} = \dim \mathfrak{z}(\mathfrak{g}_e) = \lambda_1 - 1$. This follows, for instance, from results of [Yak10]. If e is a nilpotent element in a classical Lie algebra $\mathfrak{k}$ of type other than A then it may happen that $\mathfrak{k}_e^{\mathrm{ab}}$ and $\mathfrak{z}(\mathfrak{k}_e)$ have different dimensions.

Example 1. To illustrate Corollary 1 we consider the special case where $\mathfrak{k} = \mathfrak{so}_4$. This Lie algebra has type $\mathsf{D}_2 \cong \mathsf{A}_1 \times \mathsf{A}_1$ and is isomorphic to a direct sum of two copies of $\mathfrak{sl}_2$. Therefore $\mathfrak{k}$ has three non-zero nilpotent orbits: the orbits containing root vectors e_1 and e_2 of the two simple ideals of $\mathfrak{k}$ and the regular nilpotent orbit containing $e_1 + e_2$. It is immediate that $\mathfrak{k}_{e_1} \cong \mathfrak{k}_{e_2} \cong \mathfrak{sl}_2 \oplus \Bbbk$, whilst $\mathfrak{k}_{e_1+e_2}$ is abelian and has dimension 2. In particular, $\dim \mathfrak{k}_{e_1}^{\mathrm{ab}} = \dim \mathfrak{k}_{e_2}^{\mathrm{ab}} = 1$ and $\dim \mathfrak{k}_{e_1+e_2}^{\mathrm{ab}} = 2$.

On the combinatorial side, the set $\mathcal{P}_1(4)$ contains only two non-trivial partitions, namely, $\lambda = (3,1)$ and $\mu = (2,2)$. Since $\mathfrak{k}$ is of type D and the partition $(2,2)$ has even parts only, there are two nilpotent orbits in $\mathfrak{k}$ attached to it (they are permuted by an outer automorphism of $\mathfrak{k}$ and assigned the Roman numerals I and II). It is straightforward to see that our root vectors e_1 and e_2 correspond to the partition μ, whereas $e_1 + e_2$ is attached to λ. Since $(1,2)$ is the only 2-step of λ we get $|\Delta(\lambda)| = 1$ and $s(\lambda) = \lfloor (3-1)/2 \rfloor = 1$. So $\dim \mathfrak{k}_e^{\mathrm{ab}} = 1 + 1 = 2$ by Corollary 1. On the other hand, $\Delta(\mu) = \emptyset$ and $s(\mu) = \lfloor (2-2)/2 \rfloor + \lfloor (2-0)/2 \rfloor = 1$, yielding $\dim \mathfrak{k}_{e_1}^{\mathrm{ab}} = \dim \mathfrak{k}_{e_2}^{\mathrm{ab}} = 0 + 1 = 1$. This agrees with our earlier deductions.

Example 2. Now suppose that $\mathfrak{k} = \mathfrak{so}_6$, a Lie algebra of type $\mathsf{D}_3 \cong \mathsf{A}_3$. In this case $\mathfrak{k} \cong \mathfrak{sl}_4$. The Lie algebra $\mathfrak{sl}_4$ has four non-zero nilpotent orbits which correspond to the partitions (4), $(3,1)$, $(2,2)$ and $(2,1,1)$. Using Remark 1, we see that $\dim \mathfrak{k}_e^{\mathrm{ab}}$ equals 3, 2, 1 and 1 in the respective cases.

On the other hand, the set $\mathcal{P}_1(6)$ contains four non-trivial partitions μ, namely, $(5,1)$, $(3,3)$, $(3,1,1,1)$ and $(2,2,1,1)$ and the corresponding nilpotent orbits of $\mathfrak{k}$ are associated with the partitions (4), $(3,1)$, $(2,2)$ and $(2,1,1)$ when regarded as elements of $\mathfrak{sl}_4$. Since $|\Delta(\mu)| = 1$ if μ is one of $(5,1)$, $(3,3)$ or $(2,2,1,1)$ and $\Delta(\mu) = \emptyset$ if $\mu = (3,1,1,1)$, applying Corollary 1 yields that $\dim \mathfrak{k}_e^{\mathrm{ab}}$ equals 3, 2, 1 and 1 in the respective cases. This agrees with our earlier deductions.

3. Applications to the theory of sheets in classical Lie algebras

3.1 The Kempken–Spaltenstein algorithm

Let G be a simple algebraic group over $\Bbbk$ and $m \in \mathbb{N}$. We recall that a sheet of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ is an irreducible component of the locally closed set

$$\mathfrak{g}^{(m)} := \{x \in \mathfrak{g} : \dim \mathfrak{g}_x = m\}.$$

Let $\mathcal{N}(\mathfrak{g})$ denote the the variety of all nilpotent elements in $\mathfrak{g}$. It is well known that every sheet of $\mathfrak{g}$ contains a unique nilpotent orbit; see [BK79, 5.8]. However, outside type A the sheets are not disjoint and a given nilpotent orbit of $\mathfrak{g} \not\cong \mathfrak{sl}_N$ may lie in several different sheets.

Crucial for the theory of sheets in semisimple Lie algebras is the notion of a rigid element (such elements were termed *original* by Borho). An element $x \in \mathcal{N}(\mathfrak{g})$ is called *rigid* if the adjoint G-orbit of x coincides with a sheet of $\mathfrak{g}$. Any rigid element of $\mathfrak{g}$ is necessarily nilpotent.

Let $\mathfrak{l}$ be a Levi subalgebra of $\mathfrak{g}$. The centre $\mathfrak{z}(\mathfrak{l})$ is a toral subalgebra of $\mathfrak{g}$ and for any $z \in \mathfrak{z}(\mathfrak{l})$ the centraliser $\mathfrak{g}_z$ contains $\mathfrak{l}$. We denote by $\mathfrak{z}(\mathfrak{l})_{\mathrm{reg}}$ the set of all $z \in \mathfrak{z}(\mathfrak{l})$ for which $\mathfrak{g}_z = \mathfrak{l}$. This is a non-empty Zariski open subset of $\mathfrak{z}(\mathfrak{l})$. Given a nilpotent element $e_0 \in [\mathfrak{l}, \mathfrak{l}]$, we define $\mathcal{D}(\mathfrak{l}, e_0)$ to be the G-stable set $(\mathrm{Ad}\, G)\big(e_0 + \mathfrak{z}(\mathfrak{l})_{\mathrm{reg}}\big)$ and we call $\mathcal{D}(\mathfrak{l}, e_0)$ a *decomposition class* of $\mathfrak{g}$.

Every sheet $\mathcal{S}$ of $\mathfrak{g}$ is a G-stable subset of $\mathfrak{g}$ locally closed and irreducible in the Zariski topology of $\mathfrak{g}$. By a classical result of Borho [Bor81], every sheet is a finite union of decomposition

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classes and contains a unique Zariski open such class. Furthermore, a decomposition class $\mathcal{D}(\mathfrak{l}, e_0)$ contained in $\mathcal{S}$ is open in $\mathcal{S}$ if and only if e_0 is rigid in $\mathfrak{l}$; see [Bor81, 3.7]. Conversely, every decomposition class $\mathcal{D}(\mathfrak{l}, e_0)$ with e_0 rigid in $\mathfrak{l}$ is Zariski open in a unique sheet of $\mathfrak{g}$. Furthermore, the unique nilpotent orbit in that sheet is obtained from e_0 by Lusztig–Spaltenstein induction. This result of Borho gives us a very transparent way to parameterise the sheets of $\mathfrak{g}$.

If $\mathcal{S}$ is a sheet of $\mathfrak{g}$ and $\mathcal{D}(\mathfrak{l}, e_0)$ is its open decomposition class then $\dim \mathfrak{z}(\mathfrak{l})$ is called the *rank* of $\mathcal{S}$ and abbreviated as $\mathrm{rk}(\mathcal{S})$. This notion is important as it enables us to determine the dimension of $\mathcal{S}$. Indeed, suppose that $\mathcal{S} \subset \mathfrak{g}^{(m)}$. Since the morphism

$$G \times \big(e_0 + \mathfrak{z}(\mathfrak{l}_{\mathrm{reg}})\big) \longrightarrow \mathcal{S}, \quad (g, x) \mapsto (\mathrm{Ad}\, g)x$$

is dominant, it follows from the theorem on dimensions of the fibres of a morphism and the theory of induced conjugacy classes that

$$\dim \mathcal{S} = \dim \mathfrak{g} - m + \mathrm{rk}(\mathcal{S});$$

see [Bor81, LS79] for more detail.

In this section we deal with sheets in classical Lie algebras and we keep the notation introduced in § 1. We shall be discussing the properties of various different nilpotent orbits in various different classical Lie algebras simultaneously. In order to distinguish between the various orbits we shall often appeal to their associated partitions.

Recall from § 1 the set $\mathcal{P}_\epsilon(N)$ of partitions of N associated with the nilpotent elements of $\mathfrak{k}$. Given $e \in \mathcal{N}(\mathfrak{k})$, we denote by $\lambda(e)$ the partition in $\mathcal{P}_\epsilon(N)$ corresponding to e. If $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathcal{P}_\epsilon(N)$ then we write $e(\lambda)$ for any element in $\mathcal{N}(\mathfrak{k})$ whose Jordan block sizes (arranged as in Lemma 1) are $\lambda_1, \lambda_2, \dots, \lambda_n$. The map $e \mapsto \lambda(e)$ induces a surjection from the orbit set $\mathcal{N}(\mathfrak{k})/K$ onto $\mathcal{P}_\epsilon(N)$. The fibres of this surjection are singletons unless $\mathfrak{g}$ is of type D and all parts of λ are even. In the latter case the fibre consists of two nilpotent orbits permuted by an outer automorphism of $\mathfrak{k}$ and the two orbits in the fibre are traditionally assigned the Roman numerals I and II. Since the centralisers of all elements lying in the fibres of the above surjection are isomorphic as abstract Lie algebras, the notation $e(\lambda)$ is unambiguous and will cause no confusion.

The following classification of rigid elements in $\mathcal{N}(\mathfrak{k})$ was given by Kempken and Spaltenstein:

Theorem 7 (See [Kem83, Spa82]). *Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathcal{P}_\epsilon(N)$. Then $e(\lambda) \in \mathcal{N}(\mathfrak{k})$ is rigid if and only if:*

- $\lambda_i - \lambda_{i+1} \in \{0, 1\}$ *for all* $1 \leqslant i \leqslant n$;
- *the set* $\{(i, i+1) \in \Delta(\lambda) : \lambda_i = \lambda_{i+1}\}$ *is empty.*

In the above we observe the convention $\lambda_0 = 0$ and $\lambda_i = 0$ for $i > n$. Note that $(i, i+1) \in \Delta(\lambda)$ implies $\lambda_i - \lambda_{i+1}$ is even by Lemma 1. Therefore the two conditions for $e(\lambda)$ rigid together imply $\Delta(\lambda) = \emptyset$ and we may replace second criterion for rigidity with this apparently stronger condition. Using our results on the derived subalgebra of $\mathfrak{k}_e$, we recover a result of Yakimova first proven in [Yak10, Theorem 12].

Corollary 2. *$[\mathfrak{k}_{e(\lambda)}, \mathfrak{k}_{e(\lambda)}] = \mathfrak{k}_{e(\lambda)}$ if and only if $e(\lambda)$ is rigid.*

Proof. Evidently $[\mathfrak{k}_{e(\lambda)}, \mathfrak{k}_{e(\lambda)}] = \mathfrak{k}_{e(\lambda)}$ if and only if $\dim(\mathfrak{k}_{e(\lambda)}^{\mathrm{ab}}) = 0$. Now apply Corollary 1 and Theorem 7. □

In view of Theorem 7 we have a well-defined notion of a rigid partition in $\mathcal{P}_\epsilon(N)$ and we denote the set of all such partitions by $\mathcal{P}^*_\epsilon(N)$. Relying on results of [Kem83, Spa82], Moreau describes an algorithm [Mor08] which takes $\lambda \in \mathcal{P}_\epsilon(N)$ and returns an element of $\mathcal{P}^*_\epsilon(M)$ for some $M \leqslant N$. In this section we also follow [Kem83, Spa82] and present an extended version of Moreau's algorithm which will be used later to determine when a nilpotent element of $\mathfrak{k}$ lies in a single sheet and to confirm a conjecture made by Izosimov in [Izo12].

Throughout the following, $\mathbf{i}$ shall denote a finite sequence of integers between 1 and n. The procedure is as follows. The algorithm commences with input $\lambda = \lambda^{\emptyset} \in \mathcal{P}_\epsilon(N)$ where $\emptyset$ denotes the empty sequence. At the lth iteration, the algorithm takes $\lambda^{\mathbf{i}} \in \mathcal{P}_\epsilon(N - 2\sum_{j=1}^{l-1} i_j)$ where $\mathbf{i} = (i_1, \dots, i_{l-1})$ and returns $\lambda^{\mathbf{i}'} \in \mathcal{P}_\epsilon(N - \sum_{j=1}^{l} i_j)$ where $\mathbf{i}' = (i_1, \dots, i_{l-1}, i_l)$ for some i_l. If the output $\lambda^{\mathbf{i}'}$ is a rigid partition then the algorithm terminates after the lth iteration with output $\lambda^{\mathbf{i}'}$. We shall now explicitly describe the lth iteration of the algorithm. If after the $(l-1)$th iteration the input $\lambda^{\mathbf{i}}$ is not rigid then the algorithm behaves as follows. Let i_l denote any index in the range $1 \leqslant i \leqslant n$ such that either of the following occurs.

Case 1: $\lambda^{\mathbf{i}}_{i_l} \geqslant \lambda^{\mathbf{i}}_{i_l+1} + 2$.

Case 2: $(i_l, i_l + 1) \in \Delta(\lambda^{\mathbf{i}})$ and $\lambda^{\mathbf{i}}_{i_l} = \lambda^{\mathbf{i}}_{i_l+1}$.

Note that no integer i_l will fulfil both of these criteria. If $\mathbf{i} = (i_1, \dots, i_{l-1})$ then define $\mathbf{i}' = (i_1, \dots, i_{l-1}, i_l)$. For Case 1 the algorithm has output

$$\lambda^{\mathbf{i}'} = (\lambda^{\mathbf{i}}_1 - 2, \lambda^{\mathbf{i}}_2 - 2, \dots, \lambda^{\mathbf{i}}_{i_l} - 2, \lambda^{\mathbf{i}}_{i_l+1}, \dots, \lambda^{\mathbf{i}}_n),$$

whilst for Case 2 the algorithm has output

$$\lambda^{\mathbf{i}'} = (\lambda^{\mathbf{i}}_1 - 2, \lambda^{\mathbf{i}}_2 - 2, \dots, \lambda^{\mathbf{i}}_{i_l-1} - 2, \lambda^{\mathbf{i}}_{i_l} - 1, \lambda^{\mathbf{i}}_{i_l+1} - 1, \lambda^{\mathbf{i}}_{i_l+2}, \dots, \lambda^{\mathbf{i}}_n).$$

In what follows we shall often refer to the algorithm just described as the *KS algorithm* (after Kempken and Spaltenstein). Due to its definition and the classification of rigid partitions the KS algorithm certainly terminates after a finite number of steps. In the hope of avoiding any confusion we shall use 'Case' when referring to Case 1 or Case 2 of the algorithm, and we shall use 'case' to refer to a particular situation. We shall say that a sequence $\mathbf{i} = (i_1, i_2, \dots, i_l)$ is an *admissible sequence for* λ if Case 1 or Case 2 occurs at the point i_k for the partition $\lambda^{(i_1,\dots,i_{k-1})}$ for each $k = 1, \dots, l$. We shall use the notation $|\mathbf{i}|$ to denote the length of such a sequence. An admissible sequence $\mathbf{i}$ for λ shall be called *maximal admissible for* λ if neither Case 1 nor Case 2 occurs for any index i between 1 and n for the partition $\lambda^{\mathbf{i}}$. If a sequence $\mathbf{i} = (i_1, \dots, i_l)$ is admissible for λ and $1 \leqslant j \leqslant l+1$ then we shall use the notation $\mathbf{i}_j = (i_1, \dots, i_{j-1})$. Clearly the sequence $\mathbf{i}_j$ is admissible for λ for any $1 \leqslant j \leqslant l+1$. By convention the empty sequence is admissible for any $\lambda \in \mathcal{P}_\epsilon(N)$.

LEMMA 6. *Let* $\mathbf{i}$ *be an admissible sequence for* λ*. Then* $\mathbf{i}$ *is maximal admissible if and only if* $\lambda^{\mathbf{i}}$ *is a rigid partition.*

Proof. In view of Theorem 7 this follows from the definition of maximal admissible sequences. □

Remark 2. (i) Rather that defining i_l to be any index between 1 and n such that Case 1 or Case 2 occurs, Moreau's algorithm in [Mor08] defines i_l to be the *smallest* such index. This discrepancy ensures that her algorithm is deterministic (the outcome does not depend upon a choice of indices $i_1, i_2, i_3, \dots$). In a sense, being non-deterministic is an advantage of the KS algorithm and we shall see later that it has enough power to reach and pin down all sheets of $\mathfrak{k}$ containing a given nilpotent element.

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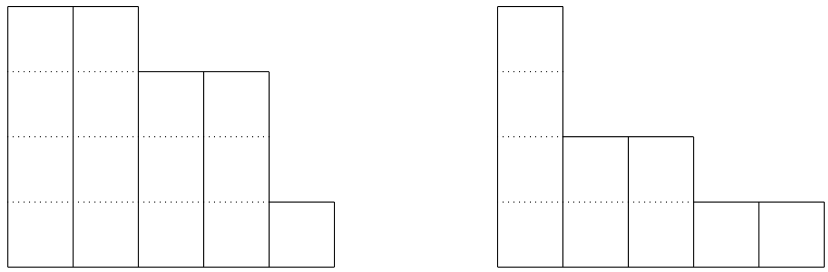

Figure 1. The Young diagrams of two singular partitions in $\mathcal{P}_1(15)$ and $\mathcal{P}_{-1}(10)$. The bad 2-steps are $(3,4)$ and $(2,3)$, respectively.

(ii) The KS algorithm is transitive in the following sense: if $\mathbf{i}$ is an admissible sequence for λ and $\mathbf{j}$ is an admissible sequence for $\lambda^{\mathbf{i}}$ then $(\mathbf{i},\mathbf{j})$ is an admissible sequence for λ, where $(\mathbf{i},\mathbf{j})$ denotes the concatenation of the two sequences $\mathbf{i}$ and $\mathbf{j}$. Furthermore, $\lambda^{(\mathbf{i},\mathbf{j})} = (\lambda^{\mathbf{i}})^{\mathbf{j}}$.

3.2 Non-singular partitions and preliminaries of the algorithm

Before placing the algorithm into the geometric context for which it was intended we shall discuss it combinatorially. We start by introducing a combinatorial notion related with the notion of a boundary of $\lambda \in \mathcal{P}_\epsilon(N)$; see Definition 1.

A 2-step $(i, i+1) \in \Delta(\lambda)$ is said to be *good* if λ_i and the boundary of $(i, i+1)$ have the opposite parity. It is worth mentioning that if $(i, i+1)$ is a good 2-step with $i > 1$ then both λ_{i-1} and λ_{i+2} must have the same parity. If a 2-step $(i, i+1) \in \Delta(\lambda)$ is not good then we say that it is *bad*. We note that $(i, i+1)$ is a bad 2-step of λ if and only if either $i > 1$ and $\lambda_{i-1} - \lambda_i \in 2\mathbb{N}$ or $\lambda_{i+1} - \lambda_{i+2} \in 2\mathbb{N}$.

We call a partition $\lambda \in \mathcal{P}_\epsilon(N)$ *singular* if it has a bad 2-step. Naturally if all 2-steps of λ are good then we call λ *non-singular*. In the next section we shall interpret these singular and non-singular partitions in geometric terms. In particular, we shall show that singular partitions correspond precisely to the nilpotent singular points on the varieties $\mathfrak{k}^{(m)}$, hence their name.

We now collect some elementary lemmas about the behaviour of the algorithm. For the rest of this subsection we assume that $\lambda \in \mathcal{P}_\epsilon(N)$ has the standard ordering $\lambda_1 \geqslant \cdots \geqslant \lambda_n$.

Lemma 7. *Suppose that $\mathbf{i} = (i)$ is a sequence of length 1. If Case 2 occurs for λ at index i then $\Delta(\lambda^{\mathbf{i}}) = \Delta(\lambda)\backslash\{(i, i+1)\}$. Furthermore, if $(i, i+1)$ is a good 2-step of λ then $s(\lambda^{\mathbf{i}}) = s(\lambda)$.*

Proof. We shall suppose that there is a 2-step

$$(j, j+1) \in \Delta(\lambda)\backslash(\Delta(\lambda^{\mathbf{i}}) \cup \{(i, i+1)\})$$

and derive a contradiction. Observe that if $j < i-2$ (respectively, $j > i+2$) then for $k \in \{j-1, j, j+1, j+2\}$ we have that $\lambda^{\mathbf{i}}_k = \lambda_k - 2$ (respectively, $\lambda^{\mathbf{i}}_k = \lambda_k$). So $(j, j+1) \in \Delta(\lambda)$ if and only if $(j, j+1) \in \Delta(\lambda^{\mathbf{i}})$. It remains to show that if $j = i \pm 1$ or $j = i \pm 2$ and $(j, j+1) \in \Delta(\lambda)$ then $(j, j+1) \in \Delta(\lambda^{\mathbf{i}})$. If $j = i \pm 1$ and $(j, j+1) \in \Delta(\lambda)$ then $\lambda_i \neq \lambda_{i+1}$, contradicting the fact that Case 2 occurs for λ at index i.

Suppose that $j = i-2$. Then $(j, j+1), (j+2, j+3) \in \Delta(\lambda)$ and hence $\lambda_{j+1} \neq \lambda_{j+2}$ and $(j+1)' = j+1$, $(j+2)' = j+2$. As a consequence $\lambda_{j+1} - \lambda_{j+2}$ is even, implying that $\lambda_{j+1} - \lambda_{j+2} \geqslant 2$ and $\lambda^{\mathbf{i}}_{j+1} \neq \lambda^{\mathbf{i}}_{j+2}$. Since for $k \in \{j-1, j, j+1\}$ the equality $\lambda^{\mathbf{i}}_k = \lambda_k - 2$ holds, we conclude that $(j, j+1) \in \Delta(\lambda^{\mathbf{i}})$. A similar argument shows that if $j = i+2$ then $(j, j+1) \in \Delta(\lambda)$ implies $(j, j+1) \in \Delta(\lambda^{\mathbf{i}})$. We conclude that $\Delta(\lambda^{\mathbf{i}}) = \Delta(\lambda)\backslash\{(i, i+1)\}$.

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Now suppose that $(i, i+1)$ is a good 2-step of λ. Since $\lambda_{i+1} - \lambda_{i+2}$ and $\lambda_{i-1} - \lambda_i$ if $i > 1$ are odd, we have that

$$\lfloor(\lambda^{\mathbf{i}}_{i+1} - \lambda^{\mathbf{i}}_{i+2})/2\rfloor = \lfloor((\lambda_{i+1} - 1) - \lambda_{i+2})/2\rfloor = \lfloor(\lambda_{i+1} - \lambda_{i+2})/2\rfloor$$

and

$$\lfloor(\lambda^{\mathbf{i}}_{i-1} - \lambda^{\mathbf{i}}_{i})/2\rfloor = \lfloor((\lambda_{i-1} - 2) - (\lambda_i - 1))/2\rfloor = \lfloor(\lambda_{i-1} - \lambda_i)/2\rfloor$$

if $i > 1$. As $\lambda^{\mathbf{i}}_j = \lambda_j$ for $j \notin \{i, i+1\}$ it follows that $s(\lambda^{\mathbf{i}}) = s(\lambda)$ as claimed. □

LEMMA 8. *If* $\mathbf{i}$ *is an admissible sequence for* λ *then* $\Delta(\lambda^{\mathbf{i}}) \subseteq \Delta(\lambda)$.

Proof. In view of Lemma 7 and Remark 2(ii) it will suffice to prove the current lemma when $\mathbf{i} = (i)$ and i is an index at which Case 1 occurs for λ. Suppose that $(j, j+1) \in \Delta(\lambda^{\mathbf{i}})$. Then since Case 1 preserves the parity of the entries of λ (that is to say, $\lambda^{\mathbf{i}}_k \equiv \lambda_k \mod 2$ for $1 \leqslant k \leqslant n$), we deduce that $j' = j$ and $(j+1)' = j+1$. If $j < i$ or $j > i+1$ then $\lambda_{j-1} - \lambda_j = \lambda^{\mathbf{i}}_{j-1} - \lambda^{\mathbf{i}}_j$ and $\lambda_{j+1} - \lambda_{j+2} = \lambda^{\mathbf{i}}_{j+1} - \lambda^{\mathbf{i}}_{j+2}$, showing that $(j, j+1) \in \Delta(\lambda)$ in these cases. If $j = i+1$ then $\lambda_{j-1} - \lambda_j = \lambda^{\mathbf{i}}_{j-1} - \lambda^{\mathbf{i}}_j + 2$ and $\lambda_{j+1} - \lambda_{j+2} = \lambda^{\mathbf{i}}_{j+1} - \lambda^{\mathbf{i}}_{j+2}$. Hence $(j, j+1) \in \Delta(\lambda)$. Finally, if $j = i$ then $\lambda_{j-1} - \lambda_j = \lambda^{\mathbf{i}}_{j-1} - \lambda^{\mathbf{i}}_j + 2$ and $\lambda_{j+1} - \lambda_{j+2} = \lambda^{\mathbf{i}}_{j+1} - \lambda^{\mathbf{i}}_{j+2}$. Thus $(j, j+1) \in \Delta(\lambda)$ in all cases and our proof is complete. □

LEMMA 9. *If* $(i, i+1)$ *is a good* 2*-step for* λ, $\mathbf{i}$ *is an admissible sequence and* $(i, i+1) \in \Delta(\lambda^{\mathbf{i}})$ *then* $(i, i+1)$ *is a good* 2*-step for* $\lambda^{\mathbf{i}}$.

Proof. It suffices to prove the lemma when $\mathbf{i} = (i_1)$ is an admissible of length 1. If Case 1 occurs at index i_1 then $\lambda^{\mathbf{i}}_j - \lambda^{\mathbf{i}}_{j+1} \equiv \lambda_j - \lambda_{j+1} \mod 2$ for all j. Since $(i, i+1)$ is good for λ it follows that $\lambda^{\mathbf{i}}_{i-1} - \lambda^{\mathbf{i}}_i$ is odd (or $i = 1$) and $\lambda^{\mathbf{i}}_{i+1} - \lambda^{\mathbf{i}}_{i+2}$ is odd, so that $(i, i+1)$ is a good 2-step for $\lambda^{\mathbf{i}}$. Now suppose that Case 2 occurs for λ at index i_1. We may assume that $i_1 \neq i$. If $i_1 = i-1$ or $i_1 = i-2$ then $(i_1, i_1+1) \in \Delta(\lambda)$ implies $\epsilon(-1)^{\lambda_{i-1}} = -1$ and $\lambda_{i-1} - \lambda_i$ is even, contrary to the assumption that the 2-step $(i, i+1)$ is good for λ. Similarly, if $i_1 = i+1$ or $i_1 = i+2$ then $\lambda_{i+1} - \lambda_{i+2}$ is even, contradicting the assumption that $(i, i+1)$ is good. It follows that $i_1 < i-2$ or $i_1 > i+2$, whence it immediately follows that $(i, i+1)$ is a good 2-step for $\lambda^{\mathbf{i}}$. □

COROLLARY 3. *If* λ *is non-singular then* $\lambda^{\mathbf{i}}$ *is non-singular for any admissible sequence* $\mathbf{i}$.

Proof. If $(i, i+1) \in \Delta(\lambda^{\mathbf{i}})$ then $(i, i+1) \in \Delta(\lambda)$ by Lemma 8. Since λ is non-singular, $(i, i+1)$ is a good 2-step for λ. By Lemma 9, $(i, i+1)$ is good for $\lambda^{\mathbf{i}}$. □

3.3 The length of admissible sequences

In this section we give a combinatorial formula for the maximal length of admissible sequences for λ. The formula shall be of central importance to our results on sheets. First we shall need some further terminology related to partitions $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathcal{P}_\epsilon(N)$.

DEFINITION 2. A sequence $1 \leqslant i_1 < i_2 < \cdots < i_k < n$ with $k \geqslant 2$ is called a 2*-cluster* of λ whenever $(i_j, i_j+1) \in \Delta(\lambda)$ and $i_{j+1} = i_j + 2$ for all j. Analogous to the terminology for 2-steps, we say that a 2-cluster $i_1, \dots, i_k$ *has a bad boundary* if either of the following conditions holds:

- $\lambda_{i_1-1} - \lambda_{i_1} \in 2\mathbb{N}$;
- $\lambda_{i_k+1} - \lambda_{i_k+2} \in 2\mathbb{N}$

(if $i_1 = 1$ the the first condition should be omitted). A *bad* 2*-cluster* is one which has a bad boundary, whilst a *good* 2*-cluster* is one without a bad boundary.

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LEMMA 10. *A good 2-cluster is maximal in the sense that it is not a proper subsequence of any 2-cluster.*

Proof. If $i_1, \ldots, i_k$ is a good 2-cluster then $\lambda_{i_1-1}-\lambda_{i_1}, \lambda_{i_k+1}-\lambda_{i_k+2} \notin 2\mathbb{N}$. The fact that (i_1, i_1+1), $(i_k, i_k+1) \in \Delta(\lambda)$ means that $\epsilon(-1)^{\lambda_{i_1}} = \epsilon(-1)^{\lambda_{i_k+1}} = -1$. Combining these few observations, we get $\epsilon(-1)^{\lambda_{i_1-1}} = \epsilon(-1)^{\lambda_{i_k+2}} = 1$, and so $(i_1-2, i_1-1) \notin \Delta(\lambda)$ and $(i_k+2, i_k+3) \notin \Delta(\lambda)$. □

We introduce the notation

$$\Delta_{\text{bad}}(\lambda) := \{\text{the bad 2-steps of } \lambda\},$$
$$\Sigma(\lambda) := \{\text{the good 2-clusters of } \lambda\};$$

and write

$$z(\lambda) = s(\lambda) + |\Delta(\lambda)| - \big(|\Delta_{\text{bad}}(\lambda)| - |\Sigma(\lambda)|\big).$$

It is immediate from the definitions that $|\Delta_{\text{bad}}(\lambda)| \geqslant |\Sigma(\lambda)|$ and $|\Delta_{\text{bad}}(\lambda)| = |\Sigma(\lambda)|$ if and only if $\Delta_{\text{bad}}(\lambda) = \emptyset$.

LEMMA 11. $|\Sigma(\lambda)| \geqslant |\Sigma(\lambda^{\mathbf{i}})|$ *for length* 1 *admissible sequences* $\mathbf{i} = (i)$, *unless Case* 2 *occurs at* i *and*

$$i-4, i-2, i, i+2, i+4$$

is a subsequence of a good 2*-cluster, in which case* $|\Sigma(\lambda)| = |\Sigma(\lambda^{\mathbf{i}})| - 1$.

Proof. We make the notation $\mathbf{i} = (i)$. In this first paragraph we deal with the possibility that Case 1 occurs for λ at index i. Let us consider some necessary conditions for $\Sigma(\lambda) \neq \Sigma(\lambda^{\mathbf{i}})$. We require that $(i-1, i)$ or $(i+1, i+2)$ lies in $\Delta(\lambda)$, that the 2-steps $(i-1, i)$ or $(i+1, i+2)$ (or both) constitutes a 2-step in a good 2-cluster, and that $\lambda_i - \lambda_{i+1} = 2$. Let us assume these conditions. If precisely one of the two pairs $(i-1, i), (i+1, i+2)$ lies in $\Delta(\lambda)$ (we may assume $(i-1, i) \in \Delta(\lambda)$) then it follows that the good 2-cluster in question has the form $i_1 \leqslant \cdots \leqslant i_k = i-1$. But $\lambda_{i_k+1} - \lambda_{i_k+2} = 2$ then implies that the 2-cluster has a bad boundary, giving a contradiction. It follows that both $(i-1, i)$ and $(i+1, i+2)$ lie in $\Delta(\lambda)$. Then we have a good 2-cluster $i_1 \leqslant \cdots \leqslant i-1 = i_l \leqslant i_{l+1} = i+1 \leqslant \cdots \leqslant i_k$. However, the sequences $i_1, i_2, \ldots, i_{l-1}$ and $i_{l+2}, \ldots, i_{k-1}, i_k$ are either of length less than or equal to 1, or are bad 2-clusters for $\lambda^{\mathbf{i}}$, so $|\Sigma(\lambda)| = |\Sigma(\lambda^{\mathbf{i}})| + 1$.

Now suppose that Case 2 occurs at index i. Similar to the previous case, $\Sigma(\lambda)$ is only affected if $(i, i+1)$ is a bad 2-step in a good 2-cluster. If precisely one of the two pairs $(i-2, i-1)$ and $(i+2, i+3)$ lies in $\Delta(\lambda)$ (we may assume $(i-2, i-1) \in \Delta(\lambda)$) then such a 2-cluster will take the form $i_1, \ldots, i_k = i$. If $k > 2$ then $i_1, \ldots, i_{k-1}$ is a good 2-cluster for $\lambda^{\mathbf{i}}$, so that $|\Sigma(\lambda^{\mathbf{i}})| = |\Sigma(\lambda)|$. If $k = 2$ (we know $k \geqslant 2$) then the 2-cluster is eradicated by the iteration of the algorithm and $|\Sigma(\lambda^{\mathbf{i}})| = |\Sigma(\lambda)| - 1$.

