

AN INVERSION FORMULA FOR THE WEIERSTRASS TRANSFORM

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1. Introduction. The Weierstrass transform $f(x)$ of a function $\phi(y)$ is defined by

$$(1.1) \quad f(x) = \int_{-\infty}^{+\infty} k\left(\frac{x-y}{2}\right) \phi(y) dy$$

where

$$k(x) = (4\pi)^{-\frac{1}{2}} e^{-x^2}$$

whenever this integral exists (7, p. 174). It is also known as the Gauss transform (11; 12). Its basic properties have been developed and studied in (7) and in particular it has been shown that the symbolic operator

$$e^{-D^2}, \quad D \equiv \frac{d}{dx}$$

will invert this transform under suitable assumptions and with certain definitions of this operator. We propose to study the definition

$$(1.2) \quad e^{-D^2} f(x) = \lim_{n \rightarrow \infty} \left(1 - \frac{D^2}{n}\right)^n f(x)$$

for $f(x)$ in C^∞ . This formula seems to have been first examined by Pollard (9) and later by Rooney (12). In so far as convergence of (1.2) is concerned, we will considerably improve the results of (12).

2. The inversion operator. Along with Rooney (12), we note that (1.2) is in reality a summability of the series arising from the following interpretation of e^{-D^2}

$$(2.1) \quad e^{-D^2} f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} D^{2n} f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} f^{(2n)}(x).$$

For a general series $\sum_{n=0}^{\infty} c_n$, the summability process is defined as

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n a_{k,n} c_k = \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{n!k}{(n-k)!n^{k+1}} s_k$$

where

$$s_k = \sum_{i=0}^k c_i, \quad a_{k,n} = \frac{n!}{(n-k)!n^k}.$$

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This process has been studied by Amerio **(1)**, Bernstein **(3)**, and Rey-Pastor **(10)**. We will refer to this as (S) summability. This follows the definition of Amerio. It is in fact regular. We will compare it with the familiar Euler (E, q) method defined as follows **(6, p. 180)**,

$$\lim_{n \rightarrow \infty} (q + 1)^{-n} \sum_{k=0}^n \binom{n}{k} q^{n-k} s_k, \quad 0 < q < \infty$$

for the series mentioned above.

THEOREM 1. *Let $\sum_{n=0}^{\infty} c_n$ be summable (E, q) for q satisfying*

$$(2.2) \quad (2q + 1) < \exp[(2q + 1)^{-1} + 1],$$

then the series is summable (S) .

Let q_0 be the unique root of the equation

$$2q_0 + 1 = \exp[(2q_0 + 1)^{-1} + 1].$$

Clearly $0 < q < q_0$. From **(6, p. 181)**, $c_n = o[(2q + 1)^n]$, $n \rightarrow \infty$. The series $\sum_{n=0}^{\infty} c_n z^n$ then converges for $|z| < (2q + 1)^{-1}$. In particular it converges for $z = Re^{i\theta}$ for any θ and $R = (2q' + 1)^{-1}$, $(2q + 1) < (2q' + 1) < (2q_0 + 1)$. By a result of Bernstein **(3, p. 358)**, the series $\sum_{n=0}^{\infty} c_n z^n$ is summable (S) up to and including the value $z = x_0$, where x_0 is defined by $x_0/R = \exp((x_0/R)^{-1} + 1)$. Clearly $(x_0/R) = 2q_0 + 1$; that is $x_0 = R(2q_0 + 1) = (2q_0 + 1)/(2q' + 1) > 1$ so that $\sum_{n=0}^{\infty} c_n$ is summable (S) .

The series is indeed summable to the same value. This along with a different proof of Theorem 1 has been shown in **(4, p. 78)**. The number q_0 is approximately 1.29 Rey-Pastor **(10)** has proven that the series

$$\sum_{n=0}^{\infty} (-1)^n a^n$$

is summable (S) to $(1 + a)^{-1}$ for $-1 < a < a_0$, $a_0 = 2q_0 + 1$, $a_0 = 3.59$ The process diverges for a outside this region. On the other hand, (E, q) sums this series to $(1 + a)^{-1}$ for $-1 < a < 2q + 1$ and thus we see that Theorem 1 is best possible.

