CHARACTERISATIONS OF GENERALISED UNISERIAL ALGEBRAS. III

by DRURY W. WALL

(Received 19th October 1964)

1. Introduction

Let A be a finite dimensional algebra with identity element over a field. A is *generalised uniserial* if every primitive left ideal and every primitive right ideal of A has only one compositions series. In the previous papers in this series (**6, 7**) generalised uniserial algebras have been characterised as algebras all of whose residue class algebras are of certain types. The purpose of this paper is to extend the earlier results by showing that in order that A be generalised uniserial it is sufficient to require weaker conditions on merely a finite sequence of residue class algebras of A.

A primitive ideal Ae of A is called dominant if it is dual to some primitive ideal fA. In a quasi-Frobenius algebra every primitive ideal is dominant and an algebra is uniserial if all of its residue class algebras are quasi-Frobenius (**6**). There are a number of types of algebras, more general than the quasi-Frobenius, in which dominant ideals exist and in which every primitive ideal is related in some way to the dominant ideals. (For details see (**5**) and (**8**).) It is known (**7**) that an algebra A is generalised uniserial if and only if every residue class algebra of A has a unique minimal faithful representation or, equivalently, is a QF-3, algebra (one in which every primitive ideal is weakly subordinate to a set of dominant ideals).

§ 2 contains the definitions and notations for the paper while in § 3 is constructed a sequence of residue class algebras A_i of A by using the socles of the dominant parts of the algebras. An algebra A is called *dominant* if there exist dominant ideals in A. § 4 contains the main result of the paper, namely, that A is generalised uniserial if every A_i is a dominant algebra. In § 5 is given an example to show that in order for A to be generalised uniserial it is not sufficient that every A/N^i be dominant, where $N = \text{Rad } A$.

2. Definitions and notations

Let A be a finite dimensional algebra with identity element 1 over a field. Let

$$1 = \sum_{i=1}^{n} \sum_{j=1}^{f_i} e_{ij}, \tag{1}$$

be a decomposition of the identity element into the sum of mutually orthogonal primitive idempotents such that $e_{ij} \cong e_{hk}$ if and only if $i = h$. If e and f are idempotents then $e \cong f$ if and only if $Ae \cong Af$ (or equivalently $eA \cong fA$). For $i = 1, \ldots, n$ denote e_{i1} by e_i and let $E_i = \sum_{j=1}^{f_i} e_{ij}$. If e is any primitive idempotent of A then the left ideal Ae is called *dominant* if there exists a primitive idempotent f such that Ae is dual to fA. If A has at least one dominant left ideal (and hence at least one dominant right ideal) then A is called *dominant*.

As in a previous paper (**9**), for a given algebra A and a fixed decomposition (1) let $\Sigma = \{i \mid Ae_i \text{ is dominant}\}$ and let $\Pi = \{i \mid e_iA \text{ is dominant}\}$. For these choices of A and (1) let

$$D(A) = \sum_{i \in \Sigma} \sum_{j=1}^{f_i} Ae_{ij} \tag{2}$$

and

$$D'(A) = \sum_{i \in \Pi} \sum_{j=1}^{f_i} e_{ij}A. \tag{3}$$

$D(A)$ is called the *left dominant part* and $D'(A)$ the *right dominant part* of A with respect to the decomposition (1).

Every primitive left ideal Ae is isomorphic to one and only one of the Ae_i and every primitive right ideal is isomorphic to one and only one of the e_iA. Thus, if A has a dominant left ideal Ae then there is some i such that Ae_i is dominant and hence $i \in \Sigma$. Hence, in the present notation, each of the following is equivalent to A being quasi-Frobenius: $\Sigma = \{1, \ldots, n\}$, $\Pi = \{1, \ldots, n\}$, $D(A) = A$, or $D'(A) = A$. Also, A is dominant if and only if one of the following hold: Σ is not empty, Π is not empty, $D(A) \neq 0$, or $D'(A) \neq 0$. (We adopt the convention that $D(A) = 0$ if Σ is the empty set.)

In the case when A is dominant but not quasi-Frobenius the sets Σ and Π need not be equal but they must have the same order. For each $i \in \Sigma$ let $\rho(i)$ be the unique integer such that Ae_i is dual to $e_{\rho(i)}A$. Then $\rho\colon \Sigma \to \Pi$ is a bijection.

For any left A-module M let $S(M)$ denote the A-socle of M, i.e., the sum of all simple A-submodules of M. Similarly let $S(M')$ denote the A-socle of a right A-module M'. It is known (**5**) that if $i \in \Sigma$ then Ae_i (and hence each $Ae_{ij}, j = 1, \ldots, f_i$) has only one minimal subideal and therefore $S(Ae_i)$ is simple as a left ideal.