Suppose that both $(i-2, i-1)$ and $(i+2, i+3)$ lie in $\Delta(\lambda)$. Then $\Sigma(\lambda)$ is unaffected unless $i_1, \ldots, i_j = i, \ldots, i_k$ is a good 2-cluster, which we shall assume henceforth. Note that $j \geqslant 2$ and $k-j \geqslant 1$ by assumption. If $j = 2$ and $k-j = 1$ then the good 2-cluster is no longer present for $\lambda^{\mathbf{i}}$ and $|\Sigma(\lambda)| = |\Sigma(\lambda^{\mathbf{i}})| - 1$. If $j > 2$ and $k-j = 1$ then $i_1, \ldots, i_{j-1}$ is a good 2-cluster for $\lambda^{\mathbf{i}}$ and $|\Sigma(\lambda)| = |\Sigma(\lambda^{\mathbf{i}})|$. The situation when $j = 2$ and $k-j > 1$ is very similar. In the final case $j > 2$, $k-j > 1$ and $i-4, i-2, i, i+2, i+4$ is a subsequence of a good 2-cluster, as in the statement of the lemma. Here both $i-2j, \ldots, i-2$ and $i, i+2, \ldots, i+2k$ are good 2-clusters for $\lambda^{\mathbf{i}}$, so that $|\Sigma(\lambda)| = |\Sigma(\lambda^{\mathbf{i}})| - 1$ as required. □

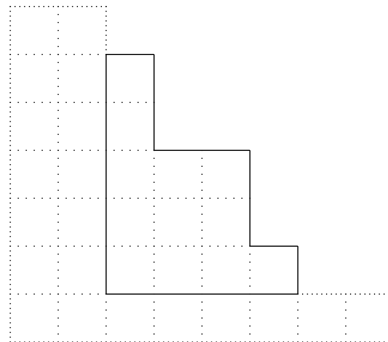

Figure 2. The dotted perimeter represents the Young diagram of the partition $\lambda = (7, 7, 6, 4, 4, 2, 1, 1) \in \mathcal{P}_{-1}(32)$. The solid perimeter represents the profile of λ of type $(3, 7)$.

Before continuing we shall need some notation. We define a construction which takes $\lambda \in \mathcal{P}_\epsilon(N)$ to $\lambda^S \in \mathcal{P}_\epsilon(N - 2k)$ for some $k \geqslant 0$. It is based entirely on application of the algorithm. The partition λ^S is called the *shell of* λ and is constructed as follows: for all $1 \leqslant i \leqslant n$ we apply Case 1 repeatedly; if $\lambda_i - \lambda_{i+1} \in 2\mathbb{N}$ and if $(i-1, i)$ or $(i+1, i+2)$ lie in $\Delta(\lambda)$ then apply Case 1 until $\lambda_i^{\mathbf{i}} - \lambda_{i+1}^{\mathbf{i}} = 2$; if we are not in the previous situation then apply Case 1 until $\lambda_i^{\mathbf{i}} - \lambda_{i+1}^{\mathbf{i}} \in \{0, 1\}$; finally apply Case 2 at every index i such that $(i, i+1)$ is a good 2-step. In order to keep the notation consistent we may regard S as the admissible sequence of indices (chosen in ascending order) used to construct λ^S.

Retain the convention $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\sum \lambda_i = N$. In order to make use of the shell λ^S we shall interest ourselves firstly in the set of partitions which equal their own shell $\lambda = \lambda^S$, and secondly in the relationship between a partition and its shell. It turns out that certain properties of a partition $\lambda = \lambda^S$ are controlled by the properties of certain special partitions constructed from λ. A *profile* μ of λ is a partition constructed in the following manner: choose indices (j, k) with $0 < j \leqslant k \leqslant n + 1$ such that $i = i'$ for all $j \leqslant i < k$, and such that $j - 1 \neq (j-1)'$ (or $j - 1 = 0$) and $k \neq k'$ (or $k = n + 1$). Define $\mu = (\mu_1, \ldots, \mu_{k-j})$ by the rule

$$\mu_i = \lambda_{i+(j-1)} - \lambda_k.$$

If $k < n + 1$ then in order to preserve the condition $i = i'$ we regard μ as an element of $\mathcal{P}_1(\sum_{i=j}^{k-1} \lambda_i - (k-j)\lambda_k)$. If $k = n + 1$ then $\lambda_k = 0$ and we may regard μ is an element of $\mathcal{P}_\epsilon(\sum_{i=j}^{n} \lambda_i)$. We say that the profile μ constructed in this manner is *of type* (j, k), and we include Figure 2 to show what is intended by the definition.

Suppose that μ is a profile of λ of type (j, k) and $\mathbf{i} = (i_1, \ldots, i_l)$ is an admissible sequence for μ. Then the *j-adjust* of $\mathbf{i}$ is the sequence

$$\jmath(\mathbf{i}) = (i_1 + (j-1), i_2 + (j-1), \ldots, i_l + (j-1)).$$

It is clear that $\jmath(\mathbf{i})$ is an admissible sequence for λ.

Proposition 4. *Suppose that λ is equal to its shell and let $\mu(1), \mu(2), \ldots, \mu(l)$ be a complete set of distinct profiles for λ, with $\mu(m)$ of type (j_m, k_m). Then the following hold:*

(1) *$z(\lambda) = \sum_{i=1}^{l} z(\mu(i))$;*
(2) *if $\mathbf{i}(m)$ is an admissible sequence for $\mu(m)$ then*

$$(\jmath_1(\mathbf{i}(1)), \jmath_2(\mathbf{i}(2)), \ldots, \jmath_l(\mathbf{i}(l)))$$

is an admissible sequence for λ, where this last sequence is obtained by concatenating the sequences $\jmath_m(\mathbf{i}(m))$.

Proof. Since $\lambda = \lambda^S$, all differences $\lambda_i - \lambda_{i+1}$ are equal to 0, 1 or 2. If $\lambda_i - \lambda_{i+1} = 2$ then necessarily $(i-1, i) \in \Delta(\lambda)$ or $(i+1, i+2) \in \Delta(\lambda)$. In either case $i = i'$, $i+1 = (i+1)'$ (or $i = n$) and it follows that there exists a profile of type (j, k) with $j \leqslant i$ and $i + 1 < k$ (or $i < k$ when $i = n$). Then each index i for which $\lambda_i - \lambda_{i+1} = 2$ contributes 1 to $s(\lambda)$ and 1 to $\sum_{j=1}^{l} s(\mu(j))$, so that $s(\lambda) = \sum_{j=1}^{l} s(\mu(j))$. The condition $\lambda = \lambda^S$ also implies that all 2-steps are bad 2-steps, so that $|\Delta(\lambda)| = |\Delta_{\mathrm{bad}}(\lambda)|$. Similarly, $\mu(m) = \mu(m)^S$ so $|\Delta(\mu(m))| = |\Delta_{\mathrm{bad}}(\mu(m))|$ for all m, and it remains to prove that $|\Sigma(\lambda)| = \sum_{i=1}^{l} |\Sigma(\mu(i))|$. This follows from the fact that all good 2-clusters $i_1 \leqslant \cdots \leqslant i_l$ fulfil $i = i'$ for all $i_1 \leqslant i \leqslant i_l + 1$, so for each such 2-cluster there exists a profile of type (j, k) with $j \leqslant i_1$ and $i_l + 1 < k$. Part (1) follows.

Part (2) actually holds even when $\lambda \neq \lambda^S$. For obvious reasons the indices of the distinct profiles do not overlap, and we may assume that $k_m < j_{m+1}$ for $m = 1, \ldots, l-1$. Then for $1 \leqslant i < l$ we set $\mathbf{j}(i) = (\jmath_1(\mathbf{i}(1)), \ldots, \jmath_i(\mathbf{i}(i)))$ and note that $\lambda_r^{\mathbf{j}(i)} = \lambda_r$ for all $r \geqslant j_{i+1}$. Using the fact that $\jmath_{i+1}(\mathbf{i}(i+1))$ is admissible for λ, we obtain by induction that $\jmath_{i+1}(\mathbf{i}(i+1))$ is an admissible sequence for $\lambda^{\mathbf{j}(i)}$. By the transitivity of the algorithm, we deduce then that $(\jmath_1(\mathbf{i}(1)), \jmath_2(\mathbf{i}(2)), \ldots, \jmath_l(\mathbf{i}(l)))$ is admissible for λ as required. □

Proposition 5. *Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition and suppose that $i = i'$ for all $1 \leqslant i \leqslant n$. Then there exists an admissible sequence for λ of length $z(\lambda)$.*

Proof. A partition λ fulfilling $i = i'$ for all $1 \leqslant i \leqslant n$ contains a good 2-cluster if and only if $1, 3, 5, \ldots, n-1$ is good 2-cluster. In this case it is the only good 2-cluster. Suppose that this is the case. Of course this implies that n is even and $\epsilon = 1$, so λ_n is odd. Construct a sequence $\mathbf{i}$ by repeatedly applying Case 1 at indices $2i - 1$ for $1 \leqslant i \leqslant n/2$, so that $\lambda_{2i-1} - \lambda_{2i} = 0$ for all such i. Then

$$|\mathbf{i}| = \sum_{i=1}^{n/2} \left\lfloor \frac{\lambda_{2i-1} - \lambda_{2i}}{2} \right\rfloor.$$

We construct an admissible sequence $\mathbf{i}'$ by subsequently applying Case 1 at indices $2i$ for $1 \leqslant i \leqslant n$, so that $\lambda_{2i} - \lambda_{2i+1} = 2$ for all such i. Our sequence $\mathbf{i}'$ has length

$$|\mathbf{i}'| = s(\lambda) - \left(\frac{n}{2} - 1\right).$$

At this point we are able to say precisely what $\lambda^{\mathbf{i}'}$ looks like. We have $\lambda^{\mathbf{i}'} = \lambda^S = (n-1, n-1, n-3, n-3, \ldots, 3, 3, 1, 1)$. Finally, we obtain $\mathbf{i}''$ by applying Case 2 precisely once at each index $2i - 1$ for $1 \leqslant i \leqslant n/2$. The partition $\lambda^{\mathbf{i}''}$ is rigid, so $\mathbf{i}''$ is maximal (Lemma 6) and

$$|\mathbf{i}''| = s(\lambda) + 1.$$

In order to complete this part of the proof we must show that $z(\lambda) = s(\lambda) + 1$. Notice that our assumptions on λ imply that every 2-step is bad. Therefore $|\Delta(\lambda)| = |\Delta_{\mathrm{bad}}(\lambda)|$, and by our original remarks $z(\lambda) = s(\lambda) + 1$ as required.

Now assume that λ has no good 2-clusters. Since $i = i'$ for all i we may apply Case 1 repeatedly at all indices to obtain a maximal admissible partition. Clearly $|\mathbf{i}| = s(\lambda)$. Once again all 2-steps are bad so that $|\Delta(\lambda)| = |\Delta_{\mathrm{bad}}(\lambda)|$, and by assumption $|\Sigma(\lambda)| = 0$. Hence $z(\lambda) = s(\lambda) = |\mathbf{i}|$ as promised. □

Theorem 8. *We have that*

$$z(\lambda) = \max |\mathbf{i}|$$

where the maximum is taken over all admissible sequences $\mathbf{i}$ *for* λ.

Proof. We begin by showing that $z(\lambda) \geqslant z(\lambda^{\mathbf{i}}) + 1$ where $\mathbf{i} = (i)$ is an admissible sequence of length 1 for λ. First assume Case 1 occurs for λ at i. Then $s(\lambda^{\mathbf{i}}) = s(\lambda) - 1$. Furthermore, if the iteration at i removes a 2-step (i.e. if $\lambda_i - \lambda_{i+1} = 2$ and either $(i-1, i) \in \Delta(\lambda)$ or $(i+1, i+2) \in \Delta(\lambda)$ or both) then that 2-step is bad. Therefore $|\Delta(\lambda)| - |\Delta(\lambda^{\mathbf{i}})| = |\Delta_{\text{bad}}(\lambda)| - |\Delta_{\text{bad}}(\lambda^{\mathbf{i}})|$. It remains to be seen that the number of good 2-clusters does not increase as we pass from λ to $\lambda^{\mathbf{i}}$. This follows from Lemma 11.

Now suppose that Case 2 occurs for λ at index i. Certainly if $(i, i+1)$ is a good 2-step then $z(\lambda^{\mathbf{i}}) = z(\lambda) - 1$, so we may assume that $(i, i+1)$ is a bad 2-step. Suppose first that this 2-step has precisely one bad boundary. We may assume that $\lambda_{i-1} - \lambda_i$ is even and $\lambda_{i+1} - \lambda_{i+2}$ is odd. We can deduce at this point that $s(\lambda^{\mathbf{i}}) = s(\lambda) - 1$ and $|\Delta(\lambda^{\mathbf{i}})| = |\Delta(\lambda)| - 1$. If $(i-2, i-1) \notin \Delta(\lambda)$ then $|\Delta_{\text{bad}}(\lambda^{\mathbf{i}})| = |\Delta_{\text{bad}}(\lambda)| - 1$. Similarly, if $(i-2, i-1) \in \Delta(\lambda)$ and $\lambda_{i-3} - \lambda_{i-2}$ is even then $|\Delta_{\text{bad}}(\lambda^{\mathbf{i}})| = |\Delta_{\text{bad}}(\lambda)| - 1$. In either of these two situations the number of good 2-clusters decreases, thanks to Lemma 11. Hence $z(\lambda) \geqslant z(\lambda^{\mathbf{i}}) + 1$ once again. We must now consider the possibility that $(i-2, i-1) \in \Delta(\lambda)$ and $\lambda_{i-3} - \lambda_{i-2}$ is odd. In this situation $s(\lambda^{\mathbf{i}}) = s(\lambda) - 1$, $|\Delta(\lambda^{\mathbf{i}})| = |\Delta(\lambda)| - 1$ and $|\Delta_{\text{bad}}(\lambda^{\mathbf{i}})| = |\Delta_{\text{bad}}(\lambda)| - 2$. Notice that $i-2, i$ is a good 2-cluster for λ but not for $\lambda^{\mathbf{i}}$, so that $|\Sigma(\lambda^{\mathbf{i}})| = |\Sigma(\lambda)| - 1$ and $z(\lambda^{\mathbf{i}}) = z(\lambda) - 1$. A similar argument works when $\lambda_{i-1} - \lambda_i$ is odd but $\lambda_{i+1} - \lambda_{i+2} = 2$.

Now we assume that $(i, i+1)$ is a bad 2-step and that both boundaries are bad. If neither $(i-2, i-1)$ nor $(i+2, i+3)$ lies in $\Delta(\lambda)$ then $s(-)$ decreases by 2, $|\Delta(-)|$ decreases by 1, and $|\Delta_{\text{bad}}(-)|$ decreases by 1 upon passing from λ to $\lambda^{\mathbf{i}}$. Certainly $|\Sigma(-)|$ may only decrease, by Lemma 11, and so $z(\lambda) \geqslant z(\lambda^{\mathbf{i}}) + 1$ in this situation. Now move on and suppose that precisely one of $i-2$ and $i+2$ lies in $\Delta(\lambda)$. We shall examine the case $(i-2, i-1) \in \Delta(\lambda)$, the other being very similar.

When $\lambda_{i-3} - \lambda_{i-2}$ is odd, $s(\lambda^{\mathbf{i}}) = s(\lambda) - 2$, $|\Delta(\lambda^{\mathbf{i}})| = |\Delta(\lambda)| - 1$ and $|\Delta_{\text{bad}}(\lambda^{\mathbf{i}})| = |\Delta_{\text{bad}}(\lambda)| - 2$ (since $(i-2, i-1)$ is no longer a bad 2-step after this iteration). Furthermore, $(i, i+1)$ cannot make up a 2-step in a good 2-cluster since $(i+2, i+3) \notin \Delta(\lambda)$ and $\lambda_{i+1} - \lambda_{i+2}$ is even, therefore $|\Sigma(\lambda)|$ remains unchanged. So consider the possibility that $(i-2, i-1)$ has two bad boundaries: that $\lambda_{i-3} - \lambda_{i-2}$ is even. Then our conclusions are exactly the same as before, except that $|\Delta_{\text{bad}}(\lambda^{\mathbf{i}})| = |\Delta_{\text{bad}}(\lambda)| - 1$. In either situation $z(\lambda^{\mathbf{i}}) \geqslant z(\lambda) - 1$.

Finally, we have the situation $(i-2, i-1), (i+2, i+3) \in \Delta(\lambda)$. Once again we must distinguish between the number of bad boundaries attached to the 2-steps $(i-2, i-1)$ and $(i+2, i+3)$. Suppose that both of these 2-steps have a single bad boundary (they have at least one). Then $i-2, i, i+2$ is a good 2-cluster. It is immediately clear upon passing from λ to $\lambda^{\mathbf{i}}$ that $s(\lambda^{\mathbf{i}}) = s(\lambda) - 2$, $|\Delta(\lambda^{\mathbf{i}})| = |\Delta(\lambda)| - 1$, $|\Delta_{\text{bad}}(\lambda^{\mathbf{i}})| = |\Delta_{\text{bad}}(\lambda)| - 3$, and $|\Sigma(\lambda^{\mathbf{i}})| = |\Sigma(\lambda)| - 1$. Once again $z(\lambda^{\mathbf{i}}) \geqslant z(\lambda) - 1$ follows. The last two situations to consider are when precisely one of the two 2-steps $(i-2, i-1)$ and $(i+2, i+3)$ has two bad boundaries, and when both of them have two bad boundaries.

Take the former situation. We may assume that $(i-2, i-1)$ has two bad boundaries, and $(i+2, i+3)$ has one (the opposite configuration is similar). Upon iterating the algorithm, $s(\lambda^{\mathbf{i}}) = s(\lambda) - 2$, $|\Delta(\lambda^{\mathbf{i}})| = |\Delta(\lambda)| - 1$ and $|\Delta_{\text{bad}}(\lambda^{\mathbf{i}})| = |\Delta_{\text{bad}}(\lambda)| - 2$. By Lemma 11, $|\Sigma(\lambda^{\mathbf{i}})| \leqslant |\Sigma(\lambda)|$. In the final case $(i-2, i-1)$ and $(i+2, i+3)$ both have two bad boundaries. The outcome is that $s(\lambda^{\mathbf{i}}) = s(\lambda) - 2$, $|\Delta(\lambda^{\mathbf{i}})| = |\Delta(\lambda)| - 1$ and $|\Delta_{\text{bad}}(\lambda^{\mathbf{i}})| = |\Delta_{\text{bad}}(\lambda)| - 1$ both decrease by 1, and by Lemma 11 either $|\Sigma(\lambda^{\mathbf{i}})| = |\Sigma(\lambda)|$ or $|\Sigma(\lambda^{\mathbf{i}})| = |\Sigma(\lambda)| + 1$.

We have eventually shown that $z(\lambda) \geqslant z(\lambda^{\mathbf{i}}) + 1$. Recall that for any maximal admissible sequence $\mathbf{i}$ the partition $\lambda^{\mathbf{i}}$ is rigid. Also notice that $z(\lambda) = 0$ for any rigid partition λ. We deduce, for any maximal admissible sequence $\mathbf{i}$ of length l, that

$$z(\lambda) \geqslant z(\lambda^{\mathbf{i}_2}) + 1 \geqslant z(\lambda^{\mathbf{i}_3}) + 2 \geqslant \cdots z(\lambda^{\mathbf{i}_{l+1}}) + l = l.$$

Here $\mathbf{i}_k$ denotes $(i_1, \ldots, i_{k-1})$. In order to complete the proof we exhibit a maximal admissible sequence of length $z(\lambda)$. This requires some reductions.

Notice first that $z(\lambda)$ decreases by 1 at each iteration when we apply Case 1 in constructing the shell λ^S. Therefore we may assume that $\lambda = \lambda^S$. Let $\mu(1), \mu(2), \ldots, \mu(l)$ be a complete set of distinct profiles for λ, as in the statement of Proposition 4. By Proposition 5 we know that for each $1 \leqslant m \leqslant l$ there is an admissible sequence of length $z(\mu(m))$ for $\mu(m)$. Using part (2) of Proposition 4 we obtain an admissible sequence for λ of length $\sum_{i=1}^{l} z(\mu(i))$, and by part (1) of the same proposition that length is equal to $z(\lambda)$. Hence a sequence of the correct length exists, and the theorem follows. □

The following corollary shall be of some importance to our later work.

Corollary 4. *For all $\lambda \in \mathcal{P}_\epsilon(N)$ the following hold:*

(1) $c(\lambda) \geqslant z(\lambda)$;
(2) $c(\lambda) = z(\lambda)$ *if and only if λ is non-singular.*

Proof. Part (1) follows from the fact that $|\Delta_{\mathrm{bad}}(\lambda)| > |\Sigma(\lambda)|$ for all partitions λ. For part (2) we observe that $|\Delta_{\mathrm{bad}}(\lambda)| - |\Sigma(\lambda)| = 0$ if and only if λ is non-singular. □

4. A geometric interpretation of the algorithm

We would like to characterise the non-singular partitions in geometric terms. This characterisation is given in the corollary to the next theorem. The remainder of this section will be spent preparing to prove that theorem. The symmetric group $\mathfrak{S}_l$ acts on the set of sequences in $\{1, \ldots, n\}$ of length l by the rule $\sigma(i_1, \ldots, i_l) = (i_{\sigma(1)}, \ldots, i_{\sigma(l)})$. Let

$$\Phi_\lambda := \{\text{the maximal admissible sequences for } \lambda\}/\!\sim$$

where $\mathbf{i} \sim \mathbf{j}$ if $\mathbf{i}$ and $\mathbf{j}$ have equal length l and are $\mathfrak{S}_l$ conjugate. What follows is the main theorem of this section.

Theorem 9. *The following are true for any $\lambda \in \mathcal{P}_\epsilon(N)$:*

(1) *$e(\lambda)$ lies in $|\Phi_\lambda|$ distinct sheets;*
(2) *$|\Phi_\lambda| = 1$ if and only if λ is non-singular.*

The next corollary explains our choice of terminology.

Corollary 5. *Suppose that $\lambda \in \mathcal{P}_\epsilon(N)$ and $\dim \mathfrak{k}_{e(\lambda)} = m$. Then the following are equivalent:*

(1) *the partition λ is non-singular;*
(2) $c(\lambda) = z(\lambda)$;
(3) *$e(\lambda)$ lies in a unique sheet.*

If the base field $\Bbbk$ has characteristic 0 or $\mathrm{char}(\Bbbk) = p \gg 0$ then (1), (2) and (3) hold if and only if $e(\lambda)$ is a non-singular point on the quasi-affine variety $\mathfrak{k}^{(m)}$.

Proof. Statements (1), (2) and (3) are equivalent by Theorems 8, Corollary 4 and Theorem 9. Now suppose that the characteristic of $\Bbbk$ is either zero or $\mathrm{char}(\Bbbk) = p \gg 0$. Then Im Hof proved in [ImH05, ch. 6] that all sheets of $\mathfrak{k}^{(m)}$ are smooth algebraic varieties. (Im Hof assumes that $\mathrm{char}(\Bbbk) = 0$, but his arguments extend easily to the case where $\mathrm{char}(\Bbbk)$ is sufficiently large.) In view of our discussion in § 3.1, Im Hof's result implies that all irreducible components of $\mathfrak{k}^{(m)}$ are smooth algebraic varieties. In this situation it follows from [Sha94, ch. II, § 2, Theorem 6] that $e = e(\lambda)$ is a non-singular point of the algebraic variety $\mathfrak{k}^{(m)}$ if and only if e belongs to a unique irreducible component of $\mathfrak{k}^{(m)}$. This completes the proof. □

We shall now assemble all of the necessary information required to prove Theorem 9. We start by recalling some facts regarding sheets and induced orbits. A short survey of these topics can be found in [Mor08]. For a full discussion in the characteristic 0 case, see [CM93] or [TY05]. Since every sheet of $\mathfrak{k}$ contains a dense decomposition class we have the following theorem.

THEOREM 10 (See [Bor81]). *There is a one-to-one correspondence between the set of sheets of $\mathfrak{k}$ and the K-conjugacy classes of pairs $(\mathfrak{l}, \mathcal{O}_\mathfrak{l})$ where $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{k}$ and $\mathcal{O}_\mathfrak{l}$ is a rigid nilpotent orbit in $\mathfrak{l}$.*

We shall say that a sheet $\mathcal{S}$ of $\mathfrak{k}$ *has data* $(\mathfrak{l}, \mathcal{O}_\mathfrak{l})$ if $\mathcal{S}$ is identified with $(\mathfrak{l}, \mathcal{O}_\mathfrak{l})$ under the above correspondence. In view of our discussion in § 3.1, this means that $\mathcal{S}$ contains an open decomposition class of the form $\mathcal{D}(\mathfrak{l}, e_0)$ with $e_0 \in \mathcal{O}_\mathfrak{l}$.

Let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ be a parabolic subalgebra of $\mathfrak{k}$ with $\mathfrak{l}$ a Levi subalgebra of $\mathfrak{k}$ and $\mathfrak{n}$ the nilradical of $\mathfrak{p}$. Let $\mathcal{O}_\mathfrak{l}$ be a nilpotent orbit in $\mathfrak{l}$. Since the orbit set $\mathcal{N}/K$ is finite there exists a unique nilpotent orbit in $\mathfrak{k}$ which meets the irreducible quasi-affine variety $\mathcal{O}_\mathfrak{l} + \mathfrak{n} \subset \mathcal{N}(\mathfrak{k})$ in a dense open subset. This orbit, denoted by $\mathrm{Ind}_\mathfrak{l}^\mathfrak{k}(\mathcal{O}_\mathfrak{l})$, is said to be *induced* from the orbit $\mathcal{O}_\mathfrak{l}$.

We record three pieces of information regarding induced orbits.

PROPOSITION 6 (See [BK79, Bor81, LS79]). *The following are true:*

1. *if $\mathcal{S}$ is a sheet with data $(\mathfrak{l}, \mathcal{O}_\mathfrak{l})$ then $\mathrm{Ind}_\mathfrak{l}^\mathfrak{k}(\mathcal{O}_\mathfrak{l})$ is the unique nilpotent orbit contained in $\mathcal{S}$;*
2. *for each nilpotent orbit $\mathcal{O} \subseteq \mathfrak{k}$ we have that $\mathcal{O} = \mathrm{Ind}_\mathfrak{k}^\mathfrak{k}(\mathcal{O})$;*
3. *if $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are Levi subalgebras of $\mathfrak{k}$, $\mathcal{O}$ is a nilpotent orbit in $\mathfrak{l}_1$ and $\mathfrak{l}_1 \subseteq \mathfrak{l}_2$, then*

$$\mathrm{Ind}_{\mathfrak{l}_2}^\mathfrak{k}(\mathrm{Ind}_{\mathfrak{l}_1}^{\mathfrak{l}_2}(\mathcal{O})) = \mathrm{Ind}_{\mathfrak{l}_1}^\mathfrak{k}(\mathcal{O}).$$

Fix a partition $\lambda \in \mathcal{P}_\epsilon(N)$. We aim to classify the K-conjugacy classes of pairs $(\mathfrak{l}, \mathcal{O})$ where $\mathfrak{l} \subseteq \mathfrak{k}$ is a Levi subalgebra and $\mathcal{O} \subseteq \mathfrak{l}$ is a rigid nilpotent orbit, such that $\mathcal{O}_{e(\lambda)} = \mathrm{Ind}_\mathfrak{l}^\mathfrak{k}(\mathcal{O})$. In view of part (1) of the above proposition this shall parameterise the set of sheets containing $e(\lambda)$. In order to begin this classification we shall require some general facts about Levi subalgebras of $\mathfrak{k}$.

Every Levi subalgebra is conjugate to a standard Levi subalgebra. If $\mathfrak{t} \subset \mathfrak{k}$ is a maximal torus and Π a fixed basis of simple roots associated with $(\mathfrak{k}, \mathfrak{t})$ then a standard Levi subalgebra is constructed from a subset $\Pi_0 \subseteq \Pi$. To each such Π_0 we attach the Levi subalgebra $\mathfrak{l}$ generated by $\mathfrak{t}$ and the roots spaces $\mathfrak{k}_{\pm\gamma}$ with $\gamma \in S$. Now order the simple roots in Π in the usual manner and let $\mathbf{i} = (i_1, \dots, i_l)$ be a sequence with $\sum_j i_j \leqslant \mathrm{rank}\, \mathfrak{k}$. Such sequences are in a bijection with the subsets of Π by letting $\Pi_\mathbf{i} = \Pi \backslash \{\alpha_{i_1 + \cdots + i_k} \colon 1 \leqslant k \leqslant l\}$. It is easy to check that in types B and C the standard Levi subalgebra constructed from $\Pi_\mathbf{i}$ is isomorphic to $\mathfrak{gl}_{i_1} \times \cdots \times \mathfrak{gl}_{i_l} \times \mathfrak{m}$ where $\mathfrak{m}$ is a classical algebra. If $\sum_j i_j = \mathrm{rank}\, \mathfrak{k} - 1$ in type D then the Levi subalgebra constructed from $\Pi_\mathbf{i}$ is actually isomorphic to $\mathfrak{gl}_{i_1} \times \cdots \times \mathfrak{gl}_{i_{l-1}} \times \mathfrak{gl}_{i_l + 1}$. If we define another sequence $\mathbf{i}' = (i_1, \dots, i_{l-1}, i_l + 1)$ then the Levi subalgebras constructed from $\mathbf{i}$ and $\mathbf{i}'$ are isomorphic. When all

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terms of $\mathbf{i}'$ are even, these standard Levi subalgebras are not conjugate and we shall label their respective conjugacy classes I and II.

When we refer to a Levi subalgebra by its isomorphism type we shall implicitly be referring to a standard Levi subalgebra constructed from a subset of Π. Let us record these conclusions formally.

Lemma 12 (See [CM93, Kem83, Mor08]). *The following are true.*

(1) *Every Levi subalgebra of $\mathfrak{k}$ is K-conjugate to a Lie algebra of the form*

$$\mathfrak{gl}_{\mathbf{i}} \times \mathfrak{m} := \mathfrak{gl}_{i_1} \oplus \cdots \oplus \mathfrak{gl}_{i_l} \oplus \mathfrak{m} \cong \mathfrak{gl}_{i_1} \times \cdots \times \mathfrak{gl}_{i_l} \times \mathfrak{m}$$

where $\mathbf{i} = (i_1, \ldots, i_l)$ is a sequence of integers with $\sum_j i_j \leqslant \operatorname{rank} \mathfrak{k}$ and where $\mathfrak{m}$ has the same type as $\mathfrak{k}$ and a standard representation of dimension $R_{\mathbf{i}} := N - 2\sum_j i_j$ (under the restriction that $R_{\mathbf{i}} \neq 2$ if $\epsilon = 1$). If $\mathfrak{k}$ has type D, $R_{\mathbf{i}} = 0$ and all parts of $\mathbf{i}$ are even, then there are two K-conjugacy classes of Levi subalgebras isomorphic to $\mathfrak{gl}_{\mathbf{i}} \times \mathfrak{m}$. They are assigned labels I *and* II. *Otherwise there is a unique K-conjugacy class of Levi subalgebras isomorphic to $\mathfrak{gl}_{\mathbf{i}} \times \mathfrak{m}$.*

(2) *If $\mathfrak{l}$ is a Levi subalgebra as in part* (1) *then the rigid nilpotent orbits in $\mathfrak{l}$ take the form*

$$\mathcal{O} = \underbrace{\mathcal{O}_0 \times \cdots \times \mathcal{O}_0}_{l \text{ times}} \times \mathcal{O}_{e(\mu)}$$

with $\mu \in \mathcal{P}^*_\epsilon(N - 2\sum_j i_j)$ a rigid partition.

Let Ψ_λ denote the set of all K-conjugacy classes of pairs $(\mathfrak{l}, \mathcal{O})$ where $\mathfrak{l} = \mathfrak{gl}_{i_1} \oplus \cdots \oplus \mathfrak{gl}_{i_l} \oplus \mathfrak{m} \cong \mathfrak{gl}_{i_1} \times \cdots \times \mathfrak{gl}_{i_l} \times \mathfrak{m}$ is a Levi subalgebra of $\mathfrak{k}$ and $\mathcal{O} = \mathcal{O}_0 \times \cdots \times \mathcal{O}_0 \times \mathcal{O}_{e(\mu)}$ a rigid nilpotent orbit in $\mathfrak{l}$, such that $\mathcal{O}_{e(\lambda)} = \operatorname{Ind}_{\mathfrak{l}}^{\mathfrak{k}}(\mathcal{O})$.

Lemma 13. *$e(\lambda)$ lies in $|\Psi_\lambda|$ distinct sheets.*

Proof. Let $\mathcal{S}$ be a sheet of $\mathfrak{k}$ with data $(\mathfrak{l}, \mathcal{O})$. By Theorem 10 and part (1) of Proposition 6 we see that $e(\lambda) \in \mathcal{S}$ if and only if $e(\lambda) = \operatorname{Ind}_{\mathfrak{l}}^{\mathfrak{k}}(\mathcal{O})$. By Lemma 12 we have $\mathfrak{l} \cong \mathfrak{gl}_{i_1} \times \cdots \times \mathfrak{gl}_{i_l} \times \mathfrak{m}$ and $\mathcal{O} = \mathcal{O}_0 \times \cdots \mathcal{O}_0 \times \mathcal{O}_{e(\mu)} \subseteq \mathfrak{l}$. □

We now briefly discuss the partitions associated with induced orbits. The result stated below may be deduced from [CM93, Corollary 7.3.3]. We warn the reader that when interpreting the proposition for the Lie algebras of type B the unique nilpotent orbit in the trivial algebra $\mathfrak{so}_1$ is labelled by the partition $\lambda = (1)$, contrary to the common convention. Furthermore, our description of labels attached to induced orbits does not quite agree with the description in [CM93]; see Remark 3 for more detail.

Recall that the natural representation of $\mathfrak{k}$ is of dimension N.

Proposition 7. *Choose a positive integer i with $2i \leqslant N$ and let $\mathfrak{l} = \mathfrak{gl}_i \oplus \mathfrak{m} \cong \mathfrak{gl}_i \times \mathfrak{m}$ be a maximal Levi subalgebra of $\mathfrak{k}$. Let $\mathcal{O} = \mathcal{O}_0 \times \mathcal{O}_\mu$ be a nilpotent orbit in $\mathfrak{l}$ where $\mathcal{O}_\mu$ has partition $\mu \in \mathcal{P}_\epsilon(N - 2i)$. Then $\mathcal{O}_{e(\lambda)} = \operatorname{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O})$ has associated partition λ where λ is obtained from μ by the following procedure: add 2 to the first i columns of μ (extending by zero if necessary); if the resulting partition lies in $\mathcal{P}_\epsilon(N)$ then we have found λ, otherwise we obtain λ by subtracting 1 from the ith column and adding 1 to the $(i+1)$th.*

Now suppose that we are in type D and λ is very even. Then either μ is very even or $N = 2i$ and $\operatorname{rank} \mathfrak{k}$ is even. If $N > 2i$ then $\mathcal{O}_{e(\lambda)}$ inherits its label from μ, whilst if $N = 2i$ then the induced orbit inherits its label from $\mathfrak{l}$.