Returning to the definition (1.2), we obtain

$$(2.3) \quad \left(1 - \frac{D^2}{n}\right)^n f(x) = \int_{-\infty}^{+\infty} K_n\left(\frac{x-y}{2}\right) k\left(\frac{x-y}{2}\right) \phi(y) dy$$

where

$$(2.4) \quad K_n(x) = \sum_{k=0}^n (-1)^k \binom{n}{k} (4n)^{-k} H_{2k}(x)$$

and $H_n(x)$ is the Hermite polynomial defined by

$$H_n(x) = (-1)^n e^{x^2} D^n e^{-x^2}.$$

We begin with

LEMMA 1.

$$(a) \int_{-\infty}^{+\infty} k\left(\frac{x}{2}\right)K_n\left(\frac{x}{2}\right)dx = 1.$$

$$(b) K_n(x) = \frac{(-n)^n \sqrt{n} n!}{(2n)!} \int_0^\infty e^{-nt} t^{-\frac{1}{2}} (1+t)^n H_{2n}\left(\frac{x\sqrt{t}}{(1+t)^{\frac{1}{2}}}\right) dt.$$

Part (a) follows from the orthogonality of Hermite polynomials. On the other hand, from Bailey (2),

$$u^{2n} H_{2n}\left(\frac{x}{u}\right) = \sum_{k=0}^n \frac{(2n)!}{k!(2n-2k)!} (1-u^2)^k H_{2n-2k}(x).$$

Let $u^2 = (t+1)/t$, multiply both sides by e^{-nt} and integrate from 0 to ∞ to get

$$\begin{aligned} \int_0^\infty e^{-nt} t^{-\frac{1}{2}} (1+t)^n H_{2n}\left(\frac{x\sqrt{t}}{(1+t)^{\frac{1}{2}}}\right) dt \\ = (-1)^n (2n)! \sum_{k=0}^n (-1)^k \frac{\Gamma(k+\frac{1}{2})}{(n-k)!(2k)!} n^{-k-\frac{1}{2}} H_{2k}(x) \end{aligned}$$

and this is part (b).

Some bounds for $K_n(x)$ are now developed.

LEMMA 2. (a) $K_n(x) = O(n^{\frac{1}{4}})$, $n \rightarrow \infty$, uniformly for x in any finite interval.

(b) $K_n(x) = O[n^{\frac{3}{4}} \exp\{(x^2/2) + n - n(1+2x^2/n)^{\frac{1}{2}} + n \log(\frac{1}{2} + \frac{1}{2}(1+2x^2/n)^{\frac{1}{2}})\}]$ uniformly for $0 \leq x \leq A\sqrt{n}$, A any constant.

(c) $K_n(x) = O(n^{-\frac{1}{4}}x^{-2}) + O(n^{-\gamma})$ where γ is some positive number and the result holds uniformly for $0 < x \leq n^\epsilon$, $\epsilon < 1/24$.

(d) $|K_n(x)| \leq (1+2x^2/n)^n$, $|x| \geq (2n+\frac{1}{2})^{\frac{1}{2}}$.

It is a consequence of a result proved in (13, p. 194) that $H_{2n}(x) = O[(2n)!/n!]$ uniformly for x in any finite interval. Thus by Lemma 1,

$$K_n(x) = O(n^{\frac{1}{2}}) \int_0^\infty e^{-nt} t^{-\frac{1}{2}} (1+t)^n dt, \quad n \rightarrow \infty.$$

An appeal to a result in (5, p. 37) finishes the proof of part (a). For part (b) we use the result, (13, p. 236),

$$e^{-x^2/2} H_{2n}(x) = O[2^n \{(2n)!\}^{\frac{1}{2}}]$$

uniformly for all x . Thus

$$K_n(x) = O(n^{\frac{3}{4}}) \int_0^\infty e^{h_n(t)} \frac{dt}{\sqrt{t(1+t)}}$$

where $h_n(t) = -nt + (n+1) \log(1+t) + (x^2 t/2(1+t))$. $h_n(t)$ has a maximum at $t = t_0$,

$$t_0 = \frac{1}{2} \left[-1 + \left(1 + \frac{2x^2}{n} \right)^{\frac{1}{2}} \right] + O\left(\frac{1}{n}\right), \quad n \rightarrow \infty$$