From the properties of socles (see (**2**, p. 63), (**8**, § 5)) and equations (2) and (3) it follows that

$$S(D(A)) = \sum_{i \in \Sigma} \sum_{j=1}^{f_i} S(Ae_{ij}) \tag{4}$$

and

$$S(D'(A)) = \sum_{i \in \Pi} \sum_{j=1}^{f_i} S(e_{ij}A) \tag{5}$$

and both sums are direct, (4) as a sum of left A-modules and (5) as a sum of right A-modules.

By applying earlier results, principally Lemmas 4 and 5 of (6), we obtain the following: (a) for each $i \in \Sigma$, $S(Ae_i)A$ is a two-sided ideal and

$$S(Ae_i)A = \sum_{j=1}^{f_i} S(Ae_{ij}) = U\{S(Af) | \; Af \; A \cong e_i\} \tag{6}$$

and

$$S(Ae_i)A = AS(e_{\rho(i)}A) \tag{7}$$

(b) $S(D(A))$ is a two-sided ideal of A and

$$S(D(A)) = S(D'(A)). \tag{8}$$

To prove (8), we use (4), (6), (7), the right dual of (6) and (5) as follows:

$$S(D(A)) = \sum_{i \in \Sigma} \sum_{j=1}^{f_i} S(Ae_{ij}) = \sum_{i \in \Sigma} S(Ae_i)A$$

$$= \sum_{i \in \Sigma} AS(e_{\rho(i)}A) = \sum_{i \in \Pi} \sum_{j=1}^{f_i} S(e_{ij}A) = S(D'(A)).$$

Thus, even though $D(A)$ and $D'(A)$ need not be equal their socles are. Also (6) shows that $S(D(A))$ is independent of the choice of the decomposition (1).

3. The sequence $\{A_i\}$

Definition 1. For any A let $A^* = A/S(D(A))$, i.e., A^* is the residue class algebra of A with respect to the two-sided ideal $S(D(A))$, which is the socle of the dominant part of A.

Definition 2. Let $\{A_i\}$ be the sequence defined inductively:

$$A_0 = A; \; A_i = A^*_{i-1} = A_{i-1}/S(D(A_{i-1})), \; i \geqq 1. \tag{9}$$

Definition 3. For each $i \geqq 1$ let h_i: $A_{i-1} \to A_i$ be the natural epimorphism with $\ker h_i = S(D(A_{i-1}))$. Let g_i: $A \to A_i$ be defined inductively:

$$g_1 = h_1; \; g_i = h_i g_{i-1}, \; i \geqq 2. \tag{10}$$

Definition 4. Let the sequence $\{Z_i\}$ of two-sided ideals of A be as follows:

$$Z_0 = \{0\}; \; Z_i = \ker g_i, \; i \geqq 1. \tag{11}$$

Lemma 1. For each $i \geqq 1$,

$$A/Z_i \cong A_i, \; Z_i/Z_{i-1} \cong S(D(A_{i-1})), \tag{12}$$

and Z_i/Z_{i-1} is semisimple both as a left A-module and as a right A-module.

Proof. Omitted.

The sequence $\{Z_i\}$ is an ascending sequence of two-sided ideals of A and therefore there exists a positive integer m such that for all $i \geqq m$,

$$Z_i = Z_m; \; A_i = A_m, \; S(D(A_i)) = S(D(A_m) = \{0\}. \tag{13}$$

There are two possibilities that can occur, either $A_m = \{0\}$ or $A_m > \{0\}$.

Lemma 2. If $A_m > \{0\}$ then A has a residue class algebra, namely A/Z_m, which is not dominant.

https://doi.org/10.1017/S001309150001316X Published online by Cambridge University Press

Proof. If $A_m > \{0\}$ then since $S(D(A_m)) = \{0\}$, A_m is a non-zero algebra with no dominant left or right ideals and hence A_m is not a dominant algebra. But, by (12), $A/Z_m \cong A_m$ and thus, Z_m is a proper two-sided ideal of A such that A/Z_m is not dominant.

4. Characterisation theorem

Theorem. If A is not generalised uniserial then A_m is not dominant.

Proof. If A is not generalised uniserial there is a primitive idempotent e such that either Ae or eA has more than one composition series. Let us consider only the case where Ae has more than one composition series. Since, as noted in § 2, there is some integer i such that $Ae \cong Ae_i$, we shall assume that e is chosen to be this e_i.