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Remark 3. The above proposition is based on [CM93, Theorem 7.3.3]. However, the reader will notice that the way in which the labels are chosen does not coincide with that theorem. The reason for this is that [CM93] contains two misprints which we must now amend.[1]

The first problem stems from comparing Lemma 5.3.5 and Theorem 7.3.3(ii) in [CM93]. We see, given the conventions of [CM93, Lemma 5.3.5], that [CM93, Theorem 7.3.3(ii)] should actually state that the label of $\mathrm{Ind}_{\mathfrak{gl}_i \oplus \mathfrak{m}}^{\mathfrak{k}}(\mathcal{O})$ is *different* from the label of $\mathcal{O}$ when $(\mathrm{rank}\,\mathfrak{k} + \mathrm{rank}\,\mathfrak{m})/2$ is odd. We could, of course, change [CM93, Theorem 7.3.3(ii)], but a better amendment is to change [CM93, Lemma 5.3.5] so that the labelling convention for very even orbits is independent of n: in the notation of [CM93] we take $a = 2$ and $b = 0$ regardless of n. With this convention the statement of [CM93, Theorem 7.3.3(ii)] is correct. However, [CM93, Theorem 7.3.3(iii)] should now state that the label of the induced orbit *coincides* with the label of a Levi subalgebra from which it is induced. This is the convention we have followed in the above proposition.

The second misprint concerns the number of conjugacy classes of maximal Levi subalgebras in [CM93, Lemma 7.3.2(ii)]. The reader will notice that when $\mathfrak{k} = \mathfrak{so}_{2\ell}$ and ℓ is odd, the longest element of the Weyl group w_0 is the negative of the outer Dynkin automorphism of the root system. Therefore if $gT = w_0 \in W = N_K(T)/T$ then $\mathrm{Ad}\, g$ exchanges the Levi subalgebras which are labelled I and II in this case. This confirms that there is just one class of Levi subalgebras of type $\mathfrak{gl}_\ell$ when ℓ is odd. When ℓ is even there are two such classes and our convention for labelling conjugacy classes of Levi subalgebras in Lemma 12 is a natural extension of [CM93, Lemma 7.3.2].

In light of the above proposition, we may explain the definition of the algorithm. We fix λ and want to decide when is it possible to find a pair consisting of a maximal Levi $\mathfrak{l} = \mathfrak{gl}_{i_1} \oplus \mathfrak{m} \cong \mathfrak{gl}_{i_1} \times \mathfrak{m}$ and a nilpotent orbit $\mathcal{O} = \mathcal{O}_0 \times \mathcal{O}_{e(\mu)}$ (with partition μ) such that $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}) = \mathcal{O}_{e(\lambda)}$. It is now clear that this occurs precisely when we have an admissible index i and a Levi subalgebra isomorphic to $\mathfrak{gl}_i \times \mathfrak{m}$. In this case $\mu = \lambda^{(i)}$ and if $\mathcal{O}_{e(\mu)}$ has a label then it is completely determined by that of $\mathcal{O}_{e(\lambda)}$. The precise statement is as follows.

Corollary 6. *Let $\lambda \in \mathcal{P}_\epsilon(N)$. Suppose that there exists a maximal Levi $\mathfrak{l} \cong \mathfrak{gl}_i \oplus \mathfrak{m} \cong \mathfrak{gl}_i \times \mathfrak{m}$. Then the following are equivalent.*

1. *i is an admissible index for λ. If $\mathfrak{k}$ has type D and there are two conjugacy classes of Levi subalgebras isomorphic to $\mathfrak{gl}_i \times \mathfrak{m}$ then $\mathfrak{l}$ belongs to the conjugacy class with the same label as $\mathcal{O}_{e(\lambda)}$.*
2. *There exists an orbit $\mathcal{O} = \mathcal{O}_0 \times \mathcal{O}_{e(\mu)}$ with $\mathcal{O}_{e(\lambda)} = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{k}}(\mathcal{O})$.*

If these two equivalent conditions hold then $\mathcal{O}_{e(\mu)}$ has partition $\mu = \lambda^{(i)}$. Furthermore, for every other orbit $\tilde{\mathcal{O}} = \mathcal{O}_0 \times \mathcal{O}_{e(\tilde{\mu})}$ in $\mathfrak{l}$ such that $\mathcal{O}_{e(\lambda)} = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{k}}(\tilde{\mathcal{O}})$, we have $(\mathfrak{l}, \mathcal{O})/K = (\mathfrak{l}, \tilde{\mathcal{O}})/K$.

Proof. Assume throughout that $\mathfrak{l} \cong \mathfrak{gl}_i \times \mathfrak{m}$ exists and let $\mathcal{O} = \mathcal{O}_0 \times \mathcal{O}_{e(\mu)} \subseteq \mathfrak{l}$. The previous proposition implies that if λ is the partition of $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{k}}(\mathcal{O}_0 \times \mathcal{O}_{e(\mu)})$ then $\lambda^{(i)} = \mu$. Suppose that (1) holds. Then the partition of $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{k}}(\mathcal{O}_0 \times \mathcal{O}_{e(\lambda^{(i)})})$ is λ. If λ is not very even then (2) follows. If we are in type D and λ is very even then according to the previous proposition either $\lambda^{(i)}$ is very even or $\mathfrak{l} \cong \mathfrak{gl}_i$ where $i = N/2 = \mathrm{rank}\,\mathfrak{k}$ is even. In the first case the orbit $\mathcal{O}_0 \times \mathcal{O}_{e(\lambda^{(i)})}$ with the same label as $\mathcal{O}_{e(\lambda)}$ induces to $\mathcal{O}_{e(\lambda)}$, whilst in the second case there is a unique orbit of the correct form (the zero orbit) and since the labels of $\mathfrak{l}$ and $\mathcal{O}_{e(\lambda)}$ coincide, it induces up to $\mathcal{O}_{e(\lambda)}$.

[1] We are grateful to Monty McGovern for this clarification.

Now suppose that (2) holds. Since $\mu = \lambda^{(i)}$ the index i is certainly admissible for λ. If there are two conjugacy classes of Levi subalgebras then again $\mathfrak{l} \cong \mathfrak{gl}_i$, and so $\mathcal{O}_{e(\lambda)} = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{k}}(\mathcal{O})$ implies that the labels of $\mathfrak{l}$ and $\mathcal{O}_{e(\lambda)}$ coincide by the last part of the previous proposition.

The statement that $\mu = \lambda^{(i)}$ is immediate from the above discussion. Fix $\mathcal{O} = \mathcal{O}_0 \times \mathcal{O}_{e(\mu)}$ fulfilling $\mathcal{O}_{e(\lambda)} = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{k}}(\mathcal{O})$. We must show for every other orbit of the form $\tilde{\mathcal{O}} = \mathcal{O}_0 \times \mathcal{O}_{e(\tilde{\mu})}$ fulfilling $\mathcal{O}_{e(\lambda)} = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{k}}(\tilde{\mathcal{O}})$ that the pair $(\mathfrak{l}, \tilde{\mathcal{O}})$ is K-conjugate to $(\mathfrak{l}, \mathcal{O})$. Since we know that $\mu = \lambda^{(i)} = \tilde{\mu}$ this is now obvious unless μ is very even and λ is not very even.

So suppose that this is the case. We claim that in this situation any admissible sequence $\mathbf{i}$ for λ has an odd term. Indeed, in order to see this it suffices to assume that $\mathbf{i} = (i)$ has length 1. Since $\lambda^{\mathbf{i}}$ is very even or empty we conclude that $(i, i+1)$ must be the only 2-step for λ. If $\lambda^{\mathbf{i}}$ is empty then $i = 1$. Assume not. Since the parts of λ which precede λ_i are all even, they must come in pairs and so i must be odd. The claim follows.

Since we are assuming that μ is very even and λ is not, the above shows that i is odd. We know that $\mathrm{rank}\,\mathfrak{m} = (N-2i)/2$ is even. From this we can be sure that $N/2 = \mathrm{rank}\,\mathfrak{k}$ is odd. Now from the tables in [Bou68] we see that the longest element w_0 of the Weyl group $W = N_K(T)/T$ is the negative of the outer diagram automorphism of the root system of $\mathfrak{k}$. Therefore if $gT = w_0$ then $\mathrm{Ad}\,g$ will preserve $\mathfrak{l}$ and exchange the orbits with partition $\lambda^{(i)}$ labelled I and II. This complete the proof. □

The following proposition uses a similar kind of induction to that in [Mor08, Proposition 3.7] and is central to our proof of Theorem 9.

Proposition 8. *Let $\mathbf{i} = (i_1, \ldots, i_l)$ be a sequence of integers with $\sum_j i_j \leqslant \mathrm{rank}\,\mathfrak{k}$. Suppose that there exists a Levi subalgebra $\mathfrak{l} \cong \mathfrak{gl}_{\mathbf{i}} \times \mathfrak{m}$. Then following are equivalent.*

(1) *$\mathbf{i}$ is an admissible sequence for λ. If $\mathfrak{k}$ has type D and there are two conjugacy classes of Levi subalgebras isomorphic to $\mathfrak{gl}_{\mathbf{i}} \times \mathfrak{m}$ then $\mathfrak{l}$ belongs to the conjugacy class with the same label as $\mathcal{O}_{e(\lambda)}$.*

(2) *There exists an orbit $\mathcal{O} = \mathcal{O}_0 \times \cdots \times \mathcal{O}_0 \times \mathcal{O}_{e(\mu)}$ with $\mathcal{O}_{e(\lambda)} = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{k}}(\mathcal{O})$.*

If these two equivalent conditions hold then $\mathcal{O}_{e(\mu)}$ has partition $\mu = \lambda^{\mathbf{i}}$. Furthermore, for every other orbit $\tilde{\mathcal{O}} \subseteq \mathfrak{l}$ with $\tilde{\mathcal{O}} = \mathcal{O}_0 \times \cdots \times \mathcal{O}_0 \times \mathcal{O}_{e(\tilde{\mu})}$ such that $\mathcal{O}_{e(\lambda)} = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{k}}(\tilde{\mathcal{O}})$, we have $(\mathfrak{l}, \mathcal{O})/K = (\mathfrak{l}, \tilde{\mathcal{O}})/K$.

Proof. The proof proceeds by induction on l. When $l = 0$ we have $\mathfrak{l} = \mathfrak{k}$ and the proposition holds by part (2) of Proposition 6 (note that $\lambda^{\emptyset} = \lambda$). If $\mathfrak{l}$ is a proper Levi subalgebra then $l > 0$. The case $l = 1$ is simply the previous corollary. The inductive step is quite similar, although to begin with we must exclude the possibility that $R_{\mathbf{i}} = 0$ and $N - 2\sum_{j=1}^{l-1} i_j = 2$ in type D. We will treat this possibility at the end.

Suppose that the proposition has been proven for all $l' < l$. Since we have excluded this anomalous case in type D we may set $\mathbf{i}' = (i_1, \ldots, i_{l-1})$ and deduce that there exists a Levi $\mathfrak{l}' \cong \mathfrak{gl}_{\mathbf{i}'} \oplus \mathfrak{m}'$ where $\mathfrak{m}'$ has a natural representation of dimension $R_{\mathbf{i}'}$ and the same type as $\mathfrak{k}$. Let M' be the closed subgroup of K with $\mathfrak{m}' = \mathrm{Lie}(M')$. We may ensure that $\mathfrak{l} \subseteq \mathfrak{l}'$ by embedding $\mathfrak{gl}_{i_l} \oplus \mathfrak{m}$ in $\mathfrak{m}'$.

Suppose that $\mathbf{i}$ is admissible and, if possible, that the label of $\mathfrak{l}$ coincides with that of $\mathcal{O}_{e(\lambda)}$. We deduce that $\mathbf{i}'$ is also admissible, and since $R_{\mathbf{i}'} > 0$ there is a unique class of Levi subalgebras isomorphic to $\mathfrak{gl}_{\mathbf{i}'} \oplus \mathfrak{m}'$. By the inductive hypothesis we deduce that there exists an orbit $\mathcal{O}' = \mathcal{O}_0 \times \cdots \times \mathcal{O}_0 \times \mathcal{O}_{e(\tau)} \subseteq \mathfrak{l}'$ with $\mathcal{O}_{e(\lambda)} = \mathrm{Ind}_{\mathfrak{l}'}^{\mathfrak{k}}(\mathcal{O}')$. We also see that this orbit is unique, that it has

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partition $\lambda^{\mathbf{i}'}$ and that if it has a label then it is the same as $\mathcal{O}_{e(\lambda)}$. Clearly i_l is admissible for $\lambda^{\mathbf{i}'}$ and, examining our labelling conventions for Levi subalgebras described preceding Lemma 12, we see that the label of the K-conjugacy class of $\mathfrak{l}$ equals the label of the M'-conjugacy class of $\mathfrak{gl}_{i_l} \oplus \mathfrak{m} \subseteq \mathfrak{m}'$. Therefore we can apply Corollary 6 to conclude that there exists an orbit $\mathcal{O} = \mathcal{O}_0 \times \mathcal{O}_{e(\mu)}$ with $\mathcal{O}_{e(\tau)} = \mathrm{Ind}_{\mathfrak{gl}_{i_l}\oplus\mathfrak{m}}^{\mathfrak{m}'}(\mathcal{O})$. We make use of Proposition 6 in the following calculation:

$$\begin{aligned}\mathcal{O}_{e(\lambda)} = \mathrm{Ind}_{\mathfrak{l}'}^{\mathfrak{k}}(\mathcal{O}') &= \mathrm{Ind}_{\mathfrak{l}'}^{\mathfrak{k}}(\mathcal{O}_0 \times \cdots \times \mathcal{O}_0 \times \mathrm{Ind}_{\mathfrak{gl}_{i_l}\oplus\mathfrak{m}}^{\mathfrak{m}'}(\mathcal{O}_0 \times \mathcal{O}_{e(\mu)}))\\ &= \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{k}}(\mathcal{O}_0 \times \cdots \times \mathcal{O}_0 \times \mathcal{O}_{e(\mu)}).\end{aligned}$$

We have shown that (1) $\Rightarrow$ (2). Before proving (2) $\Rightarrow$ (1) we shall take a quick detour to show that the final remarks in the statement of the proposition follow from (1). We certainly have $\mu = \tau^{(i_l)} = (\lambda^{\mathbf{i}'})^{(i_l)} = \lambda^{\mathbf{i}}$ by the transitivity of the algorithm. Suppose that $\tilde{\mathcal{O}} = \mathcal{O}_0 \times \mathcal{O}_0 \times \mathcal{O}_{e(\tilde{\mu})}$ is another orbit in $\mathfrak{l}$ inducing to $\mathcal{O}_{e(\lambda)}$. By the inductive hypothesis the partition of $\mathrm{Ind}_{\mathfrak{gl}_{i_l}\oplus\mathfrak{m}}^{\mathfrak{m}'}(\mathcal{O}_0 \times \mathcal{O}_{e(\tilde{\mu})})$ is $\lambda^{\mathbf{i}'}$ and so we get $\tilde{\mu} = \lambda^{\mathbf{i}} = \mu$. The uniqueness assertion is therefore obvious unless μ is very even and λ is not. In this case the argument used in the proof of Corollary 6 tells us that some term of the sequence $\mathbf{i}$ is odd. After conjugating by some element of K, we can assume that i_l is odd. The proof of uniqueness then concludes just as with the previous corollary, with $\mathfrak{gl}_{i_l} \oplus \mathfrak{m}$ playing the role of our Levi subalgebra and $\mathfrak{m}'$ playing the role of $\mathfrak{k}$.

Now we must go the other way. Keep $\mathfrak{l}$, $\mathfrak{l}'$, $\mathfrak{m}'$, etc as above. Suppose that there exists an orbit $\mathcal{O} = \mathcal{O}_0 \times \cdots \times \mathcal{O}_0 \times \mathcal{O}_{e(\mu)} \subseteq \mathfrak{l}$ with $\mathcal{O}_{e(\lambda)} = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{k}}(\mathcal{O})$. Then we set $\mathcal{O}_{e(\tau)} := \mathrm{Ind}_{\mathfrak{gl}_{i_l}\oplus\mathfrak{m}}^{\mathfrak{m}'}(\mathcal{O}_0 \times \mathcal{O}_{e(\mu)})$, $\mathcal{O}' := \mathcal{O}_0 \times \cdots \times \mathcal{O}_0 \times \mathcal{O}_{e(\tau)} \subseteq \mathfrak{l}'$, and conclude that $\mathcal{O}_{e(\lambda)} = \mathrm{Ind}_{\mathfrak{l}'}^{\mathfrak{k}}(\mathcal{O}')$ using a calculation very similar to the one above. Applying the inductive hypotheses, we get that $\mathbf{i}'$ is admissible. There is no label associated with $\mathfrak{l}'$ since $R_{\mathbf{i}'} > 0$. Now Corollary 6 tells us that i_l is an admissible index for $\tau = \lambda^{\mathbf{i}'}$ and so $\mathbf{i}$ is admissible for λ. The same corollary tells us that if the M'-conjugacy class of the Levi subalgebra $\mathfrak{gl}_{i_l} \oplus \mathfrak{m} \subseteq \mathfrak{m}'$ has a label then it coincides with that of $\mathcal{O}_{e(\tau)}$. The inductive hypothesis tells us that this label coincides with that of $\mathcal{O}_{e(\lambda)}$.

Finally, we must turn our attention to those sequences $\mathbf{i}$ in type D for which $R_{\mathbf{i}'} = 2$ (as before, $\mathbf{i}'$ stands for $\mathbf{i}$ with the last term removed). In this case there does not exist a Levi subalgebra of the form $\mathfrak{gl}_{\mathbf{i}'} \oplus \mathfrak{m}$ and the induction falls down. In order to resolve this we define $\mathbf{i}'' = (i_1, \ldots, i_{l-2})$ and let $\mathfrak{l}'' = \mathfrak{gl}_{\mathbf{i}''} \oplus \mathfrak{m}''$. Since $\mathfrak{l}$ has the form $\mathfrak{gl}_{\mathbf{i}}$ we may embed $\mathfrak{gl}_{i_{l-1}} \oplus \mathfrak{gl}_{i_l} \subseteq \mathfrak{m}''$ to get $\mathfrak{l} \subseteq \mathfrak{l}''$. Since $i_l = 1$ there is a unique conjugacy class of Levi subalgebras isomorphic to $\mathfrak{gl}_{\mathbf{i}}$. Furthermore, since the $\mathfrak{m}$ part is zero, there is only one orbit of the prescribed form in $\mathfrak{l}$. We let $\mathcal{O}$ equal the zero orbit in $\mathfrak{l}$. The proposition in this case is therefore reduced to the statement that $\mathbf{i}$ is admissible if and only if $\mathcal{O}_\lambda = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{k}}(\mathcal{O})$.

Suppose that $\mathbf{i}$ is admissible for λ. Then so is $\mathbf{i}''$ and by the inductive hypothesis there exists an orbit $\mathcal{O}'' = \mathcal{O}_0 \times \cdots \times \mathcal{O}_0 \times \mathcal{O}_{e(\tau)}$ in $\mathfrak{l}''$ with $\mathcal{O}_{e(\lambda)} = \mathrm{Ind}_{\mathfrak{l}''}^{\mathfrak{k}}(\mathcal{O}'')$. Since i_{l-1} is an admissible index for τ and $\tau^{(i_{l-1})}$ is $(1, 1)$, we conclude that $\tau = (3, 1)$ if $i_l = 1$, that $\tau = (3, 3)$ if $i_l = 2$, that $\tau = (3, 3, 2^{i_{l-1}})$ if $i_{l-1} > 2$ is even or, finally, that $\tau = (3, 3, 2^{i_{l-1}-1}, 1, 1)$ if $i_{l-1} > 2$ is odd. None of these partitions are very even, and so there is a unique orbit with partition τ. According to [CM93, Theorem 7.2.3] the induced orbit $\mathrm{Ind}_{\mathfrak{gl}_{i_{l-1}}\oplus\mathfrak{gl}_{i_l}}^{\mathfrak{gl}_{i_{l-1}+1}}(\mathcal{O}_0 \times \mathcal{O}_0)$ is the minimal nilpotent orbit in $\mathfrak{gl}_{i_{l-1}+1}$ with partition $(2, 1, \ldots, 1)$. If we induce into $\mathfrak{m}''$ then [CM93, Lemma 7.3.3(i)] tells us that $\mathrm{Ind}_{\mathfrak{gl}_{i_{l-1}+1}}^{\mathfrak{m}''}(\mathcal{O}_{\min}) = \mathcal{O}_{e(\tau)}$. Placing these ingredients together, we get

$$\begin{aligned}\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{k}}(\mathcal{O}) = \mathrm{Ind}_{\mathfrak{l}''}^{\mathfrak{k}}(\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{l}''}(\mathcal{O})) &= \mathrm{Ind}_{\mathfrak{l}''}^{\mathfrak{k}}\big(\mathcal{O}_0 \times \cdots \times \mathcal{O}_0 \times \mathrm{Ind}_{\mathfrak{gl}_{i_{l-1}}\oplus\mathfrak{gl}_{i_l}}^{\mathfrak{m}''}(\mathcal{O}_0 \times \mathcal{O}_0)\big)\\ &= \mathrm{Ind}_{\mathfrak{l}''}^{\mathfrak{k}}(\mathcal{O}'') = \mathcal{O}_{e(\lambda)}\end{aligned}$$

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as required. To go the other way, we assume that such an orbit $\mathcal{O}$ exists and go backwards through the above deductions. We will conclude that τ has one of the prescribed forms, so that $(i_{l-1}, 1)$ is an admissible sequence for $\mathbf{i}''$, and conclude that $\mathbf{i}$ is admissible. □

Corollary 7. *Let $\lambda \in \mathcal{P}_\epsilon(N)$. If $\mathbf{i} = (i_1, \dots, i_l)$ is an admissible sequence for λ (and $R_{\mathbf{i}} \neq 2$ in type* D*) then so is $\sigma(\mathbf{i})$ for every $\sigma \in \mathfrak{S}_l$.*

Proof. Since $\mathfrak{gl}_{\mathbf{i}} \times \mathfrak{m} \cong \mathfrak{gl}_{\sigma(\mathbf{i})} \times \mathfrak{m}$, this is immediate from Proposition 8. □

We now define a partial function φ from the set of all admissible sequences for λ to the set of all K-orbits of pairs $(\mathfrak{l}, \mathcal{O})$ where $\mathfrak{l} \subseteq \mathfrak{k}$ is a Levi subalgebra of $\mathfrak{k}$ and $\mathcal{O} \subset \mathfrak{l}$ is a nilpotent orbit. The map will remain undefined on sequences $\mathbf{i}$ with $R_{\mathbf{i}} = 2$ in type D. Let $\mathbf{i} = (i_1, \dots, i_l)$ be an admissible sequence for λ. Let $\mathfrak{l}$ be a Levi subalgebra isomorphic to $\mathfrak{gl}_{\mathbf{i}} \times \mathfrak{m}$. If there are two K-conjugacy classes of such Levi subalgebras then λ is very even and we require that $\mathfrak{l}$ has the same label as $\mathcal{O}_{e(\lambda)}$. Let $\varphi(\mathbf{i}) = (\mathfrak{l}, \mathcal{O})/K$ be the unique pair described in Proposition 8. The reader should take note that the admissible sequences upon which φ is undefined are not maximal, so the following makes sense.

Corollary 8. *The restriction of φ to the set of maximal admissible sequences for λ descends to a well-defined bijection from Φ_λ onto Ψ_λ. In particular, $|\Phi_\lambda| = |\Psi_\lambda|$.*

Proof. First of all, we show that φ maps the set of maximal admissible sequences for λ to Ψ_λ. Take $\mathbf{i}$ maximal admissible and $\varphi(\mathbf{i}) = (\mathfrak{l}, \mathcal{O})/K$ with $\mathcal{O} = \mathcal{O}_0 \times \cdots \times \mathcal{O}_0 \times \mathcal{O}_{e(\mu)}$. By Proposition 8 we have $\mu = \lambda^{\mathbf{i}}$ and so, by Lemma 6, $\mathcal{O}_{e(\mu)}$ is rigid. By part (2) of Lemma 12 the orbit $\mathcal{O}$ is also rigid. Furthermore, we have that $\mathcal{O}_{e(\lambda)} = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{k}}(\mathcal{O})$. Hence $\varphi(\mathbf{i}) \in \Psi_\lambda$.

We now claim that the map is well defined on Φ_λ, that is to say that $\varphi(\mathbf{i}) = \varphi(\mathbf{j})$ whenever $\mathbf{i} \sim \mathbf{j}$. Let $\varphi(\mathbf{i}) = (\mathfrak{l}_1, \mathcal{O}_1)/K$ and $\varphi(\mathbf{j}) = (\mathfrak{l}_2, \mathcal{O}_2)/K$ where $\mathfrak{l}_1 \cong \mathfrak{gl}_{\mathbf{i}} \times \mathfrak{m}_1$ and $\mathfrak{l}_2 \cong \mathfrak{gl}_{\mathbf{j}} \times \mathfrak{m}_2$. Since $\mathbf{i} = \sigma(\mathbf{j})$ for some $\sigma \in \mathfrak{S}_{|\mathbf{i}|}$ and the labels of $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are the same (if they exist), we conclude that they are K-conjugate by part (1) of Lemma 12. Thus we may assume that $\mathfrak{l}_1 = \mathfrak{l}_2$. Now the uniqueness statement at the end of Proposition 8 asserts that $(\mathfrak{l}_1, \mathcal{O}_1)/K = (\mathcal{O}_2, \mathfrak{l}_2)/K$. For the rest of the proof φ shall denote the induced map $\Phi_\lambda \to \Psi_\lambda$.

Let us prove that φ is surjective. Suppose that $(\mathfrak{l}, \mathcal{O}) \in \Psi_\lambda$ with $\mathfrak{l}$ and $\mathcal{O}$ as in the definition of Ψ_λ. Then by Proposition 8 the sequence $\mathbf{i} = (i_1, \dots, i_l)$ is admissible for λ and by Lemma 6 it is a maximal admissible. Therefore $\varphi(\mathbf{i}) = (\mathfrak{l}, \tilde{\mathcal{O}})/K$ for some orbit $\tilde{\mathcal{O}}$. Since $\mathcal{O} = \mathcal{O}_0 \times \cdots \times \mathcal{O}_0 \times \mathcal{O}_{e(\mu)}$ by construction, the uniqueness statement in Proposition 8 tells us that $(\mathfrak{l}, \mathcal{O})/K = (\mathfrak{l}, \tilde{\mathcal{O}})/K$. Hence φ sends the equivalence class of $\mathbf{i}$ in Φ_λ to $(\mathfrak{l}, \mathcal{O})/K$.

In order to prove the corollary we must show that φ is injective. Suppose that $\mathbf{i}$ and $\mathbf{j}$ are maximal admissible for λ and $\varphi(\mathbf{i}) = \varphi(\mathbf{j})$. Again we make the notation $\varphi(\mathbf{i}) = (\mathfrak{l}_1, \mathcal{O}_{e(\lambda^{\mathbf{i}})})/K$ and $\varphi(\mathbf{j}) = (\mathfrak{l}_2, \mathcal{O}_{e(\lambda^{\mathbf{j}})})/K$. Since $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are K-conjugate, the sequences $(i_1, \dots, i_{l(1)})$ and $(j_1, \dots, j_{l(2)})$ corresponding to isomorphisms

$$\mathfrak{l}_1 \cong \mathfrak{gl}_{i_1} \oplus \cdots \oplus \mathfrak{gl}_{i_{l(1)}} \oplus \mathfrak{m}_{\mathbf{i}} \cong \mathfrak{gl}_{j_1} \oplus \cdots \oplus \mathfrak{gl}_{j_{l(2)}} \oplus \mathfrak{m}_{\mathbf{j}} \cong \mathfrak{l}_2$$

must be of the same length $l = l(1) = l(2)$ and $\mathfrak{S}_l$-conjugate. This completes the proof. □

Part (1) of Theorem 9 follows quickly from the above and Lemma 13. We now prepare to prove part (2) of that theorem. Before we proceed we shall need two lemmas. Define a function $\kappa : \mathcal{P}_\epsilon(N) \longrightarrow (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ by setting

$$\kappa(\lambda)_i := \lambda_i - \lambda_{i+1} \bmod 2 \quad \text{for all } i > 0.$$

The reader should keep in mind here that $\lambda_i = 0$ for all $i > n$ by convention.

Lemma 14. *Let $M, N \in \mathbb{N}$. If $\mu \in \mathcal{P}_\epsilon^*(M)$ and $\lambda \in \mathcal{P}_\epsilon^*(N)$ then $\mu = \lambda$ if and only if $\kappa(\mu) = \kappa(\lambda)$.*

Proof. Evidently $\mu = \lambda$ if and only if $\mu_i - \mu_{i+1} = \lambda_i - \lambda_{i+1}$ for all $i > 0$. Since λ is rigid $\lambda_i - \lambda_{i+1} \in \{0, 1\}$; see Theorem 7. The statement follows. □

Lemma 15. *Suppose that λ is non-singular and that $i < j$ are admissible indices for λ. Then i is admissible for $\lambda^{(j)}$.*

Proof. If Case 1 occurs for λ at index i then $\lambda_i^{(j)} - \lambda_{i+1}^{(j)} = \lambda_i - \lambda_{i+1} \geqslant 2$ unless $j = i+1$ and Case 2 occurs for λ at index j. In this situation it follows from the non-singularity of λ that Case 1 occurs for $\lambda^{(j)}$ at index i.

Now suppose that Case 2 occurs for λ at index i. Then $(i, i+1) \in \Delta(\lambda)$ and $\lambda_i = \lambda_{i+1}$ by definition. It follows immediately that $\lambda_i^{(j)} = \lambda_{i+1}^{(j)}$. We shall show that $(i, i+1) \in \Delta(\lambda^{(j)})$ and conclude that Case 2 occurs for $\lambda^{(j)}$. If Case 1 occurs for λ at index j then $(i, i+1) \in \Delta(\lambda^{(j)})$ by the non-singularity of λ. If Case 2 occurs at j for λ then the same conclusion follows from Lemma 7. This completes the proof. □

Proposition 9. *$|\Phi_\lambda| = 1$ if and only if λ is non-singular.*

Proof. Suppose that λ is non-singular. We shall show that all maximal admissible sequences for λ have the same length and are conjugate under the symmetric group. Let $\mathbf{i}$ and $\mathbf{j}$ be two such sequences. Note that if the type is D then we can be certain that $R_{\mathbf{i}} \neq 2$ and $R_{\mathbf{j}} \neq 2$. Therefore, in any type, we might apply Corollary 7 and assume that they are both in ascending order. It is not hard to see that they are both still maximal after reordering. We shall show that they are now equal. Suppose not. Then either there exists an index k such that $i_k \neq i_j$ or one sequence is shorter than the other, say $|\mathbf{i}| < |\mathbf{j}|$ and $i_k = j_k$ for $k = 1, \ldots, |\mathbf{i}|$. In the latter situation $\mathbf{i}$ clearly fails to be maximal, so assume that we are in the former situation. We may assume without loss of generality that $i_k < j_k$. Now we may apply Lemma 15 and conclude that $\mathbf{j}$ is not maximal. This contradiction confirms that $\mathbf{i} = \mathbf{j}$ and that all maximal admissible sequences for λ are conjugate under the symmetric group.

In order to prove the converse we assume that λ is singular. Let $(i, i+1)$ be a bad 2-step with i maximal. We shall exhibit two maximal admissible sequences, $\mathbf{i}$ and $\mathbf{j}$, for λ such that $\kappa(\lambda^{\mathbf{i}})_{i+1} \neq \kappa(\lambda^{\mathbf{j}})_{i+1}$. In view of Lemma 14 the proposition shall follow. There are two possibilities: either $\lambda_{i+1} - \lambda_{i+2}$ is even, or $i > 1$ and $\lambda_{i-1} - \lambda_i$ is even. Assume the first of these possibilities, so that $\lambda_{i+1} - \lambda_{i+2}$ is even. Let

$$\mathbf{i}' = (\underbrace{i+1, i+1, \ldots, i+1}_{(\lambda_{i+1} - \lambda_{i+2})/2 \text{ times}}).$$

We have $\lambda_{i+1}^{\mathbf{i}'} = \lambda_{i+2}^{\mathbf{i}'}$. Let $\mathbf{i}$ be any maximal admissible sequence for λ extending $\mathbf{i}'$. Then $\kappa(\lambda^{\mathbf{i}})_{i+1} = 0$. Now let $\mathbf{j}' = (i)$ so that $\kappa(\lambda^{\mathbf{j}'})_{i+1} = 1$. Let $\mathbf{j}$ be any maximal admissible sequence extending $\mathbf{j}'$. By Lemmas 7 and 8, Case 2 does not occur for $\lambda^{\mathbf{j}_k}$ at any index $j_k = i$ with $k > 1$ and since $(i, i+1)$ is a maximal bad 2-step Case 2 cannot occur at index $j_k = i+2$. So $\kappa(\lambda^{\mathbf{j}})_{i+1} = \kappa(\lambda^{\mathbf{j}'})_{i+1} = 1$ which enables us to conclude that $\kappa(\lambda^{\mathbf{i}}) \neq \kappa(\lambda^{\mathbf{j}})$, $\lambda^{\mathbf{i}} \neq \lambda^{\mathbf{j}}$. Hence $|\Phi_\lambda| > 1$.

The other case is quite similar. This time we assume that $i > 1$, that $\lambda_{i-1} - \lambda_i$ is even and $\lambda_{i+1} - \lambda_{i+2}$ is odd. Our deductions will depend upon whether or not $(i-2, i-1) \in \Delta(\lambda)$.

Let us first assume that $i-2 \notin \Delta(\lambda)$. We take

$$\mathbf{i}' = (\underbrace{i-1, i-1, \ldots, i-1}_{(\lambda_{i-1}-\lambda_i)/2 \text{ times}}).$$

Let $\mathbf{i}$ be any maximal admissible sequence extending $\mathbf{i}'$. Much as before, $\kappa(\lambda^{\mathbf{i}})_{i-1} = 0$. Now let $\mathbf{j}' = (i)$ and let $\mathbf{j}$ be a maximal admissible extension of $\mathbf{j}'$. Since $(i-2, i-1) \notin \Delta(\lambda)$, Lemma 8 shows that Case 2 does not occur for $\lambda^{\mathbf{j}_k}$ at index $j_k = i-2$ for any k. Since the same can be said for $j_k = i$ at any index $k > 1$, we deduce that $\kappa(\lambda^{\mathbf{j}})_{i-1} = \kappa(\lambda)_{i-1} - 1 = 1$. But then $\lambda^{\mathbf{i}} \neq \lambda^{\mathbf{j}}$ and so $|\Phi_\lambda| > 1$ as desired.