uniformly for $0 \leq x < \infty$. Then $K_n(x) = O[n^{\frac{1}{2}}e^{h_n(t_0)}]$ and it is easy to show that

$$h_n(t_0) = n - n\left(1 + \frac{2x^2}{n}\right)^{\frac{1}{2}} + n \log \left[\frac{1}{2} + \frac{1}{2}\left(1 + \frac{2x^2}{n}\right)^{\frac{1}{2}} \right] + \frac{x^2}{2} + O(1)$$

uniformly for $0 \leq x \leq A\sqrt{n}$ for any constant A . This is part (b). For the more difficult part (c), we use the following asymptotic formula for $H_n(x)$ (13, p. 195),

$$e^{-u^2/2}H_n(u) = 2^{n/2+\frac{1}{4}}(n!)^{\frac{1}{2}}(\pi n)^{-\frac{1}{4}}(\sin \phi)^{-\frac{1}{2}} \left\{ \sin \left[\left(\frac{n}{2} + \frac{1}{4} \right) (\sin 2\phi - 2\phi) + \frac{3\pi}{4} \right] + O(n^{-1}) \right\}$$

for $u = (2n + 1)^{\frac{1}{2}} \cos \phi$ and the order condition holds uniformly for $\mu \leq \phi \leq \pi - \mu, \mu > 0$. Thus, after using Stirling's formula, we get

$$K_n(x) = \frac{(-1)^n \sqrt{n}}{\sqrt{\pi}} \int_0^\infty e^{-nt} t^{-\frac{1}{2}} (1+t)^n e^{x^2 t/2(1+t)} (\sin \phi)^{-\frac{1}{2}} \sin \left[\left(n + \frac{1}{4} \right) (\sin 2\phi - 2\phi) + \frac{3\pi}{4} \right] dt + O(n^{-\frac{1}{2}}) \int_0^\infty e^{-nt} t^{-\frac{1}{2}} (1+t)^n e^{x^2 t/2(1+t)} dt$$

with

$$x \left(\frac{t}{1+t} \right)^{\frac{1}{2}} = (4n + 1)^{\frac{1}{2}} \cos \phi.$$

The second part is $O(n^{-\frac{1}{2}}e^{h_n(t_0)})$ with $h_n(t_0)$ as before and we will now assume that $0 \leq x \leq n^\epsilon$ for ϵ as yet unspecified but less than $\frac{1}{2}$. We split the remaining integral into two parts I_1 and I_2 corresponding respectively to the ranges $(0, t_0 + n^{\delta-\frac{1}{2}})$, $(t_0 + n^{\delta-\frac{1}{2}}, \infty)$, t_0 as above and $0 < \delta < \frac{1}{2}$. $h_n(t)$ is a decreasing function in the range defining I_2 so that

$$I_2 = O(e^{h_n(t_0+n^{\delta-\frac{1}{2}})})$$

and we can show that $h_n(t_0 + n^{\delta-\frac{1}{2}}) \leq h_n(t_0) - cn^{2\delta}$ for some constant c and $0 < \delta < 1/6$. Thus $I_2 = O(e^{h_n(t_0)-cn^{2\delta}})$ and

$$K_n(x) = (-1)^n \left(\frac{n}{\pi} \right)^{\frac{1}{2}} I_1 + O(n^{-\frac{1}{2}}e^{h_n(t_0)}), \quad n \rightarrow \infty$$

$0 \leq x \leq n^\epsilon, 0 < \epsilon < \frac{1}{2}$. Now we can also show for the same range of x , $h_n(t_0) = O(n^{4\epsilon-1}) + O(1), n \rightarrow \infty$ so that for $0 < \epsilon < \frac{1}{4}$,

$$K_n(x) = (-1)^n \left(\frac{n}{\pi} \right)^{\frac{1}{2}} I_1 + O(n^{-\frac{1}{2}}), \quad n \rightarrow \infty$$

where

$$I_1 = \int_0^{t_0+n^{\delta-\frac{1}{2}}} t^{-\frac{1}{2}} e^{gn(t)} (\sin \phi)^{-\frac{1}{2}} \sin \left[\left(n + \frac{1}{4} \right) (\sin 2\phi - 2\phi) + \frac{3\pi}{4} \right] dt$$

with

$$x\left(\frac{t}{1+t}\right)^{\frac{1}{2}} = (4n+1)^{\frac{1}{2}} \cos \phi, \quad g_n(t) = -nt + n \log(1+t) + \frac{x^2 t}{2(1+t)}.$$

