SOME REMARKS ON COUNTABLY BARRELLED AND COUNTABLY QUASIBARRELLED SPACES

by SUNDAY O. IYAHEN

(Received 5th January 1967)

Barrelled and quasibarrelled spaces form important classes of locally convex spaces. In (**2**), Husain considered a number of less restrictive notions, including infinitely barrelled spaces (these are the same as barrelled spaces), countably barrelled spaces and countably quasibarrelled spaces. A separated locally convex space E with dual E' is called countably barrelled (countably quasibarrelled) if every weakly bounded (strongly bounded) subset of E' which is the countable union of equicontinuous subsets of E' is itself equicontinuous. It is trivially true that every barrelled (quasibarrelled) space is countably barrelled (countably quasibarrelled) and a countably barrelled space is countably quasibarrelled. In this note we give examples which show that (i) a countably barrelled space need not be barrelled (or even quasibarrelled) and (ii) a countably quasibarrelled space need not be countably barrelled. A third example (iii) shows that the property of being countably barrelled (countably quasibarrelled) does not pass to closed linear subspaces.

(i) Let E be the strong dual of a metrisable locally convex space. Then by ((**1**), pages 71 and 88), E need not be quasibarrelled. But E is countably barrelled, being countably quasibarrelled and complete ((**2**), Propositions 1 and 4).

(ii) Denote by c the Banach space of all convergent sequences $x = (x_1, x_2, \ldots)$ with the supremum norm, by c_0 the closed linear subspace of c consisting of sequences converging to zero and by ϕ the linear subspace consisting of all sequences containing only a finite number of non-zero entries. For each n, let f_n be the linear functional on ϕ defined by the equation $f_n(x) = nx_n$. As pointed out by Weston ((**4**), page 1), (f_n) is a pointwise bounded sequence of continuous linear functionals on ϕ (under the norm topology induced from c) which is not equicontinuous. Thus by Corollary 6 of (**2**), ϕ is not countably barrelled, though it is countably quasibarrelled, being bornological.

(iii) Since any separated locally convex space is a closed linear subspace of some barrelled space ((**3**), Theorem 1.1), to show that a closed linear subspace of a countably barrelled (countably quasibarrelled) space need not be of the same sort, it is sufficient to give an example of a separated locally convex space which is not countably quasibarrelled. Let (E, u) be c_0 with the supremum norm u and let v be the associated weak topology on c_0. For each n, let g_n be the linear map from (E, v) into (E, u) defined as follows:

$$g_n(x) = (x_1, x_2, \ldots, x_n, 0, 0, 0, \ldots).$$

Then (g_n) is a sequence of continuous linear maps from (E, v) into (E, u) such that, for each x in E, $g_n(x)$ converges to x in (E, u). Moreover, (g_n) is uniformly bounded on bounded sets, for if B is the unit ball in (E, u), the union over n of $g_n(B)$ is contained in B. But (g_n) is not equicontinuous since v is strictly coarser than u. Therefore by Corollary 6 of (**2**), (E, v) is not countably quasibarrelled.

REFERENCES

(**1**) A. Grothendieck, Sur les espaces (F) et (DF), *Summa Brasil. Math.* **3** (1954), 57-122.

(**2**) T. Husain, Two new classes of locally convex spaces, *Math. Ann.* **166** (1966), 289-299.

(**3**) Y. Komura, On linear topological spaces, *Kumamoto J. Sci.* Ser. A **5** (1962), 148-157.

(**4**) J. D. Weston, The principle of equicontinuity for topological vector spaces, *Proc. Univ. Durham Philos. Soc.* Ser. A **13** (1957), 1-5.

University of Keele
England