To conclude the proof we must consider the final possibility: $i > 1$, $\lambda_{i-1} - \lambda_i$ even, $\lambda_{i+1} - \lambda_{i+2}$ odd and $(i-2, i-1) \in \Delta(\lambda)$. We let $\mathbf{i}'$ and $\mathbf{i}$ be defined exactly as it was in the previous paragraph. We have $\lambda^{\mathbf{i}'}_{i-1} = \lambda^{\mathbf{i}'}_i$, so that Case 2 cannot occur at index $i_k = i$ for any k. Since $(i, i+1)$ is a maximal bad 2-step for λ we know that $(i+2, i+3) \notin \Delta(\lambda)$. Then Lemma 8 implies that Case 2 cannot occur at index $i_k = i+2$ for any k, yielding $\kappa(\lambda^{\mathbf{i}})_{i+1} = \kappa(\lambda)_{i+1} = 1$. Let

$$\mathbf{j}' = (i, \underbrace{i+1, i+1, \ldots, i+1}_{(\lambda_{i+1}-\lambda_{i+2})/2 \text{ times}})$$

and $\mathbf{j}$ be any maximal admissible sequence extending $\mathbf{j}'$. Since $\lambda_{i+1} - \lambda_{i+2}$ is odd, $\lambda^{\mathbf{j}_2}_{i+1} - \lambda^{\mathbf{j}_2}_{i+2}$ is even, and $\lambda^{\mathbf{j}}_{i+1} = \lambda^{\mathbf{j}}_{i+2}$. Hence $\kappa(\lambda^{\mathbf{j}}) = 0$ and $|\Phi_\lambda| > 1$ as before. $\square$

We can finally complete the proof of Theorem 9.

Proof. Part (1) follows directly from Corollary 8 and Lemma 13. For part (2) use part (1) along with Proposition 9. $\square$

Let $\mathcal{S}$ be a sheet with data $(\mathfrak{l}, \mathcal{O})$. Recall that the rank of $\mathcal{S}$ is defined to be $\dim \mathfrak{z}(\mathfrak{l})$. The importance of the rank is illustrated by the formula

$$\dim(\mathcal{S}) = \operatorname{rank}(\mathcal{S}) + \dim \operatorname{Ind}_{\mathfrak{l}}^{\mathfrak{k}}(\mathcal{O}).$$

It should be mentioned here that [Mor08, Remark 3] claims that all sheets of $\mathfrak{k}$ containing a given nilpotent element have the same rank (hence the same dimension). However, the example given in Remark 4 shows that in general this is incorrect. A corrigendum has been published in [Mor13]. The error may be traced to [Mor08, Proposition 3.11]. Using the Kempken–Spaltenstein algorithm, we can amend that proposition as follows. First of all, note that if $(\mathfrak{l}, \mathcal{O})$ is the data associated with a sheet $\mathcal{S}$ then $(\mathfrak{l}, \mathcal{O}) \in \Psi_\lambda$, so Corollary 8 tells us that $\varphi^{-1}(\mathfrak{l}, \mathcal{O})$ is a well-defined equivalence class in Φ_λ. Clearly all admissible sequences in that equivalence class have the same length, which we may denote by $|\varphi^{-1}(\mathfrak{l}, \mathcal{O})|$.

Proposition 10. *Let $\mathcal{S}$ be a sheet of $\mathfrak{k}$ with data $(\mathfrak{l}, \mathcal{O})$. Then*

$$\operatorname{rank}(\mathcal{S}) = |\varphi^{-1}(\mathfrak{l}, \mathcal{O})|.$$

Proof. If $\mathfrak{l} = \mathfrak{gl}_{i_1} \oplus \cdots \oplus \mathfrak{gl}_{i_k} \oplus \mathfrak{m}$ as in Lemma 12 then clearly $\operatorname{rank}(\mathcal{S}) = \dim \mathfrak{z}(\mathfrak{l}) = k$. On the other hand, $\varphi^{-1}(\mathfrak{l}, \mathcal{O})$ is just the equivalence class of the maximal admissible sequence $(i_1, \ldots, i_k)$ which itself has length k. $\square$

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Corollary 9. *If $\lambda \in \mathcal{P}_\epsilon(N)$ and $e = e(\lambda)$ then*

$$r(e) = \max_{e \in \mathcal{S}} \operatorname{rank}(\mathcal{S}) = z(\lambda)$$

where the maximum is taken over all sheets of $\mathfrak{k}$ containing e.

Proof. Use the above proposition and Theorem 8. □

Remark 4. Some sheets of different ranks in Lie algebras of type B, C or D may share the same nilpotent orbit. Indeed, the partition $\lambda = (4, 2, 2) \in \mathcal{P}_\epsilon(-1)$ affords three maximal admissible sequences $(1, 3)$, $(3, 1)$ and (2) of lengths 2, 2 and 1, respectively, which lead to two rigid partitions (0) and $(2, 1, 1)$. So it follows from Proposition 10 that the nilpotent element $e(\lambda) \in \mathfrak{sp}_8$ lies in two sheets $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathfrak{sp}_8$ with $\operatorname{rk}(\mathcal{S}_1) = 2$ and $\operatorname{rk}(\mathcal{S}_2) = 1$. Note that $s(\lambda) = 2$, $\Sigma(\lambda) = \emptyset$ and $|\Delta(\lambda)| = |\Delta_{\mathrm{bad}}(\lambda)| = 1$. Hence $e(\lambda)$ is singular and

$$z(\lambda) = s(\lambda) + |\Delta(\lambda)| - (|\Delta_{\mathrm{bad}}(\lambda)| - |\Sigma(\lambda)|) = 2 + 1 - (1 - 0) = 2.$$

This example shows that different sheets of classical Lie algebras containing a given nilpotent element may have different dimensions.

Remark 5. Suppose that $\operatorname{char}(\Bbbk) = 0$ and let x be an arbitrary element of $\mathfrak{k}$. Then it is immediate from Theorem 9 and Corollary 5 that the following are equivalent:

(i) x belongs to a unique sheet of $\mathfrak{k}$;
(ii) x is a smooth point of the quasi-affine variety $\mathfrak{k}^{(m)}$ where $\dim \mathfrak{k}_x = m$;
(iii) the maximal rank of the sheets of $\mathfrak{k}$ containing x equals $\dim(\mathfrak{k}_x/[\mathfrak{k}_x, \mathfrak{k}_x])$.

Indeed, if $x = x_s + x_n$ is the Jordan–Chevalley decomposition of x then it is well known (and easy to see) that $\mathfrak{l} := \mathfrak{k}_{x_s}$ is a Levi subalgebra of $\mathfrak{k}$ and $\dim(\mathfrak{k}_x/[\mathfrak{k}_x, \mathfrak{k}_x]) = \dim(\mathfrak{l}_{x_n}/[\mathfrak{l}_{x_n}, \mathfrak{l}_{x_n}])$. On the other hand, it follows from our discussion in § 3.1 and the description of the Zariski closure of a sheet given in [BK79, Theorem 5.4] that there is a rank-preserving bijection between the sheets of $\mathfrak{k}$ containing x and the sheets of $\mathfrak{l}$ containing x_n. So the problem reduces quickly to the case where $x = x_n$, and since all simple ideals of $\mathfrak{l}$ are Lie algebras of classical types, Theorem 9 and Corollary 5 apply to x_n and give the desired result. This confirms the first part of Izosimov's conjecture; see [Izo12, Conjecture 1]. The second part of his conjecture (which is also interesting and plausible) remains open. We mention for completeness that the three statements above hold true for all x in $\mathfrak{sl}_n$ and $\mathfrak{gl}_n$; see [Izo12, Proposition 3.3].

5. Commutative quotients of finite *W*-algebras and sheets

5.1 Finite *W*-algebras and related commutative algebras

From now on we assume that $\Bbbk$ is an algebraically closed field of characteristic 0 and G is a connected reductive $\Bbbk$-group. Let $\mathfrak{g} = \operatorname{Lie}(G)$ and let e be a non-zero nilpotent element in $\mathfrak{g}$. We include e in an $\mathfrak{sl}(2)$-triple $\{e, h, f\} \subset \mathfrak{g}$ and consider the *Slodowy slice* $S_e := e + \mathfrak{g}_f$, an affine subspace of $\mathfrak{g}$ transversal to the adjoint G-orbit of e. The finite W-algebra $U(\mathfrak{g}, e)$ is a non-commutative filtered deformation of the algebra $\Bbbk[S_e]$ of regular functions on S_e endowed with the Slodowy grading; see [Pre02, 5.1] for more detail. By using the Killing form of $\mathfrak{g}$ we may identify $\Bbbk[S_e]$ with the symmetric algebra $S(\mathfrak{g}_e)$.

The action of $\operatorname{ad} h$ gives $\mathfrak{g}$ a $\mathbb{Z}$-graded Lie algebra structure $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ and we have that $e \in \mathfrak{g}(2)$ and $\mathfrak{g}_e = \bigoplus_{i \geqslant 0} \mathfrak{g}_e(i)$ where $\mathfrak{g}_e(i) := \mathfrak{g}_e \cap \mathfrak{g}(i)$. Let $v_1, \dots, v_r$ be a basis of $\mathfrak{g}_e$

such that $v_i \in \mathfrak{g}_i(n_i)$ for some $n_i \geqslant 0$. According to [Pre02, Theorem 4.6] the finite W-algebra $U(\mathfrak{g}, e)$ has a $\mathbb{k}$-basis consisting of all monomials $v^{\mathbf{a}} = v_1^{a_1} \cdots v_r^{a_r}$ with $a_i \in \mathbb{Z}_{\geqslant 0}$, and assigning to $v^{\mathbf{a}}$ filtration degree $|\mathbf{a}_e| := \sum_{i=1}^r a_i(n_i+2)$ gives $U(\mathfrak{g}, e)$ an algebra filtration called the *Kazhdan filtration* of $U(\mathfrak{g}, e)$. Furthermore, the corresponding graded algebra, $\operatorname{gr}_{\mathsf{K}} U(\mathfrak{g}, e)$, is isomorphic to the symmetric algebra $S(\mathfrak{g}_e)$ with v_i having Kazhdan degree $n_i + 2$, and the following relations hold in $U(\mathfrak{g}, e)$ for all $1 \leqslant i < j \leqslant r$:

$$v_i \cdot v_j - v_j \cdot v_i = [v_i, v_j] + q_{ij}(v_1, \ldots, v_r) + \text{terms of lower Kazhdan degree}, \tag{4}$$

where q_{ij} is a polynomial of Kazhdan degree $n_i + n_j + 2$ whose constant and linear parts are both zero (here $[v_i, v_j]$ is the Lie bracket of v_i and v_j in $\mathfrak{g}_e$). We write $\mathsf{K}_l\, U(\mathfrak{g}, e)$ for of the lth component of the Kazhdan filtration of $U(\mathfrak{g}, e)$.

It is well known that the group $C(e) := G_e \cap G_f$ is reductive and its finite quotient $\Gamma := C(e)/C(e)^\circ$ identifies canonically with the component group of $\operatorname{Ad} G_e$. Besides, $\operatorname{Lie}(C(e)) = \mathfrak{g}_e(0)$. As explained in [Pre07, 2.2, 2.3], the group $C(e)$ acts on $U(\mathfrak{g}, e)$ by algebra automorphisms and preserves all components of the Kazhdan filtration of $U(\mathfrak{g}, e)$. Moreover, there exists an injective $C(e)$-module homomorphism $\Theta\colon\ \mathfrak{g}_e \to U(\mathfrak{g}, e)$ with the property that $\Theta(\mathfrak{g}_e)$ generates $U(\mathfrak{g}, e)$ as an algebra and $\operatorname{gr}_{\mathsf{K}} \Theta(\mathfrak{g}_e) \cong \mathfrak{g}_e[2]$ as $C(e)$-modules, where $\mathfrak{g}_e[2]$ stands for the $\operatorname{Ad} C(e)$-module $\mathfrak{g}_e$ with all degrees shifted by 2. To be more precise, the group $C(e) \subset G_h$ preserves both the Slodowy grading of $S(\mathfrak{g}_e)$ and the grading of $S(\mathfrak{g}_e)$ given by total degree. In view of [Pre02, Lemma 4.5] this implies that the graded linear map $\operatorname{gr}_{\mathsf{K}} \Theta(\mathfrak{g}_e) \to \mathfrak{g}_e[2]$ sending $\operatorname{gr}_{\mathsf{K}} \Theta(v) \in S(\mathfrak{g}_e)$ to its linear part is an isomorphism of $C(e)$-modules. In what follows we shall denote by $\operatorname{gr}^0_{\mathsf{K}} \Theta(v)$ the linear part of $\operatorname{gr}_{\mathsf{K}} \Theta(v)$.

To ease notation we shall sometimes suppress the notion of Θ and assume from now on that the above-mentioned identification of $\operatorname{gr}_{\mathsf{K}} U(\mathfrak{g}, e)$ and $S(\mathfrak{g}_e)$ is $C(e)$-equivariant. Thanks to [Pre07, Lemma 2.4] we then have that $v_i \cdot v_j - v_j \cdot v_i = [v_i, v_j]$ for all $v_i, v_j \in \mathfrak{g}_e(0)$, where the products of v_i and v_j are taken in $U(\mathfrak{g}, e)$ and the Lie bracket $[v_i, v_j]$ is taken in $\mathfrak{g}_e$.

To shorten notation we write $\mathfrak{c}_e$ for $\mathfrak{g}_e^{\mathrm{ab}} = \mathfrak{g}_e/[\mathfrak{g}_e, \mathfrak{g}_e]$. Since $[\mathfrak{g}_e(0), \mathfrak{g}_e] \subset [\mathfrak{g}_e, \mathfrak{g}_e]$ and $\mathfrak{g}_e(0) = \operatorname{Lie}(C(e))$, it follows from Weyl's theorem that $C(e)^\circ$ acts trivially on $\mathfrak{c}_e$. This gives rise to a natural linear action of the component group $\Gamma = G_e/G_e^\circ$ on the vector space $\mathfrak{c}_e$. We denote by $\mathfrak{c}_e^\Gamma$ the fixed point space of this action and set

$$c(e) := \dim \mathfrak{c}_e \quad \text{and} \quad c_\Gamma(e) := \dim \mathfrak{c}_e^\Gamma.$$

Let $\mathcal{S}_1, \ldots, \mathcal{S}_t$ be all pairwise distinct sheets of $\mathfrak{g}$ containing e. As we explained in § 3.1, every sheet $\mathcal{S}_i$ contains a unique Zariski open decomposition class $\mathcal{D}(\mathfrak{l}_i, e_i) = (\operatorname{Ad} G)(e_i + \mathfrak{z}(\mathfrak{l}_i)_{\mathrm{reg}})$ characterised by the property that e_i is rigid in $\mathfrak{l}_i$. We write $r_i = \dim \mathfrak{z}(\mathfrak{l}_i)$ for the rank of $\mathcal{S}_i$. It is well known that the set $X_i := \mathcal{S}_i \cap (e + \mathfrak{g}_f)$ is a connected affine variety acted upon by the reductive group $C(e)$. Working over complex numbers, Katsylo proved in [Kat83] that the subgroup $C(e)^\circ$ operates trivially on X_i, the induced action of $\Gamma = C(e)/C(e)^\circ$ on the irreducible components of X_i is transitive, and the morphism

$$G \times X_i \to \mathcal{S}_i, \quad (g, x) \mapsto (\operatorname{Ad} g) \cdot x$$

is smooth, surjective of relative dimension $\dim \mathfrak{g}_e$. Moreover, Katsylo showed that it gives rise to an open morphism $\psi_i\colon\ \mathcal{S}_i \to X_i/\Gamma$ with the following properties:

(i) the fibres of ψ_i are G-orbits;

(ii) for any open subset X of X_i/Γ the induced algebra map $\mathbb{k}[U] \to \mathbb{k}[\psi_i^{-1}(U)]^G$ is an isomorphism.

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In brief, each morphism ψ_i is a geometric quotient and $\dim(\mathcal{S}_i/G) = \dim(X_i/\Gamma) = r_i$. A purely algebraic proof of Katsylo's results can be found in [ImH05].

Denote by $U(\mathfrak{g}, e)^{\mathrm{ab}}$ the largest commutative quotient of $U(\mathfrak{g}, e)$ (it has the form $U(\mathfrak{g}, e)/I_c$ where I_c is the two-sided ideal of $U(\mathfrak{g}, e)$ generated by all commutators $u \cdot v - v \cdot u$ with u, $v \in U(\mathfrak{g}, e)$). This finitely generated $\Bbbk$-algebra is important because its maximal spectrum, $\mathcal{E}$, parameterises the one-dimensional representations of $U(\mathfrak{g}, e)$. According to [Pre10, Theorem 1.2], for any *induced* nilpotent element e of $\mathfrak{g}$ the Krull dimension of $U(\mathfrak{g}, e)^{\mathrm{ab}}$ equals $r(e) := \max\{r_1, \ldots, r_t\}$ and the number of irreducible components of $\mathcal{E}$ is greater than or equal to the total number of all irreducible components of the X_i. If $\mathfrak{g} = \mathfrak{sl}_n$ then every nilpotent element $e \in \mathfrak{g}$ lies in a unique sheet $\mathcal{S} = \mathcal{S}(e)$. Since every nilpotent element of $\mathfrak{sl}_n$ is Richardson, the sheet $\mathcal{S}(e)$ contains a dense decomposition class of the form $(\mathrm{Ad}\, G)(\mathfrak{z}(\mathfrak{l})_{\mathrm{reg}})$. Using Remark 1, it is easy to see that $\dim \mathfrak{c}_e = \dim \mathfrak{z}(\mathfrak{l})$. On the other hand, it was proved in [Pre10] that for $\mathfrak{g} = \mathfrak{sl}_n$ the algebra $U(\mathfrak{g}, e)^{\mathrm{ab}}$ is isomorphic to a polynomial algebra in $r(e) = \dim \mathfrak{z}(\mathfrak{l})$ variables. The proof relied heavily on the explicit presentation of finite W-algebras of type A obtained by Brundan and Kleshchev in [BK06].

In this section we make an attempt to classify those induced nilpotent elements $e \in \mathfrak{g}$ for which $U(\mathfrak{g}, e)^{\mathrm{ab}}$ is isomorphic to a polynomial algebra. In view of the above discussion, this can happen only if e lies in a unique sheet of $\mathfrak{g}$, which makes one wonder to what extent the converse is true. For $\mathfrak{g}$ classical, we shall apply our results on non-singular nilpotent elements to show that this is always the case (even for e rigid!), whilst for $\mathfrak{g}$ exceptional we shall rely on de Graaf's computations in [deG13] to show the same is true for *almost all* induced induced orbits.

Since the group $C(e)$ operates on $U(\mathfrak{g}, e)$ by algebra automorphisms, it acts on the variety $\mathcal{E}$ which identifies naturally with the set of all ideals of codimension 1 in $U(\mathfrak{g}, e)$. Since the group $C(e)^\circ$ preserves any two-sided ideal of $U(\mathfrak{g}, e)$ by [Pre07, p. 501], it must act trivially on $\mathcal{E}$. We thus obtain a natural action of $\Gamma = C(e)/C(e)^\circ$ on the affine variety $\mathcal{E}$. We denote by $\mathcal{E}^\Gamma$ the corresponding fixed point set and let I_Γ be the ideal of $U(\mathfrak{g}, e)^{\mathrm{ab}}$ generated by all $\phi - \phi^\gamma$ with $\phi \in U(\mathfrak{g}, e)^{\mathrm{ab}}$ and $\gamma \in \Gamma$. It is clear that $\mathcal{E}^\Gamma$ is contained in the zero locus of I_Γ. Conversely, if $\eta \in \mathcal{E}$ is such that $\phi(\eta) = 0$ for all $\phi \in I_\Gamma$, then $\gamma(\eta) = \eta$ for all $\gamma \in \Gamma$. Indeed, otherwise η and $\gamma_0^{-1}(\eta)$ would be distinct maximal ideals of $U(\mathfrak{g}, e)^{\mathrm{ab}}$ for some $\gamma_0 \in \Gamma$ and we would be able to find an element $\phi \in U(\mathfrak{g}, e)^{\mathrm{ab}}$ with $\phi(\eta) = 0$ and $\phi(\gamma_0^{-1}(\eta)) \neq 0$. But this would imply that $(\phi - \phi^{\gamma_0})(\eta) \neq 0$, a contradiction. As a result, $\mathcal{E}^\Gamma$ coincides with the zero locus of I_Γ in $\mathcal{E}$. We denote by $U(\mathfrak{g}, e)^{\mathrm{ab}}_\Gamma$ the finitely generated $\Bbbk$-algebra $U(\mathfrak{g}, e)^{\mathrm{ab}}/I_\Gamma$. The above discussion shows that

$$\mathcal{E}^\Gamma = \mathrm{Specm}\, U(\mathfrak{g}, e)^{\mathrm{ab}}_\Gamma.$$

In this section we aim to show that in most cases $U(\mathfrak{g}, e)^{\mathrm{ab}}_\Gamma$ is isomorphic to a polynomial algebra in $c_\Gamma(e)$ variables. As will be explained later, the polynomiality of $U(\mathfrak{g}, e)^{\mathrm{ab}}_\Gamma$ can be used to classify those primitive ideals I of $U(\mathfrak{g})$ whose associated variety $\mathrm{VA}(I)$ appears with multiplicity 1 in the associated cycle $\mathrm{AC}(I)$. Detailed information on such ideals is very important because the primitive quotients $U(\mathfrak{g})/I$ extend to the Dixmier algebras quantising the nilpotent orbits of $\mathfrak{g}$ in the sense of [Los10b, 5.1 and 5.3].

5.2 A sufficient condition for polynomiality

The goal of this subsection is to give a sufficient condition of polynomiality of $U(\mathfrak{g}, e)^{\mathrm{ab}}$ and $U(\mathfrak{g}, e)^{\mathrm{ab}}_\Gamma$ for an arbitrary simple Lie algebra $\mathfrak{g}$ and use it to classify those nilpotent elements in the Lie algebras of classical groups for which $U(\mathfrak{g}, e)^{\mathrm{ab}}$ is a polynomial algebra. In view of our discussion in § 5.1 we may (and will) identify $\mathfrak{g}_e$ with a $C(e)$-submodule of $U(\mathfrak{g}, e)$ containing a

PBW basis of $U(\mathfrak{g},e)$. For $k,l\in\mathbb{Z}_{\geqslant 0}$, we set $S_{\leqslant l}(\mathfrak{g}_e):=\bigoplus_{i\leqslant l}S^i(\mathfrak{g}_e)$ and denote by $S^{\langle k\rangle}(\mathfrak{g}_e)$ the linear span of all monomials $v^{\mathbf{a}}:=v_1^{a_1}\cdots v_r^{a_r}$ in $S(\mathfrak{g}_e)$ with $|\mathbf{a}|_e=k$.

Lemma 16. *Let I be a proper two-sided ideal of $U(\mathfrak{g},e)$ and let V_I and V_I' be two $C(e)$-submodules of $\mathfrak{g}_e$, identified with $\Theta(\mathfrak{g}_e)$, such that $\mathfrak{g}_e[2]=\mathrm{gr}^0_{\mathsf{K}}(V_I)\oplus\mathrm{gr}^0_{\mathsf{K}}(V_I')$ as graded $\mathrm{Ad}\,C(e)$-modules. Suppose further that*

$$\mathrm{gr}_{\mathsf{K}}(v)\in(\mathrm{gr}_{\mathsf{K}}(I)+S_{\geqslant 2}(\mathfrak{g}_e))\cap S^{\langle i\rangle}(\mathfrak{g}_e)$$

for all $v\in(V_I\cap\mathsf{K}_i\,U(\mathfrak{g},e))\backslash\mathsf{K}_{i-1}\,U(\mathfrak{g},e)$, where $i\in\mathbb{Z}_{\geqslant 0}$. Then the unital algebra $U(\mathfrak{g},e)/I$ is generated by the subspace V_I'.

Proof. Since $\mathfrak{g}_e[2]=\mathrm{gr}^0_{\mathsf{K}}(V_I)\oplus\mathrm{gr}^0_{\mathsf{K}}(V_I')$, it is easy to see that $\mathfrak{g}_e=V_I\oplus V_I'$. Let $\pi\colon\,\mathfrak{g}_e\twoheadrightarrow V_I$ and $\pi'\colon\,\mathfrak{g}_e\twoheadrightarrow V_I'$ be the $C(e)$-equivariant projections induced by the direct sum decomposition $\mathfrak{g}_e=V_I\oplus V_I'$, and denote by $\mathcal{A}$ the $\Bbbk$-span in $U(\mathfrak{g},e)$ of all $\pi'(v)^{\mathbf{i}}:=\pi'(v_1)^{i_1}\cdots\pi'(v_r)^{i_r}$ with $\mathbf{i}\in\mathbb{Z}^r_{\geqslant 0}$. We shall prove by induction on k that every monomial $v^{\mathbf{a}}\in U(\mathfrak{g},e)$ with $|\mathbf{a}|_e=k$ lies in $\mathcal{A}+I$. Then the lemma will follow.

The statement is obviously true for $k=0$. Suppose that it holds for all $k<m$. If $|\mathbf{a}|>1$ then the statement follows by induction on k. Hence we may assume further that $|\mathbf{a}|=1$, so that $v^{\mathbf{a}}=v_s$ for some $s\in\{1,\dots,r\}$ and $k=n_s+2$. Thanks to our assumption on V_I we have that

$$\pi(v_s)=u_s+\sum_{|\mathbf{i}|_e=n_s+2,\ |\mathbf{i}|\geqslant 2}\lambda_{s,\mathbf{i}}v^{\mathbf{i}}+\text{ terms of lower Kazhdan degree}\tag{5}$$

for some $u_s\in I$ and $\lambda_{s,\,\mathbf{i}}\in\Bbbk$. Therefore,

$$\pi(v_s)\equiv\sum_{|\mathbf{i}|_e=n_s+2,\ |\mathbf{i}|\geqslant 2}\lambda_{s,\mathbf{i}}v^{\mathbf{i}}\mod(\mathcal{A}+I)$$

by the induction assumption. Our aim is to show that $v^{\mathbf{i}}\in\mathcal{A}+I$ for all $\mathbf{i}$ such that $|\mathbf{i}|_e=n_s+2$ and we shall use downward induction on the total degree of $\mathbf{i}$ (this is possible since there are only finitely many $\mathbf{i}\in\mathbb{Z}^r_{\geqslant 0}$ for which $|\mathbf{i}|=n_s+2$). If $\mathbf{j}$ is such that $|\mathbf{j}|_e=n_s+2$ and $|\mathbf{j}|\geqslant|\mathbf{i}|$ whenever $|\mathbf{i}|_e=n_s+2$, then

$$v^{\mathbf{j}}=\prod_{i=1}^{r}(\pi(v_i)+\pi'(v_i))^{j_i}\equiv\pi'(v)^{\mathbf{j}}\mod\big(\mathcal{A}+I+\mathsf{K}_{n_s+1}\,U(\mathfrak{g},e)\big)$$

thanks to (5) and (4). This takes care of the induction base. Now suppose that $v^{\mathbf{i}}\in\mathcal{A}+I$ for all $\mathbf{i}$ with $|\mathbf{i}|_e=n_s+2$ and $|\mathbf{i}|>d$, and take any $\mathbf{j}\in\mathbb{Z}^r_{\geqslant 0}$ with $|\mathbf{j}|_e=n_s+2$ and $|\mathbf{j}|=d$. Since $v^{\mathbf{j}}=\prod_{i=1}^r(\pi(v_i)+\pi'(v_i))^{j_i}$, combining (5) and (4) yields that

$$v^{\mathbf{j}}\equiv\pi'(v)^{\mathbf{j}}+\sum_{|\mathbf{i}|_e=n_s+2,\ |\mathbf{i}|>d}\mu_{\mathbf{j},\,\mathbf{i}}\,v^{\mathbf{i}}\mod(\mathcal{A}+I+\mathsf{K}_{n_s+1}\,U(\mathfrak{g},e))$$

for some $\mu_{\mathbf{j},\,\mathbf{i}}\in\Bbbk$. As $\mathsf{K}_{n_s+1}\,U(\mathfrak{g},e)$ is spanned by all $v^{\mathbf{b}}$ with $|\mathbf{b}|_e<n_s+2$, we know that $\mathsf{K}_{n_s+1}\,U(\mathfrak{g},e)\subseteq\mathcal{A}+I$. Then our present induction assumption gives $v^{\mathbf{j}}\in\mathcal{A}+I$, as claimed. But then $\pi(v_s)\in\mathcal{A}+I$ in view of (5). Since $v_s=\pi(v_s)+\pi'(v_s)$ and $\pi'(v_s)\in\mathcal{A}$ by the definition of $\mathcal{A}$, we deduce that $v_s\in\mathcal{A}+I$, finishing the proof. □

It should be stressed at this point that in Lemma 16 we do not require V_I to be contained in I.

PROPOSITION 11. *Let e be any nilpotent element of $\mathfrak{g}$. Then the following are true:*

(i) *if $\mathcal{E} \neq \emptyset$, then the unital algebra $U(\mathfrak{g}, e)^{\mathrm{ab}}$ is generated by $c(e)$ elements;*
(ii) *if $\mathcal{E}^{\Gamma} \neq \emptyset$, then the unital algebra $U(\mathfrak{g}, e)^{\mathrm{ab}}_{\Gamma}$ is generated by $c_{\Gamma}(e)$ elements.*

Proof. Due to the possibility of confusion, in this proof we shall distinguish between $\mathfrak{g}_e = S^1(\mathfrak{g}_e) \subset S(\mathfrak{g}_e)$ and its isomorphic copy $\Theta(\mathfrak{g}_e) \subset U(\mathfrak{g}, e)$.

(i) The defining ideal I_c of $U(\mathfrak{g}, e)^{\mathrm{ab}}$ contains all commutators $[\Theta(u), \Theta(v)]$ with $u, v \in \mathfrak{g}_e$. Since $C(e)$ is a reductive group, $\mathfrak{g}_e$ contains a graded $\operatorname{Ad} C(e)$-submodule of dimension $c(e)$ complementary to the derived subalgebra $[\mathfrak{g}_e, \mathfrak{g}_e]$. Let M be such a submodule and recall the $C(e)$-equivariant isomorphism $\Theta(\mathfrak{g}_e) \xrightarrow{\sim} \operatorname{gr}^0_{\mathsf{K}} \Theta(\mathfrak{g}_e) = \mathfrak{g}_e[2]$ described in § 5.1. We choose for V_{I_c} and V'_{I_c} the preimages under this isomorphism of $[\mathfrak{g}_e, \mathfrak{g}_e]$ and M, respectively. It is immediate from (4) and our earlier remarks in this proof that the $C(e)$-submodules V_{I_c} and V'_{I_c} of $\Theta(\mathfrak{g}_e)$ satisfy all conditions of Lemma 16. Since $\dim V'_{I_c} = \dim M = c(e)$, the first statement follows.

(ii) Let $\widetilde{I}_c$ be the preimage of the ideal I_{Γ} of $U(\mathfrak{g}, e)^{\mathrm{ab}}$ under the canonical homomorphism $U(\mathfrak{g}, e) \twoheadrightarrow U(\mathfrak{g}, e)^{\mathrm{ab}}$. Then $\widetilde{I}_c$ is a two-sided ideal of $U(\mathfrak{g}, e)$ and $U(\mathfrak{g}, e)/\widetilde{I}_c \cong U(\mathfrak{g}, e)^{\mathrm{ab}}_{\Gamma}$ as algebras. Since $[\mathfrak{g}_e(0), M] \subseteq [\mathfrak{g}_e, \mathfrak{g}_e]$ and $M \cap [\mathfrak{g}_e, \mathfrak{g}_e] = 0$, it follows from Weyl's theorem that the connected reductive group $C(e)^{\circ}$ acts trivially on M. Therefore, M has the natural structure of a Γ-module. There exists a Γ-submodule M' of M complementary to $M^{\Gamma} := \{x \in M \colon \gamma(x) = x\}$. We choose for $V_{\widetilde{I}_c}$ and $V'_{\widetilde{I}_c}$ the preimages in $U(\mathfrak{g}, e)$ of $M' \oplus [\mathfrak{g}_e, \mathfrak{g}_e]$ and M^{Γ} under the above-mentioned isomorphism $\Theta(\mathfrak{g}_e) \xrightarrow{\sim} \operatorname{gr}^0_{\mathsf{K}} \Theta(\mathfrak{g}_e) = \mathfrak{g}_e[2]$ of $C(e)$-modules. Note that $V_{\widetilde{I}_c} = V_{I_c} \oplus N'$ where N' is the preimage of M' in $\Theta(\mathfrak{g}_e)$. Due to our choice of M' the group $C(e)^{\circ}$ acts trivially on $N' \cong M'$ and N' is spanned by the elements of the form $u - \gamma(u)$ with $u \in N'$ and $\gamma \in \Gamma$. The definition of I_{Γ} implies that $N' \subset \widetilde{I}_c$. As $I \subseteq \widetilde{I}$, our discussion in part (i) now shows that the modules $V_{\widetilde{I}_c}$ and $V'_{\widetilde{I}_c}$ satisfy all conditions of Lemma 16. Since $\dim V'_{\widetilde{I}_c} = \dim M^{\Gamma} = c_{\Gamma}(e)$ we obtain (ii). $\square$

COROLLARY 10. *Let e be an induced nilpotent element of $\mathfrak{g}$. Then the following hold:*

(i) *if $c(e) = r(e)$, then $U(\mathfrak{g}, e)^{\mathrm{ab}} \cong S(\mathfrak{c}_e)$ as $\Bbbk$-algebras and Γ-modules and $U(\mathfrak{g}, e)^{\mathrm{ab}}_{\Gamma} \cong S(\mathfrak{c}_e^{\Gamma})$ as $\Bbbk$-algebras;*
(ii) *if $\mathcal{E}^{\Gamma} \neq \emptyset$ and $\dim \mathcal{E}^{\Gamma} \geqslant c_{\Gamma}(e)$, then $U(\mathfrak{g}, e)^{\mathrm{ab}}_{\Gamma} \cong S(\mathfrak{c}_e^{\Gamma})$ is a polynomial algebra in $c_{\Gamma}(e)$ variables.*

Proof. (i) Combining [Pre10, Theorem 1.2] with the main results of [GRU10], we see that $\dim U(\mathfrak{g}, e)^{\mathrm{ab}} = r(e)$ (in particular, $\mathcal{E} \neq \emptyset$). On the other hand, since V'_{I_c} is a $C(e)$-submodule of $U(\mathfrak{g}, e)$ isomorphic to $\mathfrak{c}_e$, Proposition 11(i) implies that there exists a natural surjective $C(e)$-equivariant algebra homomorphism $\psi \colon S(\mathfrak{c}_e) \twoheadrightarrow U(\mathfrak{g}, e)^{\mathrm{ab}}$. If $c(e) = \dim S(\mathfrak{c}_e)$ equals $r(e) = U(\mathfrak{g}, e)^{\mathrm{ab}}$, the map ψ must be injective. Since $C(e)^{\circ}$ acts trivially on $\mathfrak{c}_e$ we deduce that $U(\mathfrak{g}, e)^{\mathrm{ab}} \cong S(\mathfrak{c}_e)$ as $\Bbbk$-algebras and Γ-modules. But then $\mathcal{E} \cong \mathfrak{c}_e^*$ as Γ-varieties, implying that $\mathcal{E}^{\Gamma} \cong (\mathfrak{c}_e^*)^{\Gamma}$. Since the defining ideal in $S(\mathfrak{c}_e) \cong \Bbbk[\mathfrak{c}_e^*]$ of the linear subspace $(\mathfrak{c}_e^*)^{\Gamma}$ is generated by all $f - f^{\gamma}$ with $f \in S(\mathfrak{c}_e)$ and $\gamma \in \Gamma$, its image under ψ coincides with I_{Γ}. This implies that $S(\mathfrak{c}_e^{\Gamma}) \cong U(\mathfrak{g}, e)^{\mathrm{ab}}_{\Gamma}$ as $\Bbbk$-algebras.