For $0 < \epsilon < \delta/2 < 1/24$, $g_n(t) = (-nt^2/2) + O(n^{-\frac{1}{4}-\gamma})$ uniformly for $0 \leq x \leq n^\epsilon$ where γ is some small positive number. Also

$$(\sin \phi)^{-\frac{1}{2}} = 1 + O(n^{-\frac{1}{4}-\gamma}),$$

$$\begin{aligned} \sin\left[\left(n + \frac{1}{4}\right)(\sin 2\phi - 2\phi) + \frac{3\pi}{4}\right] \\ = (-1)^n \cos\left[(4n+1)^{\frac{1}{2}} x \sqrt{t} - \frac{1}{2}(4n+1)^{\frac{1}{2}} x t^{3/2}\right] + O(n^{3\delta-7/4}) \end{aligned}$$

and the last term is $O(n^{-\frac{1}{4}-\gamma})$ for $\delta < 1/12$. Then we can show that

$$\begin{aligned} K_n(x) = \frac{n^{1/4}}{\sqrt{\pi}} \int_0^{\sqrt{n t_0 + n^\delta}} e^{-u^2/2} u^{-\frac{1}{2}} \\ \cos\left[\frac{(4n+1)^{\frac{1}{2}}}{n^{\frac{1}{4}}} x \sqrt{u} - \frac{(4n+1)^{\frac{1}{2}}}{2n^{\frac{3}{4}}} x u^{3/2}\right] du + O(n^{-\gamma}) \end{aligned}$$

uniformly for $0 \leq x \leq n^\epsilon$. The upper limit may be replaced by ∞ and the error is

$$O(n^{\frac{1}{4}} e^{-cn^{2\delta}}).$$

We now split up the integral according to the addition formula for the cosine function into J_1 and J_2 . J_1 corresponds to the integral with the cosines in the integrand. Two integrations by parts will show that $J_1 = O(n^{-\frac{1}{2}} x^{-2}) + O(n^{-\frac{1}{4}-\gamma})$. In a similar way, the second part $J_2 = O(n^{-\frac{3}{2}})$. Combining all of these results, we obtain

$$K_n(x) = O(n^{-\frac{1}{2}} x^{-2}) + O(n^{-\gamma}), \quad n \rightarrow \infty$$

uniformly for $0 < x \leq n^\epsilon$.

For part (d), we use the bound for $H_n(x)$ developed in (8, p. 158),

$$|H_n(x)| \leq 2^{3n/2} |x|^n, \quad |x| \geq (n + \frac{1}{2})^{\frac{1}{2}}$$

so that for $|x| \geq (2n + \frac{1}{2})^{\frac{1}{2}}$

$$|K_n(x)| \leq \sum_{k=0}^n \binom{n}{k} \left(\frac{1}{4n}\right)^k 2^{3k} x^{2k} = \left(1 + \frac{2x^2}{n}\right)^n.$$

This completes the proof of the lemma.

3. The main theorem. We begin with a lemma on the roots of a certain transcendental equation.

LEMMA 3. *There is one and only one solution to the pair of equations*

$$1 + (A - \frac{1}{2}) \frac{k^2}{4} - \left(1 + \frac{k^2}{2}\right)^{\frac{1}{2}} + \log\left[\frac{1}{2} + \frac{1}{2}\left(1 + \frac{k^2}{2}\right)^{\frac{1}{2}}\right] = 0$$

$$- (1 - A) \frac{k^2}{4} + \log\left(1 + \frac{k^2}{2}\right) = 0$$

outside of the trivial pair $k^2 = 0, A = \text{anything}$.

The change of variable $u = (1 + k^2/2)^{\frac{1}{2}}$ and the elimination of A results in

$$\frac{(u - 3)(u - 1)}{4} = \log\left(\frac{2u^2}{1 + u}\right)$$

and it is easy to see from this that there are only two solutions and that $u = 1$ corresponds to the trivial solution. The important solution (k_0^2, A_0) is

$$A_0 = .738 \dots, \quad k_0^2 = 49.876 \dots$$

We now prove our main result.