(ii) As $\mathcal{E}^{\Gamma} \neq \emptyset$, it follows from Proposition 11(ii) that there is a surjective algebra homomorphism $S(V'_{\widetilde{I}_c}) \twoheadrightarrow U(\mathfrak{g}, e)^{\mathrm{ab}}_{\Gamma}$. As a consequence, $c_{\Gamma}(e) = \dim V'_{\widetilde{I}_c} \geqslant \dim U(\mathfrak{g}, e)^{\mathrm{ab}}_{\Gamma}$. If $\dim U(\mathfrak{g}, e)^{\mathrm{ab}}_{\Gamma} = \dim \mathcal{E}^{\Gamma} \geqslant c_{\Gamma}(e)$, then it must be that $U(\mathfrak{g}, e)^{\mathrm{ab}}_{\Gamma} \cong S(V'_{\widetilde{I}_c}) \cong S(\mathfrak{c}_e^{\Gamma})$ as $\Bbbk$-algebras. $\square$

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5.3 Further results on polynomiality of commutative quotients

In this subsection we apply our results on non-singular nilpotent elements to give a complete description of those nilpotent elements e in classical Lie algebras $\mathfrak{g}$ for which $U(\mathfrak{g}, e)^{\mathrm{ab}}$ is a polynomial algebra. For *induced* nilpotent orbits in exceptional Lie algebras, we apply de Graaf's computations in [deG13] to obtain strong partial results in this direction which will leave undecided only six such orbits, one in type F_4, one in type E_6, one in type E_7 and three in type E_8. The very challenging case of rigid nilpotent orbits in exceptional Lie algebras requires completely different methods and is dealt with in [Pre14].

Theorem 11. *Let e be a nilpotent element in a classical Lie algebra $\mathfrak{g}$. Then the following are equivalent:*

(1) *e belongs to a unique sheet of $\mathfrak{g}$;*
(2) *$U(\mathfrak{g}, e)^{\mathrm{ab}}$ is isomorphic to a polynomial algebra in $c(e)$ variables.*

Proof. If e belongs to a unique sheet of $\mathfrak{g}$, then $c(e) = r(e)$ by Corollary 5 and Theorem 8. Since $\mathfrak{g}$ is classical, it follows from [Bry03, KP82, Los10b] that $\mathcal{E} \neq \emptyset$. But then Corollary 10(i) shows that $U(\mathfrak{g}, e)^{\mathrm{ab}} \cong S(\mathfrak{c}_e)$ as unital $\Bbbk$-algebras.

If $U(\mathfrak{g}, e)^{\mathrm{ab}}$ is isomorphic to a polynomial algebra, then the variety $\mathcal{E}$ is irreducible. If e is induced, then applying [Pre10, Theorem 1.2] yields that e belongs to a unique sheet. If e is rigid, this holds automatically as $(\mathrm{Ad}\, G)\, e$ is a sheet of $\mathfrak{g}$. This completes the proof. □

Remark 6. (a) Suppose that $\mathfrak{g}$ is a classical Lie algebra and e is a rigid nilpotent element of $\mathfrak{g}$. Then it follows from Theorem 11 and Corollary 2 that the algebra $U(\mathfrak{g}, e)$ admits a *unique* one-dimensional representation.

(b) Suppose that $\mathfrak{g}$ is a classical Lie algebra and $e = e(\lambda)$ is a nilpotent element of $\mathfrak{g}$ associated with a non-singular partition λ. Then, combining [Pre10, Theorem 1.2] with part (2) of Corollary 4 and (the proof of) Theorem 11, we deduce that e belongs to a unique sheet $\mathcal{S}(e)$ of $\mathfrak{g}$ and the variety $(e + \mathfrak{g}_f) \cap \mathcal{S}(e)$ is irreducible. Of course, the uniqueness of $\mathcal{S}(e)$ also follows from Proposition 9 which we proved by purely combinatorial arguments. The irreducibility of the variety $(e + \mathfrak{g}_f) \cap \mathcal{S}(e)$ is actually a consequence of the following more general result which follows from Im Hof's theorem on smoothness of sheets in classical Lie algebras: for any sheet $\mathcal{S}$ containing a nilpotent element $e \in \mathfrak{g}$ the affine variety $X = \mathcal{S} \cap (e + \mathfrak{g}_f)$ is smooth and irreducible.

Indeed, it is immediate from Katsylo's results mentioned in § 5.1 that $\dim \mathcal{S} = \dim X + \dim(\mathrm{Ad}\, G)\, e$. Since $\mathcal{S}$ contains both X and $(\mathrm{Ad}\, G)\, e$ the tangent space $T_e(\mathcal{S})$ contains $T_e(X) + T_e\big((\mathrm{Ad}\, G)\, e\big)$. Since $T_e(X) \subset T_e(e + \mathfrak{g}_f) = \mathfrak{g}_f$ and $T_e\big((\mathrm{Ad}\, G)\, e\big) = [e, \mathfrak{g}]$, it follows that $T_e(\mathcal{S})$ contains $T_e(X) \oplus [e, \mathfrak{g}]$. As the variety $\mathcal{S}$ is smooth and $\dim \big(T_e(X) \oplus [e, \mathfrak{g}]\big) \geqslant \dim \mathcal{S}$ it must be that $T_e(\mathcal{S}) = T_e(X) \oplus [e, \mathfrak{g}]$ and $\dim T_e(X) = \dim X$. As a consequence, e is a smooth point of X. But then e belongs to a unique irreducible component of X; see [Sha94, ch. II, § 2, Theorem 6]. On the other hand, there is a regular $\Bbbk^{\times}$-action on X attracting every point $x \in X$ to e. Therefore, all irreducible components of X contain e and hence the variety X is irreducible. Since the singular locus $\mathrm{Sing}(X)$ of X is Zariski closed and invariant under the above $\Bbbk^{\times}$-action, this argument also shows that X is a smooth variety.

Our next result relies heavily on Losev's work [Los11a]. Together with Corollary 10(ii), it will enable us to describe the variety $\mathcal{E}^{\Gamma}$ for many induced nilpotent elements $e \in \mathfrak{g}$.

Proposition 12. *Let $P = LU$ be a proper parabolic subgroup of G, where $L \subset G$ is a Levi subgroup and $U = R_u(P)$, and suppose that a nilpotent element $e = e_0 + e_1 \in \mathrm{Lie}(P)$*

with $e_0 \in \mathfrak{l} = \mathrm{Lie}(L)$ *and* $e_1 \in \mathrm{Lie}(U)$ *is induced from* e_0 *in such a way that* $G_e \subset P$*. Let* $\mathcal{E}_0 = \operatorname{Specm} U([\mathfrak{l},\mathfrak{l}], e_0)^{\mathrm{ab}}$*, and suppose further that* $\mathcal{E}_0^{\Gamma_0} \neq \emptyset$ *where* $\Gamma_0 = L_{e_0}/(L_{e_0})^\circ$*. Then* $\mathcal{E}^\Gamma \neq \emptyset$ *and* $\dim \mathcal{E}^\Gamma \geqslant \dim \mathfrak{z}(\mathfrak{l})$ *where* $\mathfrak{z}(\mathfrak{l})$ *denotes the centre of the Levi subalgebra* $\mathfrak{l} = \mathrm{Lie}(L)$*.*

Proof. (a) Let (e_0, h_0, f_0) be an $\mathfrak{sl}_2$-triple of $\mathfrak{l}$ containing e_0 (if $e_0 = 0$ then (e_0, h_0, f_0) is the zero triple). By the $\mathfrak{sl}_2$-theory, the reductive group $C(L, e_0) := L_{e_0} \cap L_{h_0}$ is a Levi subgroup of the centraliser L_{e_0}. We include e in an $\mathfrak{sl}_2$-triple (e, h, f) of $\mathfrak{g}$ and denote by λ_e the cocharacter in $X_*(G)$ with $h \in \mathrm{Lie}(\lambda_e(\Bbbk^\times))$. Note that $C(e) = G_e \cap G_h = G_e \cap Z_G(\lambda_e)$. Since $C(e) \subset P$ by our assumption on e, it follows from [Los11a, Proposition 6.1.2(4)] that the reductive group $\lambda_e(\Bbbk^\times)C(e)$ is contained in P. Since any reductive subgroup of P is conjugate under P to a subgroup of L by Mostow's theorem, we may assume without loss of generality that $\lambda_e(\Bbbk^\times)C(e) \subseteq L$. Since $C(e) \subseteq L \cap G_e$ preserves both $\mathfrak{l}$ and $\mathrm{Lie}(U)$, it must be that $C(e) \subseteq L_{e_0}$. Since the group $C(e)$ is reductive, it follows from Mostow's theorem that it is conjugate under L to a subgroup of $C(L, e_0)$. Thus no generality will be lost by assuming further that $C(e) \subseteq C(L, e_0)$.

(b) In [Los11a, 6.3], Losev used the techniques of quantum Hamiltonian reduction to define a completion $U(\mathfrak{l}, e_0)'$ of the finite W-algebra $U(\mathfrak{l}, e_0)$ and an injective algebra homomorphism $\Xi\colon U(\mathfrak{g}, e) \to U(\mathfrak{l}, e_0)'$. By construction, the reductive group $C(L, e_0)$ acts on $U(\mathfrak{l}, e_0)'$ by algebra automorphisms. Since $C(e) \subseteq C(L, e_0)$, one can see by inspection that all maps involved in Losev's construction are $C(e)$-equivariant (a related discussion can also be found in [Los12, 2.5]). This implies, in particular, that in our situation Losev's homomorphism Ξ is $C(e)$-equivariant. Here $C(e)$ operates on $U(\mathfrak{g}, e)$ as in § 5.1, and the action of $C(e)$ on $U(\mathfrak{l}, e_0)'$ is given by inclusion $C(e) \subseteq C(L, e_0)$.

(c) Given an associative algebra $\mathcal{A}$ over $\Bbbk$ and a positive integer d, we denote by $\mathcal{A}^{(d)}$ the quotient of $\mathcal{A}$ by its two-sided ideal generated by all $s_{2d}(a_1, a_2, \ldots, a_{2d})$ with $a_i \in \mathcal{A}$, where

$$s_{2d}(X_1, X_2, \ldots, X_{2d}) = \sum_{\sigma \in \mathfrak{S}_{2d}} \operatorname{sgn}(\sigma) X_{\sigma(1)} X_{\sigma(2)} \cdots X_{\sigma(2d)}.$$

According to [Los11a, Proposition 6.5.1], the inclusion $U(\mathfrak{l}, e_0) \hookrightarrow U(\mathfrak{l}, e_0)'$ induces an algebra isomorphism $U(\mathfrak{l}, e_0)'^{(d)} \cong U(\mathfrak{l}, e_0)^{(d)}$. Therefore, for every $d \in \mathbb{N}$ the map Ξ gives rise to a $C(e)$-equivariant algebra homomorphism $U(\mathfrak{g}, e)^{(d)} \to U(\mathfrak{l}, e_0)^{(d)}$. Since $U(\mathfrak{g}, e)^{(1)} \cong U(\mathfrak{g}, e)^{\mathrm{ab}}$ and $U(\mathfrak{l}, e_0)^{(1)} \cong U(\mathfrak{l}, e_0)^{\mathrm{ab}}$ as algebras, we thus obtain a $C(e)$-equivariant algebra homomorphism $\xi\colon U(\mathfrak{g}, e)^{\mathrm{ab}} \to U(\mathfrak{l}, e_0)^{\mathrm{ab}}$.

Let $\widetilde{\mathcal{E}}_0 = \operatorname{Specm} U(\mathfrak{l}, e_0)^{\mathrm{ab}}$. According to [Los11a, Theorem 6.5.2] the morphism of affine varieties $\xi^*\colon \widetilde{\mathcal{E}}_0 \to \mathcal{E}$ associated with ξ is finite. In particular, it has finite fibres. Since ξ is $C(e)$-equivariant and $C(e)^\circ$ acts trivially on both $U(\mathfrak{g}, e)^{\mathrm{ab}}$ and $U(\mathfrak{l}, e)^{\mathrm{ab}}$, the morphism ξ^* maps $\widetilde{\mathcal{E}}_0^\Gamma$ into $\mathcal{E}^\Gamma$. It follows that

$$\dim \widetilde{\mathcal{E}}_0^\Gamma = \dim \xi^*(\widetilde{\mathcal{E}}_0^\Gamma) \leqslant \dim \mathcal{E}_\Gamma.$$

(d) Write $\mathfrak{z}$ for the centre $\mathfrak{z}(\mathfrak{l})$ of the Levi subalgebra $\mathfrak{l}$. Clearly, $\mathfrak{z}$ is a toral subalgebra of $\mathfrak{g}$ and $\mathfrak{l} = \mathfrak{z} \oplus [\mathfrak{l},\mathfrak{l}]$. It follows that $U(\mathfrak{l}, e_0) \cong S(\mathfrak{z}) \otimes U([\mathfrak{l},\mathfrak{l}], e_0)$. This, in turn, implies that $U(\mathfrak{l}, e)^{\mathrm{ab}} \cong S(\mathfrak{z}) \otimes U([\mathfrak{l},\mathfrak{l}], e_0)^{\mathrm{ab}}$ as algebras. Since the subalgebra $U([\mathfrak{l},\mathfrak{l}], e_0)$ of $U(\mathfrak{l}, e_0)$ is stable under the action of $C(e)$ on $U(\mathfrak{l}, e_0)$, we have a natural action of Γ of the affine variety $\mathcal{E}_0 := \operatorname{Specm} U([\mathfrak{l},\mathfrak{l}], e_0)^{\mathrm{ab}}$. Since $C(e) \subseteq C(L, e_0)$, and $C(L, e_0)^\circ$ acts trivially on $\mathcal{E}_0$, the variety $\mathcal{E}_0^\Gamma = \{\eta \in \mathcal{E}_0 : \gamma(\eta) = \eta \text{ for all } \gamma \in \Gamma\}$ contains $\mathcal{E}_0^{\Gamma_0}$ and hence is non-empty by our assumption on e.

Note that L acts trivially on the centre $\mathfrak{z}$ of $\mathrm{Lie}(L)$ and hence so does $C(e) \subset L$. It follows that $\widetilde{\mathcal{E}}_0^\Gamma \cong \mathfrak{z}^* \times \mathcal{E}_0^\Gamma$ as affine varieties. In particular,

$$\dim \widetilde{\mathcal{E}}_0^\Gamma = \dim \mathcal{E}_0^\Gamma + \dim \mathfrak{z} \geqslant \dim \mathfrak{z}.$$

But then $\mathcal{E}^\Gamma \supseteq \xi^*(\widetilde{\mathcal{E}}^\Gamma) \neq \emptyset$ and $\dim \mathcal{E}^\Gamma \geqslant \dim \widetilde{\mathcal{E}}_0^\Gamma \geqslant \dim \mathfrak{z}(\mathfrak{l})$ as claimed. □

Remark 7. It is established in [Pre14] that the variety $\mathcal{E}^\Gamma$ is non-empty for any nilpotent element in a finite-dimensional simple Lie algebra over $\mathbb{C}$. This implies, among other things, that the condition that $\mathcal{E}_0^{\Gamma_0} \neq \emptyset$ imposed in the statement of Proposition 12 can be dropped.

5.4 Describing the varieties $\mathcal{E}^\Gamma$ for classical Lie algebras

In this subsection we assume that $\mathfrak{g}$ is either $\mathfrak{so}_N$ or $\mathfrak{sp}_N$ and e is an arbitrary nilpotent element of $\mathfrak{g}$. It is quite surprising that in this setting we have a very uniform description of the algebra $U(\mathfrak{g}, e)_\Gamma^{\mathrm{ab}}$. We call a partition $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathcal{P}_1(N)$ *exceptional* if there exists a $k \leqslant n$ such that the parts λ_k, λ_{k+1} are odd and the parts λ_i with $i \notin \{k, k+1\}$ are all even. Note that $\Delta(\lambda) = \{(k, k+1)\}$ and $\Delta_{\mathrm{bad}}(\lambda) = \emptyset$, which shows that all exceptional partitions in $\mathcal{P}_1(N)$ are non-singular. Using the KS algorithm, it is straightforward to see that any nilpotent element of $\mathfrak{g}$ associated with an exceptional partition λ is Richardson (i.e. is induced from 0).

We call a nilpotent element $e \in \mathfrak{g}$ *almost rigid* if the partition $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathcal{P}_\epsilon(N)$ of e has the property that $\lambda_i - \lambda_{i+1} \in \{0, 1\}$ for all $i \leqslant n$ (recall that $\lambda_j = 0$ for $j > n$ by convention). Since any such partition has no bad 2-steps, all almost rigid nilpotent elements of $\mathfrak{g}$ are non-singular.

Theorem 12. *Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathcal{P}_\epsilon(N)$ and let $e = e(\lambda)$ be a nilpotent element of $\mathfrak{g}$ associated with λ. Then $U(\mathfrak{g}, e)_\Gamma^{\mathrm{ab}}$ is isomorphic to a polynomial algebra in $s(\lambda)$ variables unless $\mathfrak{g}$ is of type D and $\lambda \in \mathcal{P}_\epsilon(N)$ is exceptional, in which case $U(\mathfrak{g}, e)_\Gamma^{\mathrm{ab}}$ is a polynomial algebra in $s(\lambda) + 1 = (\lambda_1 - \lambda_n + 1)/2$ variables.*

Proof. We denote by $\mathcal{O}$ the G-orbit of $e = e(\lambda)$, adopt the notation of § 2.1 pertaining to $\mathfrak{g}_e$, and choose the subspaces $V[i]$, $1 \leqslant i \leqslant n$, as in the proof of Lemma 1. In proving the theorem we may and will assume that $G = SL(V) \cap G(\Psi)$ where $G(\Psi)$ is the stabiliser in $GL(V)$ of the bilinear form $\Psi = (\,\cdot\,,\cdot\,)$.

(a) Let $I = \{1 \leqslant i \leqslant n\colon\, i' = i,\, \lambda_i > \lambda_{i+1}\}$ and set $\nu(\lambda) := |I|$. Note that $\nu(\lambda)$ is the number of *distinct* λ_i with $i = i'$. For $i \in I$ we let g_i denote the involution in $G(\Psi)$ such that $g_i(e^s v_j) = (-1)^{\delta_{i,j}} e^s(w_j)$ for all $1 \leqslant j \leqslant n$ and $0 \leqslant s \leqslant \lambda_j$, where $\delta_{i,j}$ is the Kronecker delta, and define $\widetilde{\Gamma} := \langle g_i\colon\, i \in I\rangle$, a subgroup of $G(\Psi)$. As the involutions g_i pairwise commute, $\widetilde{\Gamma}$ is an elementary abelian 2-group of order $2^{\nu(\lambda)}$. Using [Jan04, 3.8, 3.13], it is straightforward to see that the centraliser G_e is generated by $\widetilde{\Gamma} \cap SL(V)$ and G_e° (see also [McG94, Theorem 2.7′]).

Let $\overline{\mathfrak{H}}_0$ denote the image of $\mathfrak{H}_0$ in $\mathfrak{c}_e = \mathfrak{g}_e/[\mathfrak{g}_e, \mathfrak{g}_e]$. Since $\mathfrak{H}_0$ is spanned by elements that preserve every subspace $V[i]$ with $1 \leqslant i \leqslant n$, direct verification shows that the group $\widetilde{\Gamma}$ acts trivially on $\overline{\mathfrak{H}}_0$. Since G_e° acts trivially on $\mathfrak{c}_e$ and $G_e = \big(\widetilde{\Gamma} \cap SL(V)\big) \cdot G_e^\circ$ by our preceding remark, we now deduce that $\overline{\mathfrak{H}}_0 \subseteq \mathfrak{c}_e^\Gamma$. In view of Corollary 1, this yields

$$\dim \mathfrak{c}_e^\Gamma \geqslant \dim(\mathfrak{H}_0/\mathfrak{H}_0^+) = s(\lambda).$$

The proof of Corollary 1 also shows that the images of $\zeta_i^{i+1, \lambda_{i+1}-1}$ with $(i, i+1) \in \Delta(\lambda)$ in the quotient space $\bar{\mathfrak{c}}_e := \mathfrak{c}_e/\overline{\mathfrak{H}}_0$ form a $\Bbbk$-basis of $\bar{\mathfrak{c}}_e$. Note that $g_{i+1} \in \widetilde{\Gamma}$ for every 2-step $(i, i+1)$

of λ and, moreover, $g_i, g_{i+1} \in \widetilde{\Gamma}$ if $(i, i+1)$ is a 2-step of λ such that $\lambda_i \neq \lambda_{i+1}$. If $(j, j+1) \in \Delta(\lambda)$ then direct computation shows that

$$g_i(\zeta_j^{j+1,\lambda_{j+1}-1}) = \begin{cases} \zeta_j^{j+1,\lambda_{j+1}-1} & \text{if } j \notin \{i-1, i\} \\ -\zeta_j^{j+1,\lambda_{j+1}-1} & \text{if } j \in \{i-1, i\}. \end{cases} \tag{6}$$

Let $\bar{\zeta}_i^{i+1,\lambda_{i+1}-1}$ denote the image of $\zeta_i^{i+1,\lambda_{i+1}-1}$ in $\bar{\mathfrak{c}}_e$ and suppose that

$$g_i g_j \bigg(\sum_{(k,k+1)\in\Delta(\lambda)} a_k \bar{\zeta}_k^{k+1,\lambda_{k+1}-1} \bigg) = \sum_{(k,k+1)\in\Delta(\lambda)} a_k \bar{\zeta}_k^{k+1,\lambda_{k+1}-1} \quad (\forall\ i, j \in I)$$

for some $a_k \in \Bbbk$. Set $\alpha := \sum_{(k,k+1)\in\Delta(\lambda)} a_k \bar{\zeta}_k^{k+1,\lambda_{k+1}-1}$. If $a_{k_1} \neq 0$ and $a_{k_2} \neq 0$ for some $(k_1, k_1+1), (k_2, k_2+1) \in \Delta(\lambda)$ with $k_1 + 1 < k_2$, then it follows from (6) that $g_{k_1+1} g_{k_2+1}(\alpha) \neq \alpha$, a contradiction. If $a_k \neq 0$ and $a_{k+1} \neq 0$ for some $(k, k+1), (k+1, k+2) \in \Delta(\lambda)$, then $\lambda_{k-1} > \lambda_k \geqslant \lambda_{k+1} \geqslant \lambda_{k+2} > \lambda_{k+3}$ if $k > 1$ and $\lambda_1 > \lambda_2 \geqslant \lambda_3 > \lambda_4$ if $k = 1$, which implies that $g_k, g_{k+2} \in \widetilde{\Gamma}$. Since $g_k g_{k+2}(\alpha) \neq \alpha$ by (6), we now deduce that $\alpha = a_k \bar{\zeta}_k^{k+1,\lambda_{k+1}-1}$ for some $(k, k+1) \in \Delta(\lambda)$. If $a_k \neq 0$ and I is not contained in $\{k, k+1\}$ then it is straightforward to see that $g_i g_{k+1}(\alpha) \neq \alpha$ for any $i \in I \backslash \{k, k+1\}$. Therefore, $\alpha \neq 0$ implies that $I \subseteq \{k, k+1\}$ where $(k, k+1)$ is a 2-step of λ.

If $\mathfrak{g}$ is not of type C and $\alpha \neq 0$, then the above implies that $\Delta(\lambda) = \{(k, k+1)\}$ for some $k < n$ and all λ_i with $i \notin \{k, k+1\}$ are even. So λ is exceptional and h has type D in this case. Finally, if $\mathfrak{g}$ is of type C then $\det(g_i) = 1$ for all $i \in I$ and hence $\widetilde{\Gamma} = \widetilde{\Gamma} \cap SL(V)$. Furthermore, if $(k, k+1)$ is a 2-step of λ then $g_{k+1}(\alpha) = -\alpha$ by (6). So in type C it must be that $\alpha = 0$.

As a result of these deliberations we obtain that $\mathfrak{c}_e^{\Gamma} = \overline{\mathfrak{H}}_0$ and $\dim \mathfrak{c}_e^{\Gamma} = s(\lambda)$ unless λ is an exceptional partition in $\mathcal{P}_1(N)$, in which case $\dim \mathfrak{c}_e^{\Gamma} = s(\lambda) + 1$.

(b) Suppose that $\lambda_k - \lambda_{k+1} \geqslant 2$ for some $k \in \{1, \ldots, n\}$. For each $i \in \{1, \ldots, k\}$ we denote by $V'[i]$ the linear span of all $e^s(w_i)$ with $1 \leqslant s \leqslant \lambda_i - 2$ and set

$$V_k := \bigg(\bigoplus_{i=1}^{k} V'[i] \bigg) \bigoplus \bigg(\bigoplus_{i>k} V[i] \bigg),$$

a non-degenerate subspace of V with respect to Ψ. The stabiliser L_k of V_k in G is a Levi subgroup of G and $\mathfrak{l}_k := \operatorname{Lie}(L_k)$ is isomorphic to $\mathfrak{gl}_k \times \mathfrak{m}_k$ where $\mathfrak{m}_k$ is a Lie algebra of the same type as $\mathfrak{g}$.

Let t_k be the semisimple element of $\mathfrak{g}$ with $\operatorname{Ker} t_k = V_k$ such that $t_k(w_i) = -w_i$ and $t_k(e^{\lambda_i - 1} w_i) = e^{\lambda_i - 1} w_i$ for all $1 \leqslant i \leqslant k$. It is straightforward to see that t_k spans the one-dimensional centre of the Lie algebra $\mathfrak{l}_k$. Let e_k be the nilpotent element of $\mathfrak{l}_k$ with the property that $e_k(w_i) = e_k(e^{\lambda_i - 2} w_i) = e_k(e^{\lambda_i - 1} w_i) = 0$ for $1 \leqslant i \leqslant k$ and $e_k(e^s w_i) = e^{s+1} w_i$ for all (i, s) with $i > k$ and $0 \leqslant s \leqslant \lambda_i - 1$ and all (i, s) with $1 \leqslant i \leqslant k$ and $1 \leqslant s \leqslant \lambda_i - 3$. By construction, $e_k \in \mathfrak{m}_k$.

In view of Corollary 6, passing from $(\mathfrak{g}, e)$ to $(\mathfrak{l}_k, e_k)$ corresponds to applying Case 1 of the KS algorithm at index k. Hence the orbit $\mathcal{O}$ lies in the Zariski closure of the decomposition class $(\operatorname{Ad} G) \cdot (e_k + \Bbbk^{\times} t_k)$ and $e \in \operatorname{Ind}_{\mathfrak{l}_k}^{\mathfrak{g}}(\mathcal{O}_k)$ where $\mathcal{O}_k$ is the L_k-orbit of e_k.

Let W_k be the span of all $e^{\lambda_i - 1} w_i$ with $1 \leqslant i \leqslant k$ and set $\widetilde{V}_k := V_k \oplus W_k$. Let P_k be the parabolic subgroup of G which stabilises the partial flag $V \supset \widetilde{V}_k \supset W_k$ in V. Using the description of $\mathfrak{g}_e$ in § 2.1 it is immediate that $\mathfrak{g}_e \subset \operatorname{Lie}(P_k)$ which, in turn, implies that $G_e^{\circ} \subset P_k$. Since L_k is contained in P_k as well and $\widetilde{\Gamma} \cap SL(V) \subset L_k$ by the definition of $\widetilde{\Gamma}$, we now obtain $G_e = (\widetilde{\Gamma} \cap SL(V)) \cdot G_e^{\circ} \subset P_k$.

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(c) The maximality of L_k in the class of Levi subgroups of G yields that L_k is a Levi subgroup of P_k and $P_k = L_kU_k$ where $U_k = R_u(P_k)$. Furthermore, our discussion in part (b) implies that $e \in \mathrm{Lie}(P_k)$ is induced from $e_k \in \mathfrak{l}_k$ and $G_e \subset (L_k)_{e_k} \cdot U_k$.

Continuing the process described in part (b) as many times as possible and using the transitivity of induction stated in Proposition 6(3), we shall eventually arrive at a parabolic subgroup $P = LU$ of G and a nilpotent element $e_0 \in \mathfrak{l} = \mathrm{Lie}(L)$ such that $G_e \subset P$ and $e \in \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_0)$ where $\mathcal{O}_0 = (\mathrm{Ad}\, L)e_0$. From the description in part (b) it follows that $\mathfrak{l} \cong \bar{\mathfrak{l}} \oplus \mathfrak{m}$ where $\mathfrak{m}$ has the same type as $\mathfrak{g}$ and $\bar{\mathfrak{l}}$ is a Lie algebra direct sum of $s(\lambda)$ copies of various $\mathfrak{gl}_{k_i}$ with $k_i \in \mathbb{N}$. Since the process terminates at the $s(\lambda)$th step, the nilpotent element $e_0 \in \mathfrak{m}$ must be almost rigid and hence non-singular.

Let M be the special orthogonal or symplectic group with $\mathrm{Lie}(M) = \mathfrak{m}$ and denote by $\Gamma(0)$ the component group $M_{e_0}/M_{e_0}^\circ$. Let λ_0 be the partition of $e_0 \in \mathfrak{m}$. If λ_0 is not exceptional, then, combining Corollary 5 with Corollary 10(i), we deduce that $U(\mathfrak{m}, e_0)^{\mathrm{ab}}_{\Gamma(0)} \cong S(\mathfrak{c}_{e_0}^{\Gamma(0)})$. Since in the present case $\mathfrak{c}_{e_0}^{\Gamma(0)} = \{0\}$ by our discussion in part (a) (applied to $e_0 \in \mathfrak{m}$) we conclude that the maximal spectrum of $U(\mathfrak{m}, e_0)^{\mathrm{ab}}_{\Gamma(0)}$ is a single point! Note that $U([\mathfrak{l}, \mathfrak{l}], e_0) \cong U([\bar{\mathfrak{l}}, \bar{\mathfrak{l}}]) \otimes U(\mathfrak{m}, e_0)$ as $\Bbbk$-algebras and both tensor factors are stable under the natural action of the reductive part of L_{e_0} on $U([\mathfrak{l}, \mathfrak{l}], e_0)$. Proposition 12 now yields that $\mathcal{E}^\Gamma \neq \emptyset$ and $\dim \mathcal{E}^\Gamma \geqslant \dim \mathfrak{z}(\mathfrak{l})$. On the other hand, $\dim \mathfrak{z}(\mathfrak{l}) = s(\lambda) = \dim \mathfrak{c}_\Gamma(e)$ by our discussion in part (a) applied to $e \in \mathfrak{g}$. In this situation Corollary 10(ii) yields that $U(\mathfrak{g}, e)^{\mathrm{ab}}_\Gamma \cong S(\mathfrak{c}_e^\Gamma) \cong \Bbbk[X_1, \dots, X_{s(\lambda)}]$.

(d) Finally, suppose that λ_0 is exceptional. Since we only applied Step 1 of the KS algorithm to reach e_0, so must be λ. In particular, $\mathfrak{g}$ is of type D. Since e_0 is almost rigid and λ_0 is exceptional, we have that $\lambda_0 = (2, \dots, 2, 1, 1)$. Then $\Gamma(0) = \{1\}$ which enables us to apply Step 2 of the KS algorithm. After doing so we arrive at a parabolic subgroup $P' = L'U' \subset P$ such that the centre of $\mathrm{Lie}(L')$ has dimension $s(\lambda) + 1$ and $e \in \mathrm{Lie}(U')$ is a Richardson element of $\mathrm{Lie}(P')$. Since $\Gamma(0) = \{1\}$ and e_0 is a Richardson element of $\mathfrak{l} \cap \mathrm{Lie}(P')$, we also have that

$$G_e \subset L_{e_0}U = L_{e_0}^\circ U \subset (L \cap P') \cdot U \subseteq P'.$$

Let $\mathfrak{l}' = \mathrm{Lie}(L')$ and adopt the notation of Proposition 12. Since the augmentation ideal of the finite W-algebra $U([\mathfrak{l}', \mathfrak{l}'], 0) = U([\mathfrak{l}', \mathfrak{l}'])$ is $(\mathrm{Ad}\, L')$-stable, we have that $\mathcal{E}_0^{\Gamma_0} \neq \emptyset$. Applying Proposition 12 now yields that $\mathcal{E}^\Gamma \neq \emptyset$ and $\dim \mathcal{E}^\Gamma \geqslant \dim \mathfrak{z}(\mathfrak{l}') = s(\lambda) + 1$. As $\dim \mathcal{E}^\Gamma = c_\Gamma(e)$ by our discussion in part (a), Corollary 10(ii) applies to e, showing that $U(\mathfrak{g}, e)^{\mathrm{ab}}_\Gamma$ is isomorphic to a polynomial algebra in $s(\lambda) + 1$ variables.

The proof of the theorem is now complete. □

5.5 Describing the varieties $\mathcal{E}^\Gamma$ for exceptional Lie algebras

In this subsection, G is an exceptional algebraic group of adjoint type and $\mathfrak{g} = \mathrm{Lie}(G)$, a Lie algebra of type G_2, F_4, E_6, E_7 or E_8. We assume that e is an induced nilpotent element of $\mathfrak{g}$ and we embed it into an $\mathfrak{sl}_2$-triple $\{e, h, f\} \subset \mathfrak{g}$. By the $\mathfrak{sl}_2$-theory, all eigenvalues of $\mathrm{ad}\, h$ are integers and $\mathfrak{g}_e = \bigoplus_{i \geqslant 0} \mathfrak{g}_e(i)$ where $\mathfrak{g}(k)$ denotes the k-eigenspace of $\mathrm{ad}\, h$ and $\mathfrak{g}_e(k) = \mathfrak{g}_e \cap \mathfrak{g}(k)$. Since the derived subalgebra of $\mathfrak{g}_e$ is $(\mathrm{ad}\, h)$-stable, the vector space $\mathfrak{c}_e = \mathfrak{g}_e/[\mathfrak{g}_e, \mathfrak{g}_e]$ carries a natural $\mathbb{Z}_{\geqslant 0}$-grading:

$$\mathfrak{c}_e = \bigoplus_{i \geqslant 0} \mathfrak{c}_e(i), \quad \mathfrak{c}_e(i) \cong \mathfrak{g}_e(i)/[\mathfrak{g}_e, \mathfrak{g}_e] \cap \mathfrak{g}(i).$$

Let $P(e)$ be the parabolic subgroup of G with $\mathfrak{p}(e) := \bigoplus_{i \geqslant 0} \mathfrak{g}(i)$. It is well known that $P(e)$ is the optimal parabolic subgroup for the G-unstable vector $e \in \mathfrak{g}$ in the sense of the Kempf–Rousseau theory; see [Pre03], for example. In particular, $G_e \subset P(e)$.

Recall that e is called *even* if all eigenvalues of $\operatorname{ad} h$ are in $2\mathbb{Z}$. It follows from the $\mathfrak{sl}_2$-theory that any even $e \in \mathfrak{g}$ is a Richardson element of $\mathfrak{p}(e)$. For e even, we denote by $d(e)$ the number of 2s on the weighted Dynkin diagram of e (see the second column of the tables in [Car85, pp. 401–407]). It is well known (and easy to see) that $d(e)$ coincides with the dimension of the centre of the Levi subalgebra $\mathfrak{g}(0)$ of $\mathfrak{g}$.

In what follows we shall rely on the detailed information on the centralisers of nilpotent elements obtained by Lawther and Testerman in [LT11]. In fact, we shall require the extended version of [LT11] which, due to its size, is only available as a preprint; see [LT07]. We shall also rely on de Graaf's computation of $c(e) = \dim \mathfrak{c}_e$ in [deG13] and the explicit description of sheets in exceptional Lie algebras obtained by Borho [Bor81] (in type F_4) and Elashvili [Ela84] (in type E) and recently double-checked by computational methods in [deGE09].

The number of sheets containing an induced nilpotent element is given in the third column of Tables 1–6, whilst their ranks can be found in the fourth column. The numbers $c(e)$ are listed in the fifth column. This information is taken from the tables in [deGE09, deG13] and included for the reader's convenience. We should stress at this point that the last column of Tables 1–6 contains new information which will only become available *after* we establish the main results of this subsection.

From now on we shall use freely the notation from the tables of [LT07].

PROPOSITION 13. *If $\mathfrak{g}$ is an exceptional Lie algebra and e is an even nilpotent element of $\mathfrak{g}$, then $d(e) = c_\Gamma(e)$ and $U(\mathfrak{g}, e)^{\mathrm{ab}}_\Gamma \cong S(\mathfrak{c}_e^\Gamma)$ is isomorphic to a polynomial algebra in $d(e)$ variables.*

Proof. (a) Since e is a Richardson element of $\mathfrak{p}(e)$ and $G_e \subset P(e)$, Proposition 12 implies that in the present case $\mathcal{E}^\Gamma \neq \emptyset$ and $\dim \mathcal{E}^\Gamma \geqslant d(e)$.

(b) If e lies in a single sheet of $\mathfrak{g}$ then inspecting Tables 1–6 reveals that $c(e) = r(e)$ in all cases. Since e is even we must have $r(e) = d(e)$. Applying Corollary 10(i) then yields that there is a Γ-equivariant algebra isomorphism $U(\mathfrak{g}, e)^{\mathrm{ab}} \cong S(\mathfrak{c})$ and $U(\mathfrak{g}, e)^{\mathrm{ab}}_\Gamma \cong S(\mathfrak{c}_e^\Gamma)$ as algebras. On the other hand, $\mathcal{E}^\Gamma \neq \emptyset$ and $\dim \mathcal{E}^\Gamma \geqslant d(e)$ by part (a). From this it follows that $\mathfrak{c}_e^\Gamma = \mathfrak{c}_e$, that is, Γ acts trivially on $\mathfrak{c}_e$, forcing

$$U(\mathfrak{g}, e)^{\mathrm{ab}} \cong U(\mathfrak{g}, e)^{\mathrm{ab}}_\Gamma \cong \Bbbk[X_1, \dots, X_{d(e)}].$$

From now on we assume that e lies in more than one sheet of $\mathfrak{g}$. According to Tables 1–6, in this case we always have that $\Gamma \neq \{1\}$. By part (a) and our discussion in § 3.1, at least one of the sheets containing e must have rank $d(e)$ but it may happen that $r(e) > d(e)$.

(c) In this part we assume that $\Gamma \cong S_2$. Inspecting Tables 1–6, one finds out that in this case $c(e) - d(e) \in \{1, 2\}$ (the values of $d(e)$ can be found in [Car85, pp. 401–407], for example). If $c(e) = d(e) + 1$, then combining Tables 1–6 with [Car85, pp. 405–407], one observes that the Dynkin label of e is one of $\mathsf{E}_8(\mathsf{b}_4)$, $\mathsf{D}_7(\mathsf{a}_1)$, $\mathsf{E}_6(\mathsf{a}_1)$, $\mathsf{D}_5 + \mathsf{A}_2$, $\mathsf{E}_6(\mathsf{a}_3)$ if $\mathfrak{g}$ is of type E_8, one of $\mathsf{E}_7(\mathsf{a}_4)$, $\mathsf{E}_6(\mathsf{a}_3)$, A_4 if $\mathfrak{g}$ is of type E_7, and one of $\mathsf{F}_4(\mathsf{a}_1)$, $\mathsf{F}_4(\mathsf{a}_2)$ if $\mathfrak{g}$ is of type F_4.

Suppose that $\mathfrak{g}$ is of type E_8. If e has type $\mathsf{E}_8(\mathsf{b}_4)$ then $\mathfrak{c}_e(4)$ is one-dimensional by [deG13]. Then [LT07, p. 290] yields that Γ is generated by the image of $h_4(-1)$ and $\mathfrak{c}_e(4)$ is spanned by the image of v_3. As $(\operatorname{Ad} c)(v_3) = -v_3$ we deduce that Γ acts non-trivially on $\mathfrak{c}_e$ so that $c_\Gamma(e) \leqslant d(e)$. Combining Corollary 10(ii) and part (a), we now deduce that $U(\mathfrak{g}, e)^{\mathrm{ab}}_\Gamma$ is a polynomial algebra in $d(e)$ variables.

If e has type $\mathsf{D}_7(\mathsf{a}_1)$ or $\mathsf{D}_5 + \mathsf{A}_2$ then $\mathfrak{c}_e(0)$ is one-dimensional by [deG13]. On the other hand, it follows from [LT07, pp. 263, 277] that $\mathfrak{g}_e(0)$ is a one-dimensional toral subalgebra of $\mathfrak{g}$ upon which Γ acts non-trivially. Then again $c_\Gamma(e) \leqslant d(e)$ and we can argue as before to conclude that $U(\mathfrak{g}, e)^{\mathrm{ab}}_\Gamma$ is a polynomial algebra in $d(e)$ variables.

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Now suppose that e has type $\mathsf{E}_6(\mathsf{a}_1)$. This case is more subtle due to the complicated nature of the generator of Γ. From [deG13] we know that $\mathfrak{c}_e(4)$ is one-dimensional, whilst [LT07, p. 261] yields that $\mathfrak{g}_e(0)$ is simple and the largest trivial $(\operatorname{ad}\mathfrak{g}_e(0))$-submodule of $\mathfrak{g}_e(4)$ is spanned by v_8. From this it follows that $\mathfrak{c}_e(4)$ is generated by the image of v_8. By [LT07, p. 261], the group Γ is generated by the image of

$$c = n_{\substack{0122210\\1}} n_{\substack{1122110\\1}} n_{\substack{1221110\\1}} h_1(-1)h_2(-1)h_3(-1)h_5(-1)h_6(-1)h_8(-1).$$

Since the roots ${}^{0122210}_{\ \ \ 1}$, ${}^{1122110}_{\ \ \ 1}$, ${}^{1221110}_{\ \ \ 1}$ and ${}^{0111000}_{\ \ \ 0}$ are pairwise orthogonal, we have that $(\operatorname{Ad} c)(e_{\substack{0111000\\0}}) = -e_{\substack{0111000\\0}}$. Since $e_{\substack{0111000\\0}}$ occurs with a non-zero coefficient in the expression of v_8 via Chevalley generators of $\mathfrak{g}$, this implies that $(\operatorname{Ad} c)(v_8) = -v_8$. But then $c_\Gamma(e) \leqslant d(e)$ and we can argue as in the previous cases to establish the polynomiality of $U(\mathfrak{g}, e)^{\mathrm{ab}}_\Gamma$.

If e has type $\mathsf{E}_6(\mathsf{a}_3)$ then [deG13] says that $\mathfrak{c}_e = \mathfrak{c}_e(2)$ is three-dimensional. In view of [LT07, p. 234] this means that $\mathfrak{c}_e \cong \mathfrak{g}_e(2)$ as Γ-modules (in the present case the group $C(e)^\circ$ acts trivially on $\mathfrak{g}_e(2)$). As $(\operatorname{Ad} c)(v_1) = -v_1$ we see that Γ acts non-trivially on $\mathfrak{c}_e$, implying $c_\Gamma(e) \leqslant d(e)$. But then again $U(\mathfrak{g}, e)^{\mathrm{ab}}_\Gamma$ is a polynomial algebra in $d(e)$ variables.

Suppose that $\mathfrak{g}$ is of type E_7. If e is of type $\mathsf{E}_7(\mathsf{a}_4)$ then $\mathfrak{c}_e = \mathfrak{c}_e(2)$ is four-dimensional by [deG13]. As $\dim \mathfrak{g}_e(e) = 4$ by [LT07, p. 155], the image of v_1 in $\mathfrak{c}_e$ is non-zero. Since Γ is generated by the image of $c = h_4(-1)$ and $(\operatorname{Ad} c)(v_1) = -v_1$ by *loc. cit.*, we argue as before to deduce that $U(\mathfrak{g}, e)^{\mathrm{ab}}_\Gamma$ is a polynomial algebra in $d(e)$ variables. The case where $\mathfrak{g}$ is of type E_7 and e is of type $\mathsf{E}_6(\mathsf{a}_3)$ is very similar. Here $\mathfrak{c}_e = \mathfrak{c}_2(2)$ is two-dimensional by [deG13], the group Γ is again generated by the image of $c = h_4(-1)$, the image of v_1 in $\mathfrak{c}_e$ is non-zero and $(\operatorname{Ad} c)(v_1) = -v_1$ by [LT07, p. 149].

If e is of type A_4 then $d(e) = 2$ and $\mathfrak{c}_e = \mathfrak{c}_e(0) \oplus \mathfrak{c}_e(2) \oplus \mathfrak{c}_e(6)$ and all non-zero $\mathfrak{c}_e(i)$ are one-dimensional; see [deG13]. By [LT07, p. 133], the group Γ is generated by the image of

$$c = n_{\substack{011110\\1}} n_{\substack{111110\\0}} n_{\substack{122211\\1}} n_{\substack{124321\\2}} h_2(-1)h_3(-1)h_4(-1)h_6(-1),$$

and $\mathfrak{g}_e(0) = [\mathfrak{g}_e(0), \mathfrak{g}_e(0)] \oplus \operatorname{Lie}(T_1)$ with $[\mathfrak{g}_e(0), \mathfrak{g}_e(0)] \cong \mathfrak{sl}_3$ and $\operatorname{Lie}(T_1)$ spanned by the element $t \in \operatorname{Lie}(T)$ such that $\alpha_i(t) = \delta_{6,i}$ for all $1 \leqslant i \leqslant 7$. Direct computation shows that $\operatorname{Ad} c$ negates t. But then $d(e) = 2 \geqslant c_\Gamma(e)$ and we can proceed as before.

Suppose that $\mathfrak{g}$ is of type F_4. If e is of type $\mathsf{F}_4(\mathsf{a}_1)$ then $\dim \mathfrak{c}_e(4) = 1$ by [deG13]. In view of [LT07, p. 81], this shows that the image of v_2 in $\mathfrak{c}_e$ is non-zero. Since Γ is generated by the image of $c = h_4(-1)$ and $(\operatorname{Ad} c)(v_2) = -v_2$ by *loc. cit.*, the result follows. If e is of type $\mathsf{F}_4(\mathsf{a}_2)$ then $\mathfrak{c}_2 = \mathfrak{c}_e(2)$ is three-dimensional by [deG13]. As $\dim \mathfrak{g}_2(2) = 3$, the image of v_1 in $\mathfrak{c}_e$ is non-zero. It remains to note that Γ is generated by the image of $c = h_2(-1)$ and $(\operatorname{Ad} c)(v_1) = -v_1$; see [LT07, p. 80].

If $c(e) = d(e) + 2$, then combining Tables 1–6 with [Car85, pp. 405–407], one observes that the Dynkin label of e is one of $\mathsf{E}_8(\mathsf{a}_3)$, $\mathsf{E}_8(\mathsf{a}_4)$, $\mathsf{E}_8(\mathsf{a}_5)$ if $\mathfrak{g}$ is of type E_8, one of $\mathsf{E}_7(\mathsf{a}_3)$, $\mathsf{E}_6(\mathsf{a}_1)$ if $\mathfrak{g}$ is of type E_7, and $\mathsf{E}_6(\mathsf{a}_3)$ if $\mathfrak{g}$ is of type E_6.

Suppose that $\mathfrak{g}$ is of type E_8. If e has type $\mathsf{E}_8(\mathsf{a}_3)$ then [deG13] says that both $\mathfrak{c}_e(8)$ and $\mathfrak{c}_e(16)$ are one-dimensional. In view of [LT07, p. 295] this implies that the images of v_3 and v_7 in $\mathfrak{c}_e$ are linearly independent. Since *loc. cit.* also shows that Γ is generated by the image of $c = h_4(-1)$ and $(\operatorname{Ad} c)(v_i) = -v_i$ for $i = 3, 7$, we deduce that $c_\Gamma(e) \leqslant d(e)$. Arguing as before, we now conclude that in the present case $U(\mathfrak{g}, e)^{\mathrm{ab}}_\Gamma$ is a polynomial algebra in $d(e)$ variables.

If e has type $\mathsf{E}_8(\mathsf{a}_4)$ then [deG13] shows that both $\mathfrak{c}_e(4)$ and $\mathfrak{c}_e(8)$ are one-dimensional. Thanks to [LT07, p. 293] this yields that the images of v_2 and v_4 in $\mathfrak{c}_e$ are linearly independent. Since

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loc. cit. also shows that Γ is generated by the image of $c = h_4(-1)h_8(-1)$ and $(\operatorname{Ad} c)(v_i) = -v_i$ for $i = 2, 4$, we deduce that $c_\Gamma(e) \leqslant d(e)$. The result then follows as in the previous case.

Now suppose that e is of type $\mathsf{E}_8(\mathsf{a}_5)$. In this case we have to work harder. First note that $\mathfrak{c}_e = \mathfrak{c}_e(2) \oplus \mathfrak{c}_2(10)$ and $\dim \mathfrak{c}_e(10) = 2$ by [deG13]. Since e is distinguished, $\mathfrak{g}_e(2)$ maps isomorphically onto $\mathfrak{c}_e(2)$. By [LT07, pp. 288, 289], the group Γ is generated by the image of $c = h_4(-1)h_7(-1)$ and $\mathfrak{g}_e(2)$ has basis $\{v_1, v_2, e\}$ such that $(\operatorname{Ad} c)(v_1) = -v_1$ and $(\operatorname{Ad} c)(v_2) = v_2$. Since $\mathfrak{g}_e(i) \subset [\mathfrak{g}_e, \mathfrak{g}_e]$ for $i = 4, 6, 8$ by [deG13], it follows from [LT07, p. 288] that the subspaces $\mathfrak{g}_e(4) = \Bbbk v_4$, $\mathfrak{g}_e(6) = \Bbbk v_5$ and $\mathfrak{g}_e(8) = \Bbbk v_6$ are spanned by $[v_1, v_2]$, $[v_2, [v_2, v_1]]$ and $[v_2, [v_2, [v_2, v_1]]]$, respectively (one should keep in mind here that $(\operatorname{Ad} c)(v_4) = -v_4$, $(\operatorname{Ad} c)(v_5) = v_5$ and $(\operatorname{Ad} c)(v_6) = -v_6$, which is immediate from [LT07, p. 289]). Also, $[v_1, [v_1, v_2]] = 0$. Since

$$\mathfrak{g}(10) \cap [\mathfrak{g}_e, \mathfrak{g}_e] = [\mathfrak{g}_e(2), \mathfrak{g}_e(8)] + [\mathfrak{g}_e(4), \mathfrak{g}_e(6)],$$

the left-hand side is spanned by $u_1 := [v_1, [v_2, [v_2, [v_2, v_1]]]]$, $u_2 := [v_2, [v_2, [v_2, [v_2, v_1]]]]$ and $u_3 := [[v_1, v_2], [v_2, [v_2, v_1]]]$. As $[v_1, [v_2, v_1]] = 0$, the Jacobi identity yields $u_3 = u_1 - [v_2, [v_1, [v_2, [v_2, v_1]]]] = u_1$. In view of [deG13] this implies that u_1 and u_2 form a basis of $\mathfrak{g}(10) \cap [\mathfrak{g}_e, \mathfrak{g}_e]$. Note that $(\operatorname{Ad} c)(u_1) = u_1$ and $(\operatorname{Ad} c)(u_2) = -u_2$. Since it follows from [LT07, p. 289] (with the misprint in the expression for v_7 corrected in [LT11, p. 179]) that the kernel of $(\operatorname{Ad} c + \operatorname{Id})_{|\mathfrak{g}_e(10)}$ is two-dimensional, we are now able to conclude that $c_\Gamma(e) \leqslant d(e)$, which yields the desired result in the present case.

Suppose that $\mathfrak{g}$ is of type E_7. If e has type $\mathsf{E}_7(\mathsf{a}_3)$ then $\dim \mathfrak{c}_e(4) = \dim \mathfrak{c}_e(8) = 1$ by [deG13], whilst [LT07, p. 160] says that $\mathfrak{g}_e(4) = \Bbbk v_3$, $\mathfrak{g}_e(8) = \Bbbk v_6$ and Γ is generated by the image of $c = h_4(-1)$. Since $(\operatorname{Ad} c)(v_3) = -v_3$ and $(\operatorname{Ad} c)(v_6) = -v_6$, this implies that $c_\Gamma(e) \leqslant d(e)$ as desired.

If $\mathfrak{g}$ is of type E_7 and e has type $\mathsf{E}_6(\mathsf{a}_1)$ then $d(e) = 3$ by [Car85, p. 404] and $\dim \mathfrak{c}_e(0) = \dim \mathfrak{c}_e(4) = 1$ by [deG13]. By [LT07, p. 158], we have that the reductive part $\mathfrak{g}_e(0) = \operatorname{Lie}(C(e))$ is one-dimensional and Γ is generated by the image of

$$c = n_{\substack{012221\\1}} n_{\substack{112211\\1}} n_{\substack{122111\\1}} h_1(-1)h_2(-1)h_3(-1)h_5(-1)h_6(-1).$$

Direct computation shows that $\operatorname{Ad} c$ acts as $-\operatorname{Id}$ on the one-dimensional toral subalgebra $\mathfrak{g}_e(0)$ and the basis vectors $v_2, v_3 \in \mathfrak{g}_e(4)$ have non-zero weights with respect to the adjoint action of the torus $C(e)^\circ$. Since $\dim \mathfrak{g}_e(4) = 3$, it follows that the image of v_4 in $\mathfrak{c}_e$ is non-zero. As the roots $\substack{012221\\1}$, $\substack{112211\\1}$, $\substack{122111\\1}$ and $\substack{0111000\\0}$ are pairwise orthogonal, it must be that $(\operatorname{Ad} c)(e_{\substack{011100\\0}}) = -e_{\substack{011100\\1}}$. As $e_{\substack{011100\\0}}$ occurs with a non-zero coefficient in the expression of v_4 via Chevalley generators of $\mathfrak{g}$ we deduce that $(\operatorname{Ad} c)(v_4) = -v_4$. But then $c_\Gamma(e) \leqslant d(e)$ and we can argue as in the previous cases to establish the polynomiality of $U(\mathfrak{g}, e)^{\mathrm{ab}}_\Gamma$.

If $\mathfrak{g}$ is of type E_6 and e has type $\mathsf{E}_6(\mathsf{a}_3)$ then $\dim \mathfrak{c}_e(2) = 3$ and $\dim \mathfrak{c}_e(4) = 2$ by [deG13], whilst [LT07, p. 100] shows that Γ is generated by the image of $c = h_4(-1)$ and $\mathfrak{g}_e(2)$ has basis $\{v_1, v_2, e\}$ such that $(\operatorname{Ad} c)(v_1) = -v_1$ and $(\operatorname{Ad} c)(v_2) = v_2$. It is also immediate from *loc. cit.* that $[\mathfrak{g}_2(e), \mathfrak{g}_2(e)]$ has dimension 1. Since $\operatorname{Ad} c$ negates both v_5 and v_6 and these vectors are linearly independent in $\mathfrak{g}_e(4)$, we get $c_\Gamma(e) \leqslant d(e)$, which yields the desired result in the present case.

(d) Next we assume that $\Gamma \cong S_3$. In this case e is one of $\mathsf{E}_8(\mathsf{b}_5)$, $\mathsf{E}_8(\mathsf{b}_6)$ or $\mathsf{D}_4(\mathsf{a}_1)$ if $\mathfrak{g}$ is of type E_8, one of $\mathsf{E}_7(\mathsf{a}_5)$ or $\mathsf{D}_4(\mathsf{a}_1)$ if $\mathfrak{g}$ is of type E_7, has type $\mathsf{D}_4(\mathsf{a}_1)$ if $\mathfrak{g}$ is of type E_6 and has type $\mathsf{G}_2(\mathsf{a}_1)$ if $\mathfrak{g}$ is of type G_2.

If e is of type $\mathsf{E}_8(\mathsf{b}_5)$ then $d(e) = 3$ and $\mathfrak{c}_e = \mathfrak{c}_e(2) \oplus \mathfrak{c}_e(4) \oplus \mathfrak{c}_e(10)$ where $\dim \mathfrak{c}_e(2) = 4$, $\dim \mathfrak{c}_e(6) = 2$ and $\dim \mathfrak{c}_e(10) = 1$; see [deG13]. On the other hand, [LT07, pp. 285, 286] shows that $\dim \mathfrak{g}_e(2)=4$ and $\dim \mathfrak{g}_e(6)=2$, implying that the canonical homomorphism $\mathfrak{g}_e \to \mathfrak{g}_e/[\mathfrak{g}_e, \mathfrak{g}_e]$ is bijective on $\mathfrak{g}_e(2)\oplus \mathfrak{g}_e(6)$. It also follows from *loc. cit.* that Γ contains the image of $c_1 = h_1(\omega)h_2(\omega)h_5(\omega^2)$ where ω is a third primitive root of 1. Direct computation shows that $(\operatorname{Ad} c_1)(v_1)=\omega v_1$, $(\operatorname{Ad} c_1)(v_2)=\omega^2 v_2$, $(\operatorname{Ad} c_1)(v_6)=\omega^2 v_6$ and $(\operatorname{Ad} c_1)(v_7)=\omega v_7$. From this it is immediate that $c_\Gamma(e)\leqslant 7-4=d(e)$. So we can argue as before to deduce the polynomiality of $U(\mathfrak{g},e)^{\mathrm{ab}}_\Gamma$.

If e is of type $\mathsf{E}_8(\mathsf{b}_6)$ then $d(e)=2$ and $\mathfrak{c}_e=\mathfrak{c}_e(2)\oplus\mathfrak{c}_e(4)$ where $\dim \mathfrak{c}_e(2)=4$ and $\dim \mathfrak{c}_e(4)=1$; see [deG13]. By [LT07, p. 275], $\dim \mathfrak{g}_e(2)=\dim \mathfrak{g}_e(4)=4$ and Γ is generated by $c_1=h_1(\omega)h_2(\omega)h_5(\omega^2)$, where ω is a third primitive root of 1, and by

$$c_2 = n_{\substack{1000000\\0}}\, n_{\substack{0000000\\1}}\, n_{\substack{0001000\\0}}\, h_2(-1)h_3(-1)h_4(-1)h_5(-1)h_8(-1).$$

Direct verification shows that $(\operatorname{Ad} c_1)(v_1)=\omega v_1$, $(\operatorname{Ad} c_1)(v_2)=\omega^2 v_2$ and $(\operatorname{Ad} c_1)(v_3)=v_3$, which in view of *loc. cit.* implies that $\dim \mathfrak{c}_e(2)^\Gamma\leqslant 2$. Similarly, $(\operatorname{Ad} c_1)(v_6)=\omega v_6$, $(\operatorname{Ad} c_1)(v_7)=\omega^2 v_7$, $(\operatorname{Ad} c_1)(v_5)=v_5$ and $(\operatorname{Ad} c_1)(v_8)=v_8$. Since $\{v_1,v_2,v_3,e\}$ and $\{v_5,v_6,v_7,v_8\}$ are bases of $\mathfrak{g}_e(2)$ and $\mathfrak{g}_e(4)$, respectively, and $\dim\mathfrak{c}_e(4)=1$ by our earlier remark, the vectors $[v_1,v_2]$, $[v_1,v_3]$ and $[v_2,v_3]$ must form a basis of $\mathfrak{g}_2(4)\cap[\mathfrak{g}_e,\mathfrak{g}_e]$. Comparing the respective eigenvalues for $\operatorname{Ad} c_1$ yields $v_6, v_7\in[\mathfrak{g}_e,\mathfrak{g}_e]$. Using the explicit formulae for v_1 and v_2 in [LT07, p. 276], one observes that $e_{\substack{1222210\\1}}$ occurs with coefficient ± 3 in the expression of $[v_1,v_2]$ via Chevalley generators of $\mathfrak{g}$. As a consequence, $v_5+\lambda v_8\in[\mathfrak{g}_e,\mathfrak{g}_e]$ for some $\lambda\in\Bbbk$, implying that $\mathfrak{c}_e(4)$ is generated by the image of $v_8=e_{\substack{1221000\\1}}$. Since the roots

$$\substack{1000000\\0},\ \substack{0000000\\1},\ \substack{0001000\\0},\ \substack{1221000\\1}$$

are pairwise orthogonal, we have that $(\operatorname{Ad} c_2)(v_8)=-v_8$. But then $c_\Gamma(e)=\dim\mathfrak{c}_e(2)^\Gamma\leqslant 2=d(e)$ and we can argue as in the previous cases.

Now suppose that e has type $\mathsf{D}_4(\mathsf{a}_1)$. Then $d(e)=1$ by [Car85, pp. 402, 403, 405]. If $\mathfrak{g}$ is of type E_8 then $\mathfrak{c}_e=\mathfrak{c}_e(2)$ has dimension 3 by [deG13] and $[\mathfrak{g}_e(0),\mathfrak{g}_e(2)]$ has codimension 3 in $\mathfrak{g}_e(2)$ by [LT07, pp. 190, 191]. This implies that $\mathfrak{c}_e$ is generated by the images of v_{25}, $v_{26}=e_2+e_5$ and $v_{27}=e$. By *loc. cit.*, the group Γ is generated by the images of $c_1=n_{\substack{1110000\\1}}\,n_{\substack{1111000\\0}}\,h_2(-1)$ and $c_2=(n_{\substack{1221100\\1}}\,n_{\substack{1122100\\1}}\,h_1(-1)h_2(-1)h_6(-1))^g$ where

$$g = x_{\substack{0010000\\0}}(\tfrac{1}{3})\,n_{\substack{0010000\\0}}\,h_1(4)h_2(-4)h_3(16)h_4(-48)h_5(16)h_6(-8)\,x_{\substack{0010000\\0}}(-\tfrac{1}{3}).$$

Since $\operatorname{Ad} c_1$ fixes e and permutes the lines $\Bbbk e_2$ and $\Bbbk e_5$, it must permute e_2 and e_5. But then $(\operatorname{Ad} c_1)(v_{26})=v_{26}$. Similarly, $\operatorname{Ad} c_1$ must permute $e_{\substack{0010000\\1}}$ and $e_{\substack{0011000\\0}}$. Since the roots $\substack{1110000\\1}$, $\substack{1111000\\0}$, $\substack{0110000\\0}$ are pairwise orthogonal and $e_{\substack{0110000\\0}}$ occurs with a non-zero coefficient in the expression of v_{25} via Chevalley generators of $\mathfrak{g}$, it must be that $(\operatorname{Ad} c_1)(v_{25})=-v_{25}\pm 2v_{26}$. Note that $(\operatorname{Ad} g)(e_2)$ is a linear combination of e_2 and $e_{\substack{0010000\\1}}$ and $(\operatorname{Ad} g)(e_5)$ is a linear combination of e_5 and $e_{\substack{0011000\\0}}$. From this it is immediate that $(\operatorname{Ad} c_2)(v_{26})$ is a linear combination of e_2, $e_{\substack{0010000\\1}}$, e_3 and $e_{\substack{0110000\\0}}$. In particular, $(\operatorname{Ad} c_2)(v_{26})\neq v_{26}$. In conjunction with our earlier remarks, this implies that $\mathfrak{c}_e^\Gamma$ is spanned by the image of e. Therefore, $c_\Gamma(e)=d(e)$ and we can argue as before to deduce the polynomiality of $U(\mathfrak{g},e)^{\mathrm{ab}}_\Gamma$.

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The cases where e is of type $\mathsf{D}_4(\mathsf{a}_1)$ and $\mathfrak{g}$ is of type E_6 or E_7 are very similar because here e, c_1 and c_2 have the same expressions as in the previous case; see [LT07, pp. 93, 124]. If $\mathfrak{g}$ is of type E_7, then $\mathfrak{c}_e = \mathfrak{c}_e(2)$ is three-dimensional, whilst in type E_6 we have that $\mathfrak{c}_e = \mathfrak{c}_e(0) \oplus \mathfrak{c}_e(2)$ where $\dim \mathfrak{c}_e(2) = 3$ and $\mathfrak{c}_e(0) \cong \mathfrak{g}_e(0)$ as vector spaces; see [deG13]. Arguing as in the E_8-case, we obtain that $\mathfrak{c}_e(2)^\Gamma$ is generated by the image of e. This takes care of the E_7-case and reduces the E_6-case to verifying that the group generated by c_1 and c_2 acts fixed point freely on the two-dimensional toral subalgebra $\mathfrak{g}_e(0)$. Since $\mathfrak{g}_e(0)$ is described explicitly in [LT07, p. 93], the latter is easily seen by a direct computation (we leave the details to the interested reader).

Finally, suppose that $\mathfrak{g}$ is of type G_2 and e has type $\mathsf{G}_2(\mathsf{a}_1)$. Then $\mathfrak{c}_e = \mathfrak{c}_e(2)$ is three-dimensional by [deG13] and Γ contains the image of $c_1 = h_1(\omega)$ where ω is a primitive third root of 1; see [LT07, p. 66]. Since $\mathfrak{c}_e(2) \cong \mathfrak{g}_e(2)$ has basis $\{e_{11}, e_{21}, e\}$ and e_{11}, e_{21} are short root vectors, it is straightforward to see that $\mathfrak{c}_e^\Gamma$ is spanned by the image of e. Then $c_\Gamma(e) = d(e)$ and we can argue as in the previous cases.

(d) If $\Gamma \cong S_4$ then $\mathfrak{g}$ is of type F_4 and e has type $\mathsf{F}_4(\mathsf{a}_3)$. In this case $\mathfrak{c}_e = \mathfrak{c}_e(2) \cong \mathfrak{g}_e(2)$ is six-dimensional, whilst $\Gamma \cong C(e)$ is generated by $c_1 = h_1(\omega)h_3(\omega)$, $c_2 = n_{1000}\, n_{0010}\, h_2(-1)h_3(-1)$ and $c_3 = (n_{0011}\ h_3(-\frac{2}{3})h_4(\frac{2}{3}))^u$ where ω is a third primitive root of 1 and $u = x_{0011}(-\frac{1}{2})x_{0001}(1)x_{0010}(-1)$; see [LT07, p. 77]. Straightforward verification shows that $(\operatorname{Ad} c_1)(v_i) = \omega v_i$ for $i = 2, 4$, $(\operatorname{Ad} c_1)(v_i) = \omega^{-1} v_i$ for $i = 1, 3$ and $(\operatorname{Ad} c_1)(v_5) = v_5$. Since $\mathfrak{g}_e(2)$ has basis $\{v_1, v_2, v_3, v_4, v_5, e\}$, it follows that $\mathfrak{g}_e(2)^\Gamma \subseteq \operatorname{span}\{v_5, e\}$. By [LT07, p. 77], $e = e_{0100} + e_{1120} + e_{1111} + e_{0121}$ and $v_5 = e_{0100} + e_{1120}$. Since $c_2 \in G_e$ and $\operatorname{Ad} c_2$ permutes the lines $\Bbbk e_{0100}$ and $\Bbbk e_{1120}$, it must be that $(\operatorname{Ad} c_2)(v_5) = v_5$.