THEOREM 2. *Suppose $A < A_0$, where A_0 is defined in Lemma 3 and that*

$$\int_{-\infty}^{+\infty} e^{-A(y^2 - 2x_0y)/4} |\phi(y)| dy < \infty.$$

If, moreover, $\phi(x_0 +)$ and $\phi(x_0 -)$ exist and $f(x)$ is the Weierstrass transform of ϕ , then

$$\lim_{n \rightarrow \infty} \left(1 - \frac{D^2}{n}\right)^n f(x) \Big|_{x=x_0} = \frac{1}{2}[\phi(x_0 +) + \phi(x_0 -)].$$

Let $\Lambda = \frac{1}{2}[\phi(x_0 +) + \phi(x_0 -)]$. By Lemma 1, we can write

$$\left(1 - \frac{D^2}{n}\right)^n f(x) - \Lambda = \int_{-\infty}^{+\infty} k\left(\frac{x_0 - y}{2}\right) K_n\left(\frac{x_0 - y}{2}\right) [\phi(y) - \Lambda] dy.$$

The integral is decomposed into I_1, I_2, I_3 corresponding respectively to the ranges $(-\infty, x_0 - \eta), (x_0 - \eta, x_0 + \eta), (x_0 + \eta, \infty), \eta > 0$. Again decompose I_3 into I_3', I_3'', I_3''' corresponding respectively to $(x_0 + \eta, x_0 + k_0 n^\epsilon), (x_0 + k_0 n^\epsilon, x_0 + k_0 \sqrt{n}), (x_0 + k_0 \sqrt{n}, \infty)$ with ϵ, k_0 as before. Now, by Lemma 2,

$$|I_3'''| \leq \int_{k_0 \sqrt{n}}^{\infty} e^{g(u)} e^{-Au^2/4} |\phi(u + x_0) - \Lambda| du$$

where $g(u) = -(1 - A)u^2/4 + n \log(1 + (u^2/2n))$. $g(u)$ has a maximum for

$$u = u_0 = \sqrt{n} \left(2 \frac{(1 + A)}{(1 - A)}\right)^{\frac{1}{2}}.$$

Then

$$u_0 < \sqrt{n} \left(2 \frac{(1 + A_0)}{(1 - A_0)}\right)^{\frac{1}{2}} < k_0 \sqrt{n}.$$

Also $g'(u) < 0$ in the range of integration so that

$$I_3''' = O[e^{\rho(k_0\sqrt{n})}] \int_{k_0\sqrt{n}}^{\infty} e^{-Au^2/4} |\phi(u + x_0) - \Lambda| du.$$

But

$$g(k_0\sqrt{n}) = n \left[- (1 - A) \frac{k_0^2}{4} + \log \left(1 + \frac{k_0^2}{2} \right) \right] < n \left[- (1 - A_0) \frac{k_0^2}{4} + \log \left(1 + \frac{k_0^2}{2} \right) \right] = 0$$

and thus $I_3''' = o(1)$, $n \rightarrow \infty$. On the other hand, by Lemma 2

$$I_3'' = O(n^{\frac{3}{4}}) \int_{k_0 n^\epsilon}^{k_0\sqrt{n}} e^{d(u)} e^{-Au^2/4} |\phi(u + x_0) - \Lambda| du.$$

Now

$$(3.1) \quad d(u) = n + (A - \frac{1}{2}) \frac{u^2}{4} - n \left(1 + \frac{u^2}{2n} \right)^{\frac{1}{2}} + n \log \left[\frac{1}{2} + \frac{1}{2} \left(1 + \frac{u^2}{2n} \right)^{\frac{1}{2}} \right] = (A - A_0) \frac{u^2}{4} + h(u).$$