Unfortunately, this means that we have to examine $(\operatorname{Ad} c_3)(v_5)$ which is rather more complicated. Suppose for a contradiction that $(\operatorname{Ad} c_3)(v_5) = v_5$. Then $(\operatorname{Ad} c_3)(e_{1111} + e_{0121}) = e_{1111} + e_{0121}$. Note that $(\operatorname{Ad} u)^{-1}(e_{1111} + e_{0121}) \equiv e_{1111} + e_{0121} \mod V$ where $V = \operatorname{span}\{e_{0122}, e_{1121}, e_{1122}\}$. It follows that

$$\operatorname{Ad}\big(n_{0011}\ h_3(-\tfrac{2}{3})h_4(\tfrac{2}{3})u^{-1}\big)(e_{1111} + e_{0121}) \equiv \lambda e_{1111} + \mu e_{0110} \mod n_{0011}(V)$$

for some $\lambda, \mu \in \Bbbk^\times$. Since $n_{0011}(V) = \operatorname{span}\{e_{0100}, e_{1110}, e_{1100}\}$ we have that

$$\operatorname{Ad}\big(u^{-1}c_3\big)(e_{1111} + e_{0121}) = \lambda e_{1111} + \mu e_{0110} + a e_{0100} + b e_{1110} + c e_{1100}$$

for some $a, b, c \in \Bbbk$. If $a \neq 0$ then e_{0100} would occur with a non-zero coefficient in the expression of $(\operatorname{Ad} c_3)(v_5) = (\operatorname{Ad} u)(\lambda e_{1111} + \mu e_{0110} + a e_{0100} + b e_{1110} + c e_{1100})$ via Chevalley generators of $\mathfrak{g}$ contrary to our assumption that $\operatorname{Ad} c_3$ fixes v_5. Hence $a = 0$. But then e_{0110} occurs with coefficient $\mu \neq 0$ in the expression of $(\operatorname{Ad} c_3)(v_5)$ via Chevalley generators of $\mathfrak{g}$, a contradiction. We thus conclude that $\mathfrak{g}_e(2)^\Gamma = \Bbbk e$, which yields $c_\Gamma(e) = 1 = d(e)$.

(e) Finally, suppose that $\Gamma \cong S_5$. Then $\mathfrak{g}$ is of type E_8 and e has type $\mathsf{E}_8(\mathsf{a}_7)$. By [LT07, p. 251], the group $\Gamma \cong C(e)$ contains $c_1 = h_2(\zeta)h_3(\zeta^4)h_4(\zeta)h_6(\zeta^4)h_7(\zeta)h_8(\zeta^2)$, where ζ is a fifth primitive root of 1, and

$$c_2 = n_{\substack{0100000\\0}}\, n_{\substack{0010000\\0}}\, n_{\substack{0000000\\1}}\, n_{\substack{0000100\\0}}\, n_{\substack{0000010\\0}}\, n_{\substack{0000001\\0}}\, h,$$

where $h = h_1(-1)h_3(-1)h_5(-1)h_6(-1)h_8(-1)$. By [deG13], we have that $\mathfrak{c}_e = \mathfrak{c}_e(2) \cong \mathfrak{g}_e(2)$. Direct computation shows that the basis $\{v_1, v_2, \ldots, v_9, e\}$ of $\mathfrak{g}_e(2)$ described in [LT07, p. 256] consists of eigenvectors for $\operatorname{Ad} c_1$. More precisely, one has $(\operatorname{Ad} c_1)(v_i) = \zeta v_i$ for $i = 2, 5$, $(\operatorname{Ad} c_1)(v_i) = \zeta^2 v_i$ for $i = 1, 8$, $(\operatorname{Ad} c_1)(v_i) = \zeta^3 v_i$ for $i = 3, 6$, $(\operatorname{Ad} c_1)(v_i) = \zeta^4 v_i$ for $i = 4, 7$ and $(\operatorname{Ad} c_1)(v_9) = v_9$. Therefore, $\mathfrak{g}_e(2)^\Gamma \subseteq \operatorname{span}\{v_9, e\}$. Since

$$v_9 = e_{\substack{0001000\\0}} + e_{\substack{1121100\\1}} + e_{\substack{1111111\\0}} + e_{\substack{1121110\\1}},$$

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we see that $(\operatorname{Ad} c_2)(v_9)$ is a non-zero linear combination of $e_{\substack{0111100\\0}}$, $e_{\substack{1111110\\1}}$, $e_{\substack{1221000\\1}}$ and $e_{\substack{1111111\\1}}$. But then $(\operatorname{Ad} c_2)(v_9) \neq v_9$, forcing $\mathfrak{g}_e(2)^\Gamma = \Bbbk e$ and implying that $c_\Gamma(e) = 1 = d(e)$. So we can argue as before to show that $U(\mathfrak{g}, e)^{\mathrm{ab}}_\Gamma$ is a polynomial algebra in $d(e)$ variables. The proof of the proposition is now complete. □

Next we shall investigate the case where e is an induced nilpotent element of $\mathfrak{g}$ which is not even and lies in a single sheet of $\mathfrak{g}$.

Proposition 14. *Suppose that e is not even, induced, and lies in a single sheet $\mathcal{S}(e)$ of $\mathfrak{g}$. Assume further that e is not of type $\mathsf{E}_7(\mathsf{a}_2)$ or $\mathsf{E}_6(\mathsf{a}_3)+\mathsf{A}_1$ if $\mathfrak{g}$ is of type E_8, not of type $\mathsf{D}_6(\mathsf{a}_2)$ if $\mathfrak{g}$ is of type E_7 or E_8, not of type $\mathsf{A}_3+\mathsf{A}_1$ if $\mathfrak{g}$ is of type E_6, and not of type $\mathsf{C}_3(\mathsf{a}_1)$ if $\mathfrak{g}$ is of type F_4. Then $c(e) = \operatorname{rk} \mathcal{S}(e)$ and $U(\mathfrak{g}, e)^{\mathrm{ab}}$ is a polynomial algebra in $c(e)$ variables. Furthermore, $U(\mathfrak{g}, e)^{\mathrm{ab}}_\Gamma \cong S(\mathfrak{c}_e^\Gamma)$ as algebras and the value of $c_\Gamma(e) = \dim \mathfrak{c}_e^\Gamma$ is given in the sixth column of Tables 1–6.*

Proof. If e satisfies the above conditions then $c(e) = \operatorname{rk} \mathcal{S}(e)$ by [deG13, Proposition 2]. Corollary 10(i) then shows that $U(\mathfrak{g}, e)^{\mathrm{ab}} \cong S(\mathfrak{c}_e)$ and $U(\mathfrak{g}, e)^{\mathrm{ab}}_\Gamma \cong S(\mathfrak{c}_e^\Gamma)$ as $\Bbbk$-algebras. Since the value of $c(e)$ is computed in [deG13] in all cases, it remains to determine the value of $c_\Gamma(e)$. We thus may assume from now on that $\Gamma \neq \{1\}$. Inspecting Tables 1–6, one observes that this happens only if $\mathfrak{g}$ is of type E_8 or E_7 and $\Gamma \cong S_2$.

(a) Suppose that $\mathfrak{g}$ is of type E_8. Then e is one of $\mathsf{E}_7(\mathsf{a}_3)$, $\mathsf{E}_6(\mathsf{a}_1)+\mathsf{A}_1$, $\mathsf{D}_6(\mathsf{a}_1)$, $\mathsf{A}_4+2\mathsf{A}_1$, $\mathsf{D}_5(\mathsf{a}_1)$ or $\mathsf{A}_4+\mathsf{A}_1$.

If e is of type $\mathsf{E}_7(\mathsf{a}_3)$ then [deG13] shows that $\mathfrak{c}_e = \mathfrak{c}_e(2)\oplus\mathfrak{c}_e(4)\oplus\mathfrak{c}_e(6)\oplus\mathfrak{c}_e(8)$ and each non-zero $\mathfrak{c}_e(i)$ is one-dimensional. Since $[\mathfrak{c}_e(1), \mathfrak{c}_e(1)]$ is one-dimensional by [deG13], using the explicit expressions for the v_i given in [LT07, p. 272] it is straightforward to see that $[v_1, v_2] \in \Bbbk^\times v_3$ and $[v_3, v_5] = \pm v_9$. This implies that $\mathfrak{c}_e$ has basis consisting of the images of e, v_5, v_8 and v_{10}. By [LT07, p. 271], the group Γ is generated by the image of $c = h_4(-1)$. As $(\operatorname{Ad} c)(v_5) = -v_5$, $(\operatorname{Ad} c)(v_8) = v_8$ and $(\operatorname{Ad} c)(v_{10}) = -v_{10}$ we deduce that $c_\Gamma(e) = 2$ in the present case.

If e is of type $\mathsf{E}_6(\mathsf{a}_1)+\mathsf{A}_1$ then [deG13] shows that $\mathfrak{c}_e = \mathfrak{c}_e(0)\oplus\mathfrak{c}_e(2)\oplus\mathfrak{c}_e(4)$ and each non-zero $\mathfrak{c}_e(i)$ is one-dimensional. It follows from [LT07, pp. 271, 272] that $\mathfrak{g}_e(0)$ is a one-dimensional toral subalgebra spanned by the element $t \in \operatorname{Lie}(T)$ such that $\alpha_i(t) = \delta_{7,i}$ for $1 \leqslant i \leqslant 8$. The explicit expressions for the v_i in *loc. cit.* show that $[v_1, v_2] = \pm v_3$, $[v_1, v_6] = \pm v_7$ and $[v_2, v_5] = \pm v_8$. It follows that the images of t, e and v_9 form a basis of $\mathfrak{c}_e$. It is also shown in *loc. cit.* that Γ is generated by the image of

$$c = n_{\substack{1244321\\2}} n_{\substack{1343321\\2}} n_{\substack{2343221\\2}} h_1(-1)h_2(-1)h_3(-1)h_5(-1)h_6(-1)h_8(-1).$$

It is straightforward to see that $(\operatorname{Ad} c)(t) = -t$. Since the roots $\substack{1244321\\2}$, $\substack{1343321\\2}$, $\substack{2343221\\2}$ and $\substack{0111000\\0}$ are pairwise orthogonal, we have that $(\operatorname{Ad} c)(e_{\substack{0111000\\0}}) = -e_{\substack{0111000\\0}}$. Since $e_{\substack{0111000\\0}}$ occurs with a non-zero coefficient in the expression of v_9 via Chevalley generators of $\mathfrak{g}$, this implies that $(\operatorname{Ad} c)(v_9) = -v_9$. As a result, $\mathfrak{c}_e^\Gamma$ is spanned by the image of e and hence $c_\Gamma(e) = 1$.

If e is of type $\mathsf{D}_6(\mathsf{a}_1)$ then [deG13] shows that $\mathfrak{c}_e = \mathfrak{c}_e(2)\oplus\mathfrak{c}_e(10)$ where $\dim \mathfrak{c}_e(2) = 2$ and $\dim \mathfrak{c}_e(10) = 1$. By [LT07, pp. 256, 257], the Lie algebra $\mathfrak{g}_e(0)$ is semisimple and $\mathfrak{g}_e(2)^{\operatorname{ad} \mathfrak{g}_e(0)}$ is spanned by v_5 and e, whilst $\mathfrak{g}_e(10)$ is spanned by v_{26} and v_{27}. Since it is easy to see that $[v_1, v_{20}] = \pm v_{26}$, the images of e, v_5 and v_{27} form a basis of $\mathfrak{c}_e$. By *loc. cit.*, the group Γ is generated by the image of $c = n_{\substack{0011111\\1}} n_{\substack{0111111\\0}} h_2(i)h_3(i)$, where i is a fourth primitive root of 1, and $\operatorname{Ad} c$ fixes v_{27}. Note that $v_5 = e_2 + e_3$ and $\operatorname{Ad} c$ permutes the lines $\Bbbk e_2$ and $\Bbbk e_3$. As $e = e_2 + e_3 +$ (sum of other root vectors) and c fixes e, it must be that $(\operatorname{Ad} c)(v_5) = v_5$. This shows that $c_\Gamma(e) = c(e) = 3$.

If e is of type $\mathsf{A}_4 + 2\mathsf{A}_1$ then $\mathfrak{c}_e = \mathfrak{c}_e(0)$ is one-dimensional by [deG13]. On the other hand, it is immediate from [LT07, p. 221] that $\mathfrak{g}_e(0) = [\mathfrak{g}_e(0), \mathfrak{g}_e(0)] \oplus \operatorname{Lie}(T_1)$ where $[\mathfrak{g}_e(0), \mathfrak{g}_e(0)] \cong \mathfrak{sl}_2$ and $\operatorname{Lie}(T_1)$ is a one-dimensional toral subalgebra spanned by the element $t \in \operatorname{Lie}(T)$ such that $\alpha_i(t) = \delta_{5,i}$ for all $1 \leqslant i \leqslant 8$. This implies that $\mathfrak{c}_e$ is spanned by the image of t. Since Γ is generated by the image of

$$c = n_{\substack{1222210\\1}} n_{\substack{1222111\\1}} n_{\substack{1354321\\3}} n_{\substack{2354321\\2}} h_1(-1)h_3(-1)h_4(-1)h_6(-1),$$

it is straightforward to check that $(\operatorname{Ad} c)(t) = -t$. So $c_\Gamma(e) = 0$ in the present case and hence $U(\mathfrak{g}, e)^{\mathrm{ab}}_\Gamma \cong \Bbbk$ (i.e. $\mathcal{E}^\Gamma$ is a single point!).

If e is of type $\mathsf{D}_5(\mathsf{a}_1)$ then [deG13] shows that $\mathfrak{c}_e = \mathfrak{c}_e(2) \oplus \mathfrak{c}_e(4)$ and $\dim \mathfrak{c}_e(2) = \dim \mathfrak{c}_e(4) = 1$. It follows from [LT07, pp. 219, 220] that $\mathfrak{g}_e(0) \cong \mathfrak{sl}_4$, the subspace $\mathfrak{g}_e(2)^{\operatorname{ad} \mathfrak{g}_e(0)}$ has basis $\{v_{15}, e\}$, and $\mathfrak{g}_e(4)$ is spanned by v_{17}. Direct computation shows that $[v_1, v_8] = \pm v_{15}$, implying that the images of e and v_{17} span $\mathfrak{c}_e$. By [LT07, p. 219], the group Γ is generated by the image of $c = n_{\substack{1233210\\1}} n_{\substack{1232210\\2}} h_1(-1)h_2(-1)h_4(-1)$. Since the roots $\substack{1233210\\1}$, $\substack{1232210\\2}$, $\substack{1110000\\0}$ are pairwise orthogonal, $(\operatorname{Ad} c)(e_{\substack{1110000\\0}}) = -e_{\substack{1110000\\0}}$. Since $v_{17} = e_{\substack{1110000\\0}} +$ (sum of other root vectors), we have $(\operatorname{Ad} c)(v_{17}) = -v_{17}$. As a result, $c_\Gamma(e) = 1$.

If e is of type $\mathsf{A}_4 + \mathsf{A}_1$ then $\mathfrak{c}_e = \mathfrak{c}_e(0)$ is one-dimensional by [deG13]. On the other hand, [LT07, p. 214] shows that $\mathfrak{g}_e(0) = [\mathfrak{g}_e(0), \mathfrak{g}_e(0)] \oplus \operatorname{Lie}(T_1)$ where $[\mathfrak{g}_e(0), \mathfrak{g}_e(0)] \cong \mathfrak{sl}_3$ and $\operatorname{Lie}(T_1)$ is spanned by the element $t \in \operatorname{Lie}(T)$ such that $\alpha_5(t) = 3$, $\alpha_7(t) = -5$ and $\alpha_i(t) = 0$ for $i = 1, 2, 3, 4, 6, 8$. Since Γ is generated by the image of

$$c = n_{\substack{0111110\\1}} n_{\substack{1111110\\0}} n_{\substack{1222100\\1}} n_{\substack{1243210\\2}} h_1(-1)h_3(-1)h_4(-1)h_6(-1)h_8(-1),$$

it is straightforward to check that $\operatorname{Ad} c$ negates t. We thus deduce that $c_\Gamma(e) = 0$ and hence $\mathcal{E}^\Gamma$ is a single point!

(b) Now suppose that $\mathfrak{g}$ is of type E_7. Then e is one of $\mathsf{D}_5(\mathsf{a}_1)$, $\mathsf{A}_4 + \mathsf{A}_1$, $\mathsf{D}_4(\mathsf{a}_1) + \mathsf{A}_1$ or $\mathsf{A}_2 + \mathsf{A}_1$.

If e is of type $\mathsf{D}_5(\mathsf{a}_1)$ then $\mathfrak{c}_e = \mathfrak{c}_e(0) \oplus \mathfrak{c}_e(2) \oplus \mathfrak{c}_e(4)$ and all non-zero $\mathfrak{c}_e(i)$ are one-dimensional; see [deG13]. By [LT07, p. 140], we have that $\mathfrak{g}_e(0) = [\mathfrak{g}_e(0), \mathfrak{g}_e(0)] \oplus \operatorname{Lie}(T_1)$, where $[\mathfrak{g}_e(0), \mathfrak{g}_e(0)] \cong \mathfrak{sl}_2$, and $\mathfrak{g}_e(2)^{\operatorname{ad} \mathfrak{g}_e(0)}$ has basis $\{e, v_7\}$. It is also shown in *loc. cit.* that Γ is generated by the image of $c = n_{\substack{123321\\1}} n_{\substack{123221\\2}} h_1(-1)h_2(-1)h_4(-1)$ and $\mathfrak{g}_e(1)$ is spanned by v_{23} which is fixed by the adjoint action of G_e. It is straightforward to see that $\operatorname{Lie}(T_1)$ is spanned by the element $t \in \operatorname{Lie}(T)$ such that $\alpha_i(t) = \delta_{6,i}$ for $1 \leqslant i \leqslant 7$ and $[v_1, v_4] = \pm v_7$. Since $\operatorname{Ad} c$ negates t we deduce that $\mathfrak{c}_e^\Gamma$ is spanned by the images of e and v_{23} and so $c_\Gamma(e) = 2$.

If e is of type $\mathsf{A}_4 + \mathsf{A}_1$ then $\mathfrak{c}_e = \mathfrak{c}_e(0)$ is two-dimensional by [deG13]. On the other hand it is immediate from [LT07, p. 138] that $\mathfrak{g}_e(0)$ is a two-dimensional subalgebra of $\operatorname{Lie}(T)$ spanned by t_1 and t_2 such that $\alpha_i(t_1) = \delta_{5,i}$ and $\alpha_i(t_2) = \delta_{7,i}$ for $1 \leqslant i \leqslant 7$. Since in the present case Γ is generated by the image of

$$c = n_{\substack{011111\\1}} n_{\substack{111111\\0}} n_{\substack{121210\\1}} n_{\substack{124321\\2}} h_3(-1)h_4(-1)h_6(-1),$$

one checks by direct computation that $\operatorname{Ad} c$ negates both t_1 and t_2. Therefore, $c_\Gamma(e) = 0$ and hence $\mathcal{E}^\Gamma$ is a single point!

If e is of type $\mathsf{D}_4(\mathsf{a}_1)+\mathsf{A}_1$ then $\mathfrak{c}_e=\mathfrak{c}_e(2)$ is two-dimensional by [deG13]. By [LT07, pp. 129, 130], we know that $\mathfrak{g}_e(0)$ is semisimple, $\mathfrak{g}_e(2)^{\operatorname{ad}\mathfrak{g}_e(0)}$ has basis $\{v_9, v_{10}, v_{11}, e\}$, and Γ is generated by the image of $c=n_{\substack{111000\\1}}n_{\substack{111100\\0}}h_2(-1)$. It also follows from *loc. cit.* that $[\mathfrak{g}_e(1),\mathfrak{g}_e((1)]^{\operatorname{ad}\mathfrak{g}_e(0)}$ is spanned by $[v_1,v_2]$ and $[v_3,v_4]$. Since $\dim\mathfrak{c}_e(2)=2$, the above yields that $[v_1,v_2]$ and $[v_3,v_4]$ are linearly independent. It is straightforward to see that $(\operatorname{Ad}c)(v_1)\in\Bbbk^\times v_3$ and $(\operatorname{Ad}c)(v_2)\in\Bbbk^\times v_4$. So $\operatorname{Ad}c$ must permute the lines $\Bbbk^\times[v_1,v_2]$ and $\Bbbk^\times[v_3,v_4]$. Since $\operatorname{Ad}c$ acts on $\mathfrak{g}_e(2)^{\operatorname{ad}\mathfrak{g}_e(0)}$ as an involution, this implies that it has eigenvalues ±1 on the subspace $[\mathfrak{g}_e(1),\mathfrak{g}_e((1)]^{\operatorname{ad}\mathfrak{g}_e(0)}=\Bbbk[v_1,v_2]\oplus\Bbbk[v_3,v_4]$. Since the roots $\substack{111000\\1}$, $\substack{111100\\0}$, $\substack{000001\\0}$ are pairwise orthogonal, $\operatorname{Ad}c$ fixes $v_{11}=e_7$. Since $\operatorname{Ad}c$ also fixes $e=e_2+e_5+(\text{sum of other root vectors})$ and permutes the lines $\Bbbk e_2$ and $\Bbbk e_5$, it must be that $(\operatorname{Ad}c)(e_2+e_5)=e_2+e_5$. So $\operatorname{Ad}c$ fixes $v_{10}=e_2+e_5$. But then the (-1)-eigenspace of $\operatorname{Ad}c$ on $\mathfrak{g}_e(2)^{\operatorname{ad}\mathfrak{g}_e(0)}$ is one-dimensional. In conjunction with our earlier remarks, this gives $c(e)=c_\Gamma(e)=2$.

If e is of type $\mathsf{A}_2+\mathsf{A}_1$ then $\mathfrak{c}_e=\mathfrak{c}_e(0)$ is one-dimensional by [deG13], whilst [LT07, p. 112] shows that $\mathfrak{g}_e(0)=[\mathfrak{g}_e(0),\mathfrak{g}_e(0)]\oplus\operatorname{Lie}(T_1)$ where $[\mathfrak{g}_e(0),\mathfrak{g}_e(0)]\cong\mathfrak{sl}_4$ and $\operatorname{Lie}(T_1)$ is spanned by the element $t\in\operatorname{Lie}(T)$ such that $\alpha_i(t)=\delta_{4,i}$ for $1\leqslant i\leqslant 7$. Furthermore, the group Γ is generated by the image of

$$c=n_{\substack{112111\\1}}n_{\substack{112210\\1}}n_{\substack{134321\\2}}h_3(-1)h_5(-1)h_7(-1).$$

Direct verification shows that $\operatorname{Ad}c$ negates t. Since $\mathfrak{c}_e$ is spanned by the image of t we conclude that $c_\Gamma(e)=0$ in the present case and hence $\mathcal{E}^\Gamma$ is a single point!

This completes the proof of the proposition. □

Now we deal with those non-even, induced nilpotent elements which lie in more than one sheet of $\mathfrak{g}$. We first recall that if a nilpotent orbit $\mathcal{O}\subset\mathfrak{g}$ is induced from a nilpotent orbit $\mathcal{O}_L\subset\operatorname{Lie}(L)$, where $P=LU$ is a proper parabolic subgroup of G with unipotent radical U, then the adjoint action of G induces a surjective morphism

$$\pi\colon\ G\times^P\big(\overline{\mathcal{O}}_L+\operatorname{Lie}(U)\big)\to\overline{\mathcal{O}},\quad (g,x)\longmapsto(\operatorname{Ad}g)\,x,$$

sometimes referred to as a *generalised Springer map*; see [Fu10] for more detail. It is immediate from [Los11a, Proposition 6.1.2(4)] that π is birational (that is, generically injective) if and only if $G_e\subset P$ for some $e\in\mathcal{O}\cap(\mathcal{O}_L+\operatorname{Lie}(U))$.

Proposition 15. *Suppose that e is not even, induced, and lies in more than one sheet of $\mathfrak{g}$. Assume further that e is not of type $\mathsf{E}_7(\mathsf{a}_5)$ if $\mathfrak{g}$ is of type E_8. Then the following hold:*

- (i) *there exists a parabolic subgroup $P=LU$ of G such that the pair (P,e) satisfies all conditions of Proposition 12 and the centre of L has dimension $r(e)$;*
- (ii) *$U(\mathfrak{g},e)^{\mathrm{ab}}_\Gamma$ is isomorphic to a polynomial algebra in $r(e)=c_\Gamma(e)$ variables.*

Proof. Inspecting Tables 1–6, one observes that if e not even, induced, and lies in more than one sheet of $\mathfrak{g}$ then $\mathfrak{g}$ has type E_8 or E_7 and all sheets containing e have the same rank equal to $r(e)$. Part (i) then follows from [Fu10, Proposition 3.1], which implies that in our situation there exists at least one birational morphism $\pi\colon\ G\times^P\big(\overline{\mathcal{O}}_L+\operatorname{Lie}(U)\big)\to\overline{\mathcal{O}}$ with $e\in\mathcal{O}$ (the proof of [Fu10, Proposition 3.1] relies on Fu's earlier results obtained in [Fu07]). In view of Proposition 12 and part (ii) of Corollary 10, it thus suffices to show that the inequality $r(e)\geqslant c_\Gamma(e)$ holds for all nilpotent elements e as above and $\mathcal{E}_0^{\Gamma_0}\neq\emptyset$ (the notation of Proposition 12).

Suppose that $\mathfrak{g}$ is of type E_8. Then e is one of $\mathsf{E}_7(\mathsf{a}_4)$, $\mathsf{D}_7(\mathsf{a}_2)$, $\mathsf{A}_3+\mathsf{A}_2$.

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If e has type $\mathsf{E}_7(\mathsf{a}_4)$ then $r(e) = 2$ by [deGE09] and $\mathfrak{c}_e = \mathfrak{c}_e(2)$ is three-dimensional by [deG13]. According to [LT07, pp. 259, 260], the reductive group $C(e)^\circ$ acts trivially on $\mathfrak{g}_e(2)$ which has basis $\{v_3, v_4, v_5, e\}$, and Γ is generated by the image of $c = h_4(-1)$ which negates v_1, v_2, v_3 and fixes v_4, v_5 and e. It is also immediate from *loc. cit.* that $[\mathfrak{g}_e(1), \mathfrak{g}_e(1)] = \Bbbk[v_1, v_2]$. Since $\dim \mathfrak{c}_e(2) = 3$ and $\dim \mathfrak{g}_e(2) = 4$ it must be that $[v_1, v_2] \neq 0$. Since $\mathrm{Ad}\, c$ fixes $[v_1, v_2]$ we now deduce that the image of v_3 in $\mathfrak{c}_e$ is non-zero and hence $c_\Gamma(e) = r(e) = 2$. By [Fu10, 3.4], we can take for $P = LU$ a parabolic subgroup of G with $\mathcal{D}L$ of type D_6 and for $e_0 \in \mathrm{Lie}(L)$ a rigid nilpotent element associated with the partition $(3, 2^2, 1^5)$. But then $\mathcal{E}_0^{\Gamma_0} = \mathcal{E}_0 \neq \emptyset$ because $\mathcal{E}_0$ is a single point by Remark 6(a).

If e has type $\mathsf{D}_7(\mathsf{a}_2)$ then $r(e) = 2$ by [deGE09], whilst [deG13] shows that $\mathfrak{c}_e = \mathfrak{c}(0) \oplus \mathfrak{c}_e(2) \oplus \mathfrak{c}_e(6)$ and all non-zero $\mathfrak{c}_e(i)$ are one-dimensional. On the other hand, [LT07, pp. 267, 268] yields that Γ is generated by the image of $c = n_{\substack{2354321\\3}}\, n_{\substack{2454321\\2}}\, h_4(-1)h_5(-1)$ and $\mathfrak{g}_e(0)$ is spanned by the element $t \in \mathrm{Lie}(T)$ such that $\alpha_i(t) = \delta_{1,i}$ for all $1 \leqslant i \leqslant 8$. It is straightforward to see that $\mathrm{Ad}\, c$ negates t. It follows that $r(e) = 2 \geqslant c_\Gamma(e)$. By [Fu07, Example 5.12], e is Richardson and we can take for $P = LU$ a parabolic subgroup of G with $\mathcal{D}L$ of type $\mathsf{A}_3 + \mathsf{A}_3$ and for e_0 the zero nilpotent element of $\mathfrak{l} = \mathrm{Lie}(L)$. Then $\mathcal{E}_0^{\Gamma_0}$ contains the augmentation ideal of $U([\mathfrak{l}, \mathfrak{l}], e_0) = U([\mathfrak{l}, \mathfrak{l}])$. In particular, $\mathcal{E}_0^{\Gamma_0} \neq \emptyset$.

If e has type $\mathsf{A}_3 + \mathsf{A}_2$ then $r(e) = 1$ and $\mathfrak{c}_e = \mathfrak{c}_e(0) \oplus \mathfrak{c}_e(2)$ where $\dim \mathfrak{c}_e(0) = \dim \mathfrak{c}_e(2) = 1$; see [deGE09, deG13]. By [LT07, p. 201], the group Γ is generated by the image of

$$c = n_{\substack{0011100\\1}}\, n_{\substack{0111100\\0}}\, n_{\substack{0122110\\1}}\, h_4(-1)h_5(-1)h_6(-1)h_7(-1)$$

and $\mathfrak{g}_e(0) = [\mathfrak{g}_e(0), \mathfrak{g}_e(0)] \oplus \mathrm{Lie}(T_1)$ where $[\mathfrak{g}_e(0), \mathfrak{g}_e(0)] \cong \mathfrak{so}_5$. Furthermore, $\mathrm{Lie}(T_1)$ is spanned by the element $t \in \mathrm{Lie}(T)$ such that $\alpha_1(t) = -3$, $\alpha_5(t) = 2$, $\alpha_8(t) = -2$ and $\alpha_i(t) = 0$ for $i = 2, 3, 4, 6, 7$. It is straightforward to check that $\mathrm{Ad}\, c$ negates t. But then $r(e) = 1 \geqslant c_\Gamma(e)$. By [Fu10, 3.4], we can take for $P = LU$ a parabolic subgroup of G with $\mathcal{D}L$ of type D_7 and for $e_0 \in \mathrm{Lie}(L)$ a rigid nilpotent element associated with the partition $(2^2, 1^{10})$. Then Remark 6(a) yields $\mathcal{E}_0^{\Gamma_0} \neq \emptyset$.

Now suppose that $\mathfrak{g}$ is of type E_7. Then e has type $\mathsf{A}_3 + \mathsf{A}_2$. This case is almost identical to the previous case. Here we again have that $r(e) = 1$ and $\mathfrak{c}_e = \mathfrak{c}_e(0) \oplus \mathfrak{c}_e(2)$ where $\dim \mathfrak{c}_e(0) = \dim \mathfrak{c}_e(2) = 1$; see [deGE09, deG13]. Also, [LT07, p. 131] shows that the group Γ is generated by the image of

$$c = n_{\substack{001110\\1}}\, n_{\substack{011110\\0}}\, n_{\substack{012211\\1}}\, h_4(-1)h_5(-1)h_6(-1)h_7(-1),$$

$\mathfrak{g}_e(0) = [\mathfrak{g}_e(0), \mathfrak{g}_e(0)] \oplus \mathrm{Lie}(T_1)$ where $[\mathfrak{g}_e(0), \mathfrak{g}_e(0)] \cong \mathfrak{sl}_2$, and $\mathrm{Lie}(T_1)$ is spanned by the element $t \in \mathrm{Lie}(T)$ such that $\alpha_1(t) = -3$, $\alpha_5(t) = 2$ and $\alpha_i(t) = 0$ for $i = 2, 3, 4, 6, 7$. As before, we check that $\mathrm{Ad}\, c$ negates t, yielding $r(e) = 1 \geqslant c_\Gamma(e)$. By [Fu10, 3.3], we can take for $P = LU$ a parabolic subgroup of G with $\mathcal{D}L$ of type D_6 and for $e_0 \in \mathrm{Lie}(L)$ a rigid nilpotent element associated with the partition $(3, 2^2, 1^5)$. Then Remark 6(a) shows that $\mathcal{E}_0^{\Gamma_0} \neq \emptyset$. This completes the proof. $\square$

For the remaining seven induced orbits in the Lie algebras of exceptional types our results are weaker. It turns out that the variety $\mathcal{E}^\Gamma$ is non-empty in all cases and we can determine its dimension, but our methods seem insufficient for describing the irreducible components of $\mathcal{E}^\Gamma$. We state our results as a remark:

Remark 8. (a) If $\mathfrak{g}$ has type F_4 and e is of type $\mathsf{C}_3(\mathsf{a}_1)$ then [deGE09] shows that e lies in a single sheet of rank 1. So $\dim \mathcal{E} = 1$ by [Pre10, Theorem 1.2]. On the other hand, $\mathfrak{c}_e = \mathfrak{c}_e(2)$ is three-dimensional by [deG13] whilst [LT07, p. 76] yields that $\mathfrak{c}_e \cong \mathfrak{g}_e(2)$ and $\Gamma \cong S_2$ is generated

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by the image $c = h_4(-1)$. Also, $\mathfrak{g}_e(2)$ has basis $\{v_1, v_2, e\}$ and $\operatorname{Ad} c$ negates v_1 and fixes v_2. This gives $c_\Gamma(e) = 2$. It follows from [Fu10, 3.1] that e is induced from a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ with $[\mathfrak{l}, \mathfrak{l}]$ of type B_3 and a nilpotent element $e_0 \in [\mathfrak{l}, \mathfrak{l}]$ corresponding to the rigid partition $(2^2, 1^3) \in \mathcal{P}_1(7)$. Moreover, the conditions of Proposition 12 are satisfied and $\mathcal{E}_0^{\Gamma_0} \neq \emptyset$ by Theorem 12. Therefore, $\mathcal{E}^\Gamma \neq \emptyset$ and $\dim \mathcal{E}^\Gamma \geqslant \mathfrak{z}(\mathfrak{l}) = 1$. As a consequence, $\dim \mathcal{E}^\Gamma = \dim \mathcal{E} = 1$. In view of Proposition 11(ii), the variety $\mathcal{E}^\Gamma$ is isomorphic to a non-empty one-dimensional closed subset of the affine plane $\mathbb{A}^2$.