By hypothesis $h(k_0\sqrt{n}) = 0$ and also $h(0) = 0$. It is not difficult to show that $h(u) \leq 0$ for $0 \leq u \leq k_0\sqrt{n}$. Then from (3.1)

$$\max d(u) \leq \frac{1}{4} (A - A_0) (k_0 n^\epsilon)^2.$$

Therefore

$$I_3'' = O(n^{\frac{3}{4}} e^{(k_0^2/4)(A-A_0)n^2\epsilon}) \int_{k_0 n^\epsilon}^{k_0\sqrt{n}} e^{-Au^2/4} |\phi(u + x_0) - \Lambda| du$$

and this is $o(1)$. Now by Lemma 2,

$$I_3' = O(n^{-\frac{1}{4}}) \int_{\eta}^{k_0 n^\epsilon} e^{-u^2/4} u^{-2} |\phi(u + x_0) - \Lambda| du + O(n^{-\gamma}) \int_{\eta}^{k_0 n^\epsilon} e^{-u^2/4} |\phi(u + x_0) - \Lambda| du$$

and this is also $o(1)$, $n \rightarrow \infty$. Thus $I_3 = o(1)$. The proof that $I_1 = o(1)$ is similar. There remains I_2 . Since $K_n(x)$ is even,

$$(3.2) \quad I_2 = \int_{x_0}^{x_0+\eta} k \left(\frac{x_0 - y}{2} \right) K_n \left(\frac{x_0 - y}{2} \right) [\phi(y) - \phi(x_0+)] dy + \int_{x_0-\eta}^{x_0} k \left(\frac{x_0 - y}{2} \right) K_n \left(\frac{x_0 - y}{2} \right) [\phi(y) - \phi(x_0-)] dy.$$

For the first integral, given $\delta > 0$ choose η such that $|\phi(y) - \phi(x_0+)| < \delta$ for $x_0 \leq y \leq x_0 + \eta$. Then

$$\int_{x_0}^{x_0+\eta} k\left(\frac{x_0 - y}{2}\right) \left| K_n\left(\frac{x_0 - y}{2}\right) \right| |\phi(y) - \phi(x_0 +)| dy < \delta \int_0^\eta e^{-u^2/4} \left| K_n\left(\frac{u}{2}\right) \right| du$$

and this last integral is split up according to the ranges $(0, n^{-\frac{1}{4}})$, $(n^{-\frac{1}{4}}, \eta)$. By Lemma 2,

$$\int_0^{n^{-\frac{1}{4}}} e^{-u^2/4} |K_n\left(\frac{u}{2}\right)| du = O(n^{\frac{1}{4}}) \int_0^{n^{-\frac{1}{4}}} e^{-u^2/4} du = O(1), \quad n \rightarrow \infty.$$

Also

$$\begin{aligned} \int_{n^{-\frac{1}{4}}}^\eta e^{-u^2/4} |K_n\left(\frac{u}{2}\right)| du &= O(n^{-\frac{1}{4}}) \int_{n^{-\frac{1}{4}}}^\eta e^{-u^2/4} u^{-2} du + O(n^{-\gamma}) \int_{n^{-\frac{1}{4}}}^\eta e^{-u^2/4} du \\ &= O(n^{-\frac{1}{4}}) \int_{n^{-\frac{1}{4}}}^\eta u^{-2} du + O(n^{-\gamma}) = O(1). \end{aligned}$$

Thus the first integral in (3.2) is $\delta O(1), n \rightarrow \infty$. The same holds true for the second integral and therefore

$$\overline{\lim} |I_2| \leq M\delta, \quad M \text{ a constant.}$$

This proves the theorem.

This result should be compared with the theorem on convergence of (2.1). The series inverts approximately when $A = \frac{1}{2}$ (referring to the A of Theorem 2). See (11) and (4, p. 12). Not all functions which are Weierstrass transforms can be inverted by our inversion formula. For example (7, p. 178), let

$$f(x) = e^{ax^2} = \int_{-\infty}^{+\infty} k\left(\frac{x - y}{2}\right) (1 + 4a)^{-\frac{1}{2}} e^{(a/1+4a)y^2} dy$$

for $-\frac{1}{4} < a < \infty$. The series (2.1) is now

$$e^{ax^2} \sum_{n=0}^\infty \frac{a^n}{n!} H_{2n}(\sqrt{-a} x)$$

which for $x = 0$ is

$$\sum_{n=0}^\infty (-1)^n \frac{(2n)!}{(n!)^2} a^n.$$

This is $(1 + 4a)^{-\frac{1}{2}}$ for $|4a| < 1$. It can be shown (4, p. 107) that this series is summable (S) to $(1 + 4a)^{-\frac{1}{2}}$ for $(4a) < \exp((1/4a) + 1)$ and diverges otherwise. That is, it diverges for $a > .897 \dots$

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