(b) If $\mathfrak{g}$ has type E_6 and e is of type $\mathsf{A}_3 + \mathsf{A}_1$ then $\Gamma = \{1\}$ and e lies in a single sheet of rank 1 by [deGE09]. Then $\mathcal{E}^\Gamma = \mathcal{E} \neq \emptyset$ and $\dim \mathcal{E} = 1$ thanks to [Pre10, Theorem 1.2]. By [deG13], we have that $\mathfrak{c}_e = \mathfrak{c}_e(0) \oplus \mathfrak{c}_e(2)$ is two-dimensional. As in the previous case we now deduce that $\mathcal{E}^\Gamma = \mathcal{E}$ is isomorphic to a non-empty one-dimensional closed subset of the affine plane $\mathbb{A}^2$.

(c) If $\mathfrak{g}$ has type E_7 and e is of type $\mathsf{D}_6(\mathsf{a}_2)$ then $\Gamma = \{1\}$ and e lies in a single sheet of rank 2; see [deGE09]. On the other hand, $\mathfrak{c}_e = \mathfrak{c}_e(2)$ is three-dimensional by [deG13]. In view of [Pre10, Theorem 1.2] and Proposition 11(ii), this means that $\mathcal{E}^\Gamma = \mathcal{E}$ is isomorphic to a non-empty closed two-dimensional subset of the affine space $\mathbb{A}^3$.

(d) If $\mathfrak{g}$ has type E_8 and e is of type $\mathsf{E}_6(\mathsf{a}_3) + \mathsf{A}_1$ then e lies in a single sheet of rank 1 by [deGE09] and $\mathfrak{c}_e = \mathfrak{c}_e(2)$ is three-dimensional by [deG13]. On the other hand, [LT07, pp. 245, 246] yields that $\Gamma \cong S_2$ is generated by the image of $c = h_4(-1)$, the subspace $\mathfrak{g}_e(2)$ has basis $\{v_5, v_6, v_7, e\}$, and $v_7 = \pm[v_1, v_4]$. From this it is immediate that the images of v_5, v_6 and e under the natural epimorphism $\mathfrak{g}_e(2) \twoheadrightarrow \mathfrak{c}_e$ form a basis of $\mathfrak{c}_e$. Direct computations show that c negates v_5 and fixes v_6, yielding $c_\Gamma(e) = 2$.

By [Pre10, Theorem 1.2] the variety $\mathcal{E}$ is non-empty and has dimension $r(e) = 1$, whereas [Fu10, 3.4] implies that e is induced from a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ with $[\mathfrak{l}, \mathfrak{l}]$ of type E_7 and a nilpotent element $e_0 \in [\mathfrak{l}, \mathfrak{l}]$ of type $2\mathsf{A}_2 + \mathsf{A}_1$ in such a way that all conditions of Proposition 12 are satisfied (one should keep in mind here that $\Gamma_0 = \{1\}$ by [Car85, p. 403] and $\mathcal{E}_0 \neq \emptyset$ by [GRU10]). Hence $\dim \mathcal{E}^\Gamma \geqslant 1$. We conclude that $\mathcal{E}^\Gamma$ is isomorphic to a non-empty one-dimensional closed subset of the affine plane $\mathbb{A}^2$.

(e) If $\mathfrak{g}$ has type E_8 and e is of type $\mathsf{D}_6(\mathsf{a}_2)$ then again e lies in a single sheet of rank 1 by [deGE09] and $\mathfrak{c}_e = \mathfrak{c}_e(2)$ is three-dimensional by [deG13]. It follows from [Fu10, 3.4] and Theorem 12 that e is induced from a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ with $[\mathfrak{l}, \mathfrak{l}]$ of type D_7 and a nilpotent element $e_0 \in [\mathfrak{l}, \mathfrak{l}]$ attached to the rigid partition $(3, 2^4, 1^3) \in \mathcal{P}_1(14)$ in such a way that all conditions of Proposition 12 are satisfied. This implies that $\mathcal{E}^\Gamma \neq \emptyset$ and $\dim \mathcal{E}^\Gamma \geqslant \dim \mathfrak{z}(\mathfrak{l}) = 1$. On the other hand, combining [deG13] and [LT07, pp. 243, 244], we deduce that $\mathfrak{c}_e = \mathfrak{c}_e(2) \cong \mathfrak{g}_e(2)$ and the images of v_1, v_2 and e form a basis of $\mathfrak{c}_e$.

The group $\Gamma \cong S_2$ is generated by the image of $c = n_{\substack{0011111\\1}} n_{\substack{0111111\\0}} h_4(-1)h_5(-1)$ and it is straightforward to see that $\operatorname{Ad} c$ permutes the lines $\Bbbk_2$ and $\Bbbk e_3$ and fixes $e_{\substack{0011000\\0}}$. Since $e = e_2 + e_3 + (\text{sum of other root vectors})$, $\operatorname{Ad} c$ must permute e_2 and e_3. From this it is immediate that $\operatorname{Ad} c$ negates v_1 and fixes v_2. As a consequence, $c_\Gamma(e) = 2$. Since $\dim \mathcal{E} = 1$ by [Pre10, Theorem 1.2], our earlier remarks now show that $\mathcal{E}^\Gamma$ is isomorphic to a non-empty closed one-dimensional subset of the affine plane $\mathbb{A}^2$.

(f) If $\mathfrak{g}$ has type E_8 and e is of type $\mathsf{E}_7(\mathsf{a}_5)$ then e lies in two sheets both of which have rank 1; see [deGE09]. Also, $\mathfrak{c}_e = \mathfrak{c}_e(2)$ is six-dimensional by [deG13]. It follows from [Fu10, 3.4] that e is induced from a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ with $[\mathfrak{l}, \mathfrak{l}]$ of type $\mathsf{E}_6 + \mathsf{A}_1$ and a nilpotent element $e_0 \in [\mathfrak{l}, \mathfrak{l}]$ of type $3\mathsf{A}_1 + 0$ in such a way that all conditions of Proposition 12 are satisfied (it is important here that $\Gamma_0 = \{1\}$ by [Car85, p. 402] and $\mathcal{E}_0 \neq \emptyset$ by [GRU10]). This implies

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that $\mathcal{E}^\Gamma \neq \emptyset$ and $\dim \mathcal{E}^\Gamma \geqslant \dim \mathfrak{z}(\mathfrak{l}) = 1$. Since $\dim \mathcal{E} = r(e) = 1$ by [Pre10, Theorem 1.2] this yields $\dim \mathcal{E}^\Gamma = \dim \mathcal{E} = 1$.

By [LT07, pp. 247, 248], the subspace $\mathfrak{g}_e(2) \cong \mathfrak{c}_e$ is spanned by v_1, v_2, v_3, v_4, v_5 and e, and $\Gamma \cong S_3$ is generated by the images of $c_1 = h_2(\omega)h_3(\omega)h_5(\omega)$ and $c_2 = n_{\alpha_2}n_{\alpha_3}n_{\alpha_5}h_3(-1)h_4(-1)$ where ω is a primitive third root of 1. Direct computations show that $(\operatorname{Ad} c_1)(v_i) = \omega^{-1}v_i$ for $i = 1, 4$, $(\operatorname{Ad} c_1)(v_i) = \omega v_i$ for $i = 2, 3$, and $(\operatorname{Ad} c_1)(v_5) = v_5$. Since $v_5 = e_4 + e_{\substack{0111000\\1}}$ and $\operatorname{Ad} c_2$ permutes the lines $\Bbbk e_4$ and $\Bbbk e_{\substack{0111000\\1}}$ and fixes $e = e_4 + e_{\substack{0111000\\1}} + (\text{sum of other root vectors})$, it must be that $(\operatorname{Ad} c_2)(v_5) = v_5$. But then $c_\Gamma(e) = 2$ and we conclude that $\mathcal{E}^\Gamma$ is isomorphic to a non-empty one-dimensional closed subset of the affine plane $\mathbb{A}^2$.

(g) If $\mathfrak{g}$ is of type E_8 and e has type $\mathsf{E}_7(\mathsf{a}_2)$ then $\Gamma = \{1\}$ and e lies in a single sheet of rank 3; see [deGE09]. Since $\mathfrak{c}_e = \mathfrak{c}_e^\Gamma$ is four-dimensional by [deG13] and $\mathcal{E} \neq \emptyset$ has dimension $r(e) = 3$ by [Pre10, Theorem 1.2] we conclude that $\mathcal{E}^\Gamma = \mathcal{E}$ is isomorphic to a non-empty three-dimensional closed subset of the affine space $\mathbb{A}^4$.

5.6 Applications to completely prime primitive ideals

Let e be an induced nilpotent element of $\mathfrak{g}$ and let $\mathcal{X}_\mathcal{O}$ be the set of all primitive ideals I of the universal enveloping algebra $U(\mathfrak{g})$ with $\operatorname{VA}(I) = \overline{\mathcal{O}}$. Here $\mathcal{O}$ is the adjoint G-orbit of e and $\operatorname{VA}(I)$ denotes the associated variety of I, i.e. the zero locus in $\mathfrak{g}$ of the ideal $\operatorname{gr}(I)$ of $S(\mathfrak{g}) = \operatorname{gr}(U(\mathfrak{g}))$ where, as usual, we identify the maximal spectrum of $S(\mathfrak{g})$ with $\mathfrak{g}$ by means of the Killing form of $\mathfrak{g}$.

Let J denote the defining ideal of $\overline{\mathcal{O}}$. Since $A := S(\mathfrak{g})/\operatorname{gr}(I)$ is a finitely generated $S(\mathfrak{g})$-module and J is the only minimal prime ideal of $S(\mathfrak{g})$ containing the annihilator $\operatorname{Ann}_{S(\mathfrak{g})}A$ by Joseph's theorem, it follows from [Mat89, Theorem 6.4], for instance, that there exist prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_l$ containing J and a finite chain $\{0\} = A_0 \subset A_1 \subset \cdots \subset A_l = A$ of $S(\mathfrak{g})$-submodules of A such that $A_i/A_{i-1} \cong S(\mathfrak{g})/\mathfrak{p}_i$ for $1 \leqslant i \leqslant l$. The *multiplicity* of $\mathcal{O}$ in $U(\mathfrak{g})/I$, denoted $\operatorname{mult}_\mathcal{O}(U(\mathfrak{g})/I)$, is defined as

$$\operatorname{mult}_\mathcal{O}(U(\mathfrak{g})/I) := \operatorname{Card}\{i\colon 1 \leqslant i \leqslant l, \mathfrak{p}_i = J\}.$$

It is well known that this number is independent of the choices made; see [Jan04, 9.6] for more detail. The results of the previous subsection can be applied to characterise those primitive ideals $I \in \mathcal{X}_\mathcal{O}$ for which $\operatorname{mult}_\mathcal{O}(U(\mathfrak{g})/I) = 1$; we call such ideals *multiplicity-free*. The characterisation we obtain can be regarded as a generalisation of Mœglin's theorem [Mœg87] on completely prime primitive ideals of $U(\mathfrak{sl}_n)$ to simple Lie algebras of other types (that theorem was recently reproved by Brundan [Bru11] by using the theory of finite W-algebras).

The rest of this subsection is devoted to proving Theorem 5. First we note that for $\mathfrak{g} = \mathfrak{sl}_n$ the statement of Theorem 5 is equivalent to Mœglin's theorem thanks to (1) and the main results of [Pre11].

(a) Suppose that $\mathfrak{g}$ is one of $\mathfrak{so}_N$ or $\mathfrak{sp}_N$ and e is associated with a partition $\lambda \in \mathcal{P}_\epsilon(N)$. In what follows we shall use the notation introduced in the course of proving Theorem 12.

Repeating the construction used in part (b) of the proof of Theorem 12 as many times as possible, we arrive at a pair $(\mathfrak{p}, e_0)$, where $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$ with a Levi subalgebra $\mathfrak{l} = \bar{\mathfrak{l}} \oplus \mathfrak{m}$ and e_0 is an almost rigid nilpotent element of $\mathfrak{m}$, such that e is induced from e_0 (regarded as an element of $\mathfrak{l}$). Let P be the parabolic subgroup of G with $\operatorname{Lie}(P) = \mathfrak{p}$.

Recall Losev's homomorphism $\Xi\colon U(\mathfrak{g}, e) \to U(\mathfrak{l}, e_0)'$ from part (b) of the proof of Proposition 12. Since $I \in \mathcal{X}_\mathcal{O}$ is multiplicity-free, we have that $I = Q_e \otimes_{U(\mathfrak{g},e)} \Bbbk_\eta$ for some one-dimensional Γ-invariant representation of η of $U(\mathfrak{g}, e)$. By [Los11a, Theorem 6.5.2], the morphism $\xi^*\colon \operatorname{Specm} U(\mathfrak{l}, e_0)^{\mathrm{ab}} \to \operatorname{Specm} U(\mathfrak{g}, e)^{\mathrm{ab}}$ induced by Ξ is finite. As explained in the proof of

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Proposition 12, the inclusion $G_e \subset P_e$ implies that Γ acts on $\widetilde{\mathcal{E}}_0 = \operatorname{Specm} U(\mathfrak{l}, e_1)^{\mathrm{ab}}$ and ξ^* maps the Zariski closed subset $\widetilde{\mathcal{E}}_0^{\Gamma_0}$ into $\mathcal{E}^{\Gamma}$ (as before, Γ_0 stands for the component group of L_{e_0}). Since the variety $\mathcal{E}^{\Gamma}$ is irreducible of dimension $s(\lambda)$ by Theorem 12 and $\dim(\widetilde{\mathcal{E}}_0^{\Gamma_0}) \geqslant \dim \mathfrak{z}(\mathfrak{l}) = s(\lambda)$, we deduce that $\xi^*\big(\widetilde{\mathcal{E}}_0^{\Gamma_0}\big) = \mathcal{E}^{\Gamma}$ (one should keep in mind here that, being a finite morphism, ξ^* is closed and has finite fibres). As $(\operatorname{Ker}\eta)/I_c$ lies in $\mathcal{E}^{\Gamma}$, we obtain that $\eta = \eta_0 \circ \xi$ for some one-dimensional Γ_0-invariant representation η_0 of $U(\mathfrak{l}, e_0)$ where $\xi\colon U(\mathfrak{g}, e)^{\mathrm{ab}} \to U(\mathfrak{l}, e_0)^{\mathrm{ab}}$ is the homomorphism of $\Bbbk$-algebras induced by Ξ.

Let $I_0 \subset U(\mathfrak{l})$ be the annihilator of $\widetilde{E} := Q_0 \otimes_{U(\mathfrak{l},e_0)} \Bbbk_{\eta_0}$ where Q_0 is a generalised Gelfand–Graev $\mathfrak{l}$-module associated with e_0. By construction, I_0 is a multiplicity-free primitive ideal of $U(\mathfrak{l})$. Since $\eta = \eta_0 \circ \xi$, it follows from [Los11a, Corollary 6.4.2] that I is obtained from I_0 by parabolic induction. More precisely,

$$I = \operatorname{Ann}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \widetilde{E})$$

where we regard $\widetilde{E}$ as a $\mathfrak{p}$-module with the trivial action of the nilradical of $\mathfrak{p}$.

(b) Suppose that λ is not exceptional. As any almost rigid nilpotent element of $\mathfrak{m}$ is nonsingular, combining Corollary 5 with Borho's classification of sheets, one observes that there exists a unique (up to conjugacy in P) parabolic subalgebra $\mathfrak{p}_1 = \mathfrak{l}_1 \oplus \mathfrak{n}_1$ of $\mathfrak{g}$ contained in $\mathfrak{p}$ and a rigid nilpotent element $e_1 \in \mathfrak{l}_1$ such that $e_0 \in \mathfrak{l}$ is induced from e_1 (here $\mathfrak{n}_1$ is the nilradical of $\mathfrak{p}_1$ and $\mathfrak{l}_1$ is a Levi subalgebra of $\mathfrak{p}_1$ contained in $\mathfrak{l}$). Then e is induced from e_1 by Proposition 6(3).

Let Ξ_0, ξ_0^* and ξ_0 be the analogues of the maps Ξ, ξ^* and ξ associated with the finite W-algebras $U(\mathfrak{l}, e_0)$ and $U(\mathfrak{l}_1, e_1)$. It follows from Theorem 11 that

$$U(\mathfrak{l}, e_0)^{\mathrm{ab}} \cong U(\bar{\mathfrak{l}})^{\mathrm{ab}} \otimes U(\mathfrak{m}, e_0) \cong S\big(\mathfrak{z}(\bar{\mathfrak{l}})\big) \otimes U(\mathfrak{m}, e_0)^{\mathrm{ab}}$$

is a polynomial algebra, hence a domain. By [Los11a, Theorem 6.5.2], the morphism $\xi_0^*\colon \operatorname{Specm} U(\mathfrak{l}_1, e_1)^{\mathrm{ab}} \to \operatorname{Specm} U(\mathfrak{l}, e_0)^{\mathrm{ab}}$ induced by Ξ_0 is finite, hence closed and has finite fibres. As $\operatorname{Specm} U(\mathfrak{l}, e_0)^{\mathrm{ab}}$ is an irreducible variety, the map ξ_0^* must be surjective. So there exists a one-dimensional representation η_1 of $U(\mathfrak{l}_1, e_1)$ such that $\eta_0 = \eta_1 \circ \xi_0$.

Let I_1 be the annihilator in $U(\mathfrak{l}_1)$ of $E := Q_1 \otimes_{U(\mathfrak{l}_1,e_1)} \Bbbk_{\eta_1}$, where Q_1 denotes a generalised Gelfand–Graev $\mathfrak{l}_1$-module associated with e_1, a completely prime primitive ideal of $U(\mathfrak{l}_1)$. Since $\eta_0 = \eta_1 \circ \xi_0$, applying [Los11a, Corollary 6.4.2] once again, we deduce that $I_0 = \operatorname{Ann}_{U(\mathfrak{l})}\big(U(\mathfrak{l}) \otimes_{U(\mathfrak{p}_1)} E\big)$ where E is regarded as a $\mathfrak{p}_1$-module with the trivial action of $\mathfrak{n}_1$. In a slight abuse of the notation introduced in § 1.6, we then have

$$I = \mathfrak{I}_{\mathfrak{p}}^{\mathfrak{g}}(I_0) = \mathfrak{I}_{\mathfrak{p}}^{\mathfrak{g}}\big(\mathfrak{I}_{\mathfrak{p}_1}^{\mathfrak{p}}(I_1)\big) = \mathfrak{I}_{\mathfrak{p}_1}^{\mathfrak{g}}(I_1)$$

where the last equality follows from [BGR73, 10.4] (which, in turn, follows from transitivity of induction). As a consequence, $I = \operatorname{Ann}_{U(\mathfrak{g})}\big(U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_1)} E\big)$, proving Theorem 5 in the present case.

(c) Now suppose that λ is exceptional. In this case, we choose for $\mathfrak{p}_1$ the parabolic subalgebra $\operatorname{Lie}(P')$ where P' is the parabolic subgroup introduced in part (d) of Theorem 12. Then e_0 is a Richardson element of $\mathfrak{p}_1$ and the map $\xi_0^*\colon \operatorname{Specm} U(\mathfrak{l}_1, 0)^{\mathrm{ab}} \to \operatorname{Specm} U(\mathfrak{l}, e_0)^{\mathrm{ab}}$ is still surjective (here $\mathfrak{l}_1$ is a Levi subalgebra of $\mathfrak{p}_1$). Therefore, $\eta_0 = \eta_1 \circ \xi_0$ for some one-dimensional representation of $U(\mathfrak{l}_1, 0) = U(\mathfrak{l}_1)$. Applying [Los11a, Corollary 6.4.2] and repeating almost verbatim the argument from part (b) we now obtain that $I = \mathfrak{I}_{\mathfrak{p}_1}^{\mathfrak{g}}(I_1)$ for some ideal I_1 of codimension 1 in $U(\mathfrak{l}_1)$. In other words, $I = \operatorname{Ann}_{U(\mathfrak{g})}\big(U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_1)} \Bbbk_\lambda\big)$ for some one-dimensional representation λ of $\mathfrak{p}_1$.

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(d) Finally, suppose that $\mathfrak{g}$ is exceptional. If e is even, then we choose for $\mathfrak{p}$ the Jacobson–Morozov parabolic subalgebra $\mathfrak{p}(e)$ and for e_0 the zero element of the Levi subalgebra of $\mathfrak{l} = \mathfrak{g}(0)$ of $\mathfrak{p}(e)$. Since $U(\mathfrak{l}, e_0) = U(\mathfrak{l})$ and it follows from Proposition 13 that ξ^* is still surjective, we argue as in part (a) to deduce that $I = \mathfrak{I}^{\mathfrak{g}}_{\mathfrak{p}(e)}(I_0)$ for some ideal I_0 of codimension 1 in $U(\mathfrak{l})$. As a consequence, $I = \mathrm{Ann}_{U(\mathfrak{g})}\big(U(\mathfrak{g}) \otimes_{U(\mathfrak{p}(e))} \Bbbk_\lambda\big)$ for some one-dimensional representation λ of $\mathfrak{p}(e)$.

If e satisfies the conditions of Proposition 14 then it lies in a unique sheet $\mathcal{S}(e)$ which contains an open decomposition class $\mathcal{D}(\mathfrak{l}, e_0)$ such that e_0 is a rigid nilpotent element of $\mathfrak{l} = \mathrm{Lie}(L)$. Furthermore, we may assume that there exists a parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is the nilradical of $\mathfrak{p}$, such that $e \in \mathfrak{p}$ is induced from $e_0 \in \mathfrak{l}$. In view of Proposition 14 we can now repeat verbatim our arguments from part (b) to deduce that the map $\xi^* \colon \mathrm{Specm}\, U(\mathfrak{l}, e_0)^{\mathrm{ab}} \to \mathcal{E}$ is surjective. Thanks to [Los11a, Corollary 6.4.2] this yields that $I = \mathfrak{I}^{\mathfrak{g}}_{\mathfrak{p}}(I_0)$ for some completely prime primitive ideal I_0 of $U(\mathfrak{l})$ with $\mathrm{VA}(I_0) = \overline{(\mathrm{Ad}\, L)\, e_0}$.

Finally, if e satisfies the conditions of Proposition 15 then we choose a Levi subalgebra $\mathfrak{l} = \mathrm{Lie}(L)$ and $e_0 \in \mathfrak{l}$ according to the recipe described in Fu's paper [Fu10] (see the proof of Proposition 15 for detail). Then there exists a parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ such that e_0 is rigid in $\mathfrak{l}$ and $e \in \mathfrak{p}$ is induced from e_0. Since the map $\xi \colon \mathrm{Specm}\, U(\mathfrak{l}, e_0)^{\mathrm{ab}} \to \mathcal{E}$ is Γ-equivariant by our choice of $\mathfrak{p}$ and e_0, combining [Los11a, Theorem 6.5.2] with Proposition 15 yields that $\xi^*\big(\widetilde{\mathcal{E}}^{\Gamma_0}\big) = \mathcal{E}^{\Gamma}$. Then, arguing as in part (a), we obtain that $I = \mathfrak{I}^{\mathfrak{g}}_{\mathfrak{p}}(I_0)$ for some multiplicity-free primitive ideal I_0 of $U(\mathfrak{l})$ with $\mathrm{VA}(I_0) = \overline{\mathcal{O}}_0$, where $\mathcal{O}_0 = (\mathrm{Ad}\, L)\, e_0$.

This completes the proof of Theorem 5. We note that for $\mathfrak{g} = \mathfrak{sl}_n$ one can argue as in part (c) (with $\mathfrak{l}_1$ replaced by $\mathfrak{g}$) to obtain yet another proof of Mœglin's theorem.

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TABLE 0. Unresolved cases.

F_4	E_6	E_7	E_8	E_8	E_8	E_8
$\mathsf{C}_3(\mathsf{a}_1)$	$\mathsf{A}_3+\mathsf{A}_1$	$\mathsf{D}_6(\mathsf{a}_2)$	$\mathsf{E}_6(\mathsf{a}_3)+\mathsf{A}_1$	$\mathsf{D}_6(\mathsf{a}_2)$	$\mathsf{E}_7(\mathsf{a}_2)$	$\mathsf{E}_7(\mathsf{a}_5)$

TABLE 1. Data for the induced orbits in type E_8.

Dynkin label	Type of Γ	Number of sheets	Ranks of sheets	$\dim \mathfrak{c}_e$	$\dim \mathfrak{c}_e^\Gamma$
E_8	1	1 (even)	8	8	8
$\mathsf{E}_8(\mathsf{a}_1)$	1	1 (even)	7	7	7
$\mathsf{E}_8(\mathsf{a}_2)$	1	1 (even)	6	6	6
$\mathsf{E}_8(\mathsf{a}_3)$	S_2	2 (even)	6,5	7	5
E_7	1	1	4	4	4
$\mathsf{E}_8(\mathsf{a}_4)$	S_2	2 (even)	5,4	6	4
$\mathsf{E}_8(\mathsf{b}_4)$	S_2	2 (even)	4,3	5	4
$\mathsf{E}_7(\mathsf{a}_1)$	1	1	5	5	5
$\mathsf{E}_8(\mathsf{a}_5)$	S_2	2 (even)	4,3	5	3
$\mathsf{E}_8(\mathsf{b}_5)$	S_3	3 (even)	4,4,3	7	3
D_7	1	1	2	2	2
$\mathsf{E}_7(\mathsf{a}_2)$	1	1	3	4	4*
$\mathsf{E}_8(\mathsf{a}_6)$	S_3	3 (even)	3,3,2	6	2
$\mathsf{D}_7(\mathsf{a}_1)$	S_2	2 (even)	3,2	4	3
$\mathsf{E}_6+\mathsf{A}_1$	1	1	2	2	2
$\mathsf{E}_7(\mathsf{a}_3)$	S_2	1	4	4	2
$\mathsf{E}_8(\mathsf{b}_6)$	S_3	3 (even)	2,2,1	5	2
$\mathsf{E}_6(\mathsf{a}_1)+\mathsf{A}_1$	S_2	1	3	3	1
A_7	1	1	1	1	1
E_6	1	1 (even)	4	4	4
$\mathsf{D}_7(\mathsf{a}_2)$	S_2	2	2,2	3	2
D_6	1	1	2	2	2
$\mathsf{E}_6(\mathsf{a}_1)$	S_2	2 (even)	3,3	4	3
$\mathsf{D}_5+\mathsf{A}_2$	S_2	2 (even)	2,1	3	2
$\mathsf{E}_7(\mathsf{a}_4)$	S_2	2	2,2	3	2
$\mathsf{A}_6+\mathsf{A}_1$	1	1	1	1	1

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TABLE 2. Data for the induced orbits in type E_8 (continued).

Dynkin label	Type of Γ	Number of sheets	Ranks of sheets	dim $\mathfrak{c}_e$	dim $\mathfrak{c}_e^\Gamma$
$\mathsf{D}_6(\mathsf{a}_1)$	S_2	1	3	3	3
A_6	1	1 (even)	2	2	2
$\mathsf{E}_8(\mathsf{a}_7)$	S_5	4 (even)	2,2,1,1	10	1
$\mathsf{D}_5+\mathsf{A}_1$	1	1	2	2	2
$\mathsf{E}_7(\mathsf{a}_5)$	S_3	2	1,1	6	2^*
$\mathsf{D}_6(\mathsf{a}_2)$	S_2	1	1	3	2^*
$\mathsf{E}_6(\mathsf{a}_3)+\mathsf{A}_1$	S_2	1	1	3	2^*
D_5	1	1 (even)	3	3	3
$\mathsf{E}_6(\mathsf{a}_3)$	S_2	2 (even)	2,2	3	2
$\mathsf{D}_4+\mathsf{A}_2$	S_2	1 (even)	2	2	2
A_5	1	1	1	1	1
$\mathsf{D}_5(\mathsf{a}_1)+\mathsf{A}_1$	1	1	1	1	1
$\mathsf{A}_4+\mathsf{A}_2+\mathsf{A}_1$	1	1	1	1	1
$\mathsf{A}_4+\mathsf{A}_2$	1	1 (even)	1	1	1
$\mathsf{A}_4+2\mathsf{A}_1$	S_2	1	1	1	0
$\mathsf{D}_5(\mathsf{a}_1)$	S_2	1	2	2	1
$\mathsf{A}_4+\mathsf{A}_1$	S_2	1	1	1	0
$\mathsf{D}_4+\mathsf{A}_1$	1	1	1	1	1
$\mathsf{D}_4(\mathsf{a}_1)+\mathsf{A}_2$	S_2	1 (even)	1	1	1
A_4	S_2	1 (even)	2	2	2
$\mathsf{A}_3+\mathsf{A}_2$	S_2	2	1,1	2	1
D_4	1	1 (even)	2	2	2
$\mathsf{D}_4(\mathsf{a}_1)$	S_3	2 (even)	1,1	3	1
$2\mathsf{A}_2$	S_2	1 (even)	1	1	1
A_3	1	1	1	1	1
A_2	S_2	1 (even)	1	1	1

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Table 3. Data for the induced orbits in type E_7.

Dynkin label	Type of Γ	Number of sheets	Ranks of sheets	$\dim \mathfrak{c}_e$	$\dim \mathfrak{c}_e^\Gamma$
E_7	1	1 (even)	7	7	7
$\mathsf{E}_7(\mathsf{a}_1)$	1	1 (even)	6	6	6
$\mathsf{E}_7(\mathsf{a}_2)$	1	1 (even)	5	5	5
$\mathsf{E}_7(\mathsf{a}_3)$	S_2	2 (even)	5,4	6	4
E_6	1	1 (even)	4	4	4
D_6	1	1	3	3	3
$\mathsf{E}_6(\mathsf{a}_1)$	S_2	2 (even)	4,3	5	3
$\mathsf{E}_7(\mathsf{a}_4)$	S_2	2 (even)	3,2	4	3
$\mathsf{D}_6(\mathsf{a}_1)$	1	1	4	4	4
$\mathsf{D}_5+\mathsf{A}_1$	1	1	3	3	3
A_6	1	1 (even)	2	2	2
D_5	1	1 (even)	3	3	3
$\mathsf{E}_7(\mathsf{a}_5)$	S_3	3 (even)	3,3,2	6	2
$\mathsf{D}_6(\mathsf{a}_2)$	1	1	2	3	3*
$\mathsf{E}_6(\mathsf{a}_3)$	S_2	2 (even)	2,2	3	2
$\mathsf{A}_5+\mathsf{A}_1$	1	1	1	1	1
$(\mathsf{A}_5)'$	1	1	1	1	1
$\mathsf{D}_5(\mathsf{a}_1)+\mathsf{A}_1$	1	1 (even)	2	2	2
$\mathsf{D}_5(\mathsf{a}_1)$	S_2	1	3	3	2
$\mathsf{A}_4+\mathsf{A}_2$	1	1 (even)	1	1	1
$\mathsf{A}_4+\mathsf{A}_1$	S_2	1	2	2	0
$(\mathsf{A}_5)''$	1	1 (even)	3	3	3
$\mathsf{D}_4+\mathsf{A}_1$	1	1	1	1	1
A_4	S_2	2 (even)	2,2	3	2
$\mathsf{A}_3+\mathsf{A}_2+\mathsf{A}_1$	1	1 (even)	1	1	1
$\mathsf{A}_3+\mathsf{A}_2$	S_2	2	1,1	2	1
$\mathsf{D}_4(\mathsf{a}_1)+\mathsf{A}_1$	S_2	1	2	2	2
D_4	1	1 (even)	2	2	2
$\mathsf{A}_3+2\mathsf{A}_1$	1	1	1	1	1
$\mathsf{D}_4(\mathsf{a}_1)$	S_3	2 (even)	1,1	3	1
$(\mathsf{A}_3+\mathsf{A}_1)''$	1	1 (even)	2	2	2
A_3	1	1	1	1	1
$2\mathsf{A}_2$	1	1 (even)	1	1	1
$\mathsf{A}_2+3\mathsf{A}_1$	1	1 (even)	1	1	1
$\mathsf{A}_2+\mathsf{A}_1$	S_2	1	1	1	0
A_2	S_2	1 (even)	1	1	1
$(3\mathsf{A}_1)''$	1	1 (even)	1	1	1

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Table 4. Data for the induced orbits in type E_6.

Dynkin label	Type of Γ	Number of sheets	Ranks of sheets	$\dim \mathfrak{c}_e$	$\dim \mathfrak{c}_e^\Gamma$
E_6	1	1 (even)	6	6	6
$\mathsf{E}_6(\mathsf{a}_1)$	1	1 (even)	5	5	5
D_5	1	1 (even)	4	4	4
$\mathsf{E}_6(\mathsf{a}_3)$	S_2	2 (even)	4,3	5	3
$\mathsf{D}_4+\mathsf{A}_1$	1	1	1	1	1
A_5	1	1	2	2	2
$\mathsf{D}_5(\mathsf{a}_1)$	1	1	3	3	3
$\mathsf{A}_4+\mathsf{A}_1$	1	1	2	2	2
A_4	1	1 (even)	3	3	3
$\mathsf{D}_4(\mathsf{a}_1)$	S_3	3 (even)	2,2,1	5	1
$\mathsf{A}_3+\mathsf{A}_1$	1	1	1	2	2^*
A_3	1	1	2	2	2
$\mathsf{A}_2+2\mathsf{A}_1$	1	1	1	1	1
$2\mathsf{A}_2$	1	1	2	2	2
$\mathsf{A}_2+\mathsf{A}_1$	1	1	1	1	1
A_2	S_2	1 (even)	1	1	1
$2\mathsf{A}_1$	1	1	1	1	1

Table 5. Data for the induced orbits in type F_4.

Dynkin label	Type of Γ	Number of sheets	Ranks of sheets	$\dim \mathfrak{c}_e$	$\dim \mathfrak{c}_e^\Gamma$
F_4	1	1 (even)	4	4	4
$\mathsf{F}_4(\mathsf{a}_1)$	S_2	2 (even)	3,3	4	3
$\mathsf{F}_4(\mathsf{a}_2)$	S_2	2 (even)	2,2	3	2
B_3	1	1 (even)	2	2	2
C_3	1	1	2	2	2
$\mathsf{F}_4(\mathsf{a}_3)$	S_4	3 (even)	2,1,1	6	1
$\mathsf{C}_3(\mathsf{a}_1)$	S_2	1	1	3	2^*
B_2	S_2	1 (even)	1	1	1
$\widetilde{\mathsf{A}}_2$	1	1 (even)	1	1	1
A_2	S_2	1 (even)	1	1	1

Table 6. Data for the induced orbits in type G_2.

Dynkin label	Type of Γ	Number of sheets	Ranks of sheets	$\dim \mathfrak{c}_e$	$\dim \mathfrak{c}_e^\Gamma$
G_2	1	1 (even)	2	2	2
$\mathsf{G}_2(\mathsf{a}_1)$	S_3	2 (even)	1,1	3	1

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