Let $L_0 = \{0\}$. Then, L_0 is the trivial initial term in any composition series for Ae. If Ae has only one minimal (non-zero) subideal let it be L_1. In this case, every composition series for Ae would contain L_0 and L_1 as the first and second terms. If for any integer h, $h \geqq 1$, there are subideals $L_0, L_1, \ldots, L_{h-1}$ of Ae which appear as the first h terms in every composition series of Ae and if there is only one subideal L of Ae such that $L_{h-1} \subset L$ and L/L_{h-1} is a simple left A-module then let $L_n = L$. Since, Ae is assumed to have more than one composition series there is some smallest h such that L_h is not defined.

For this h let $M = L_{h-1}$. Thus, there exist subideals M' and M^* of Ae such that $M' \neq M^*$, $M \subset M'$, $M \subset M^*$ and both M/M and M^*/M are simple left A-modules.

(a) Assume that there is some integer t such that $M \subset Z_t$ and consider the smallest such t. Let $t = k-1$. Then $M \subset Z_{k-1}$ and $M \not\subset Z_{k-2}$. Thus, from (11) and (12), it follows that

$$g_{k-1}(M) = 0,\ g_{k-2}(M) \neq 0,\ g_{k-2}(M) \subset g_{k-2}(Z_{k-1}). \tag{14}$$

But $g_{k-2}(Z_{k-1}) = Z_{k-1}/Z_{k-2}$ and is a semi-simple left A-module and therefore, by (14) $g_{k-2}(M)$ is also a semi-simple left A-module.

Since e was chosen as one of the e_i of the decomposition (1) and since the constructions of § 3 are independent of (1), the results of Lemma 1 and equation (4) give

$$Ae \cap Z_{k-1} = M \tag{15}$$

and $Ae/M = Ae/(Ae \cap Z_{k-1}) = g_{k-1}(Ae) \cong A_{k-1}e'$ where e' is a primitive idempotent of A_{k-1}.

But, considered as a primitive left ideal of A_{k-1}, Ae/M has more than one minimal subideal, i.e., both M'/M and M^*/M are minimal subideals of Ae/M and hence neither M'/M nor M^*/M can be contained in $S(D(A_{k-1}))$. Thus,

$$Ae/M \cap S(D(A_{k-1})) = \{0\}. \tag{16}$$

Therefore, by Lemma 1 and (15), $[Ae/(Z_{k-1} \cap Ae)] \cap [Z_k/Z_{k-1}] = \{0\}$ and hence $[Ae/(Z_{k-1} \cap Ae)] \cap [(Ae \cap Z_k)/(Ae \cap Z_{k-1})] = \{0\}$. Thus,

$$(Ae \cap Z_k)/(Ae \cap Z_{k-1}) = \{0\},$$

and

$$Ae \cap Z_k = Ae \cap Z_{k-1}. \tag{17}$$

Therefore $Ae/M \cong g_{k-1}(Ae) \cong g_k(Ae)$. Inductively, it follows that for any $r>k$

$$g_r(Ae) \cong g_k(Ae) \cong Ae/M. \tag{18}$$

Thus, in each A_r, for $r>k$, there is a left ideal isomorphic to Ae/M and hence $A_r \neq \{0\}$. In particular, the A_m of (13) is not zero and hence A_m is not dominant.

(b) Assume that there exists no k such that $M \subset Z_k$. Then $Z_m \neq A$ and $A/Z_m \neq \{0\}$ and thus $A_m \neq \{0\}$ and A_m is not dominant.

Corollary. A is generalised uniserial if and only if for all n, A/Z_n is a dominant algebra.

The corollary generalises the earlier result [7] that A is generalised uniserial if and only if every residue class algebra of A is QF-3($UMFR$). An algebra is QF-3 if every primitive ideal is weakly subordinate to a set of dominant ideals or, equivalently if it has a unique minimal faithful representation ($UMFR$). Thus every QF-3 algebra is a dominant algebra. However the converse is false.

5. Example

Let A be the set of all matrices of the form in (19)

$$\begin{bmatrix} \alpha_1 & 0 & 0 & 0 & 0 \\ \alpha_5 & \alpha_2 & 0 & 0 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 \\ 0 & 0 & \alpha_6 & \alpha_4 & 0 \\ 0 & 0 & \alpha_7 & 0 & \alpha_4 \end{bmatrix} \tag{19}$$

where the α_i are taken from a field F. Then the set of matrices of the form (19) with all but the first column zero is a dominant left ideal of A and hence, A is a dominant algebra. On the other hand, the set of matrices of the form (19) with all but the third column zero form a primitive left ideal of A which is neither dominant nor weakly subordinate to any set of dominant ideals of A. Therefore A is not a QF-3 algebra.

In studying an algebra it is often useful to study the residue class algebra A/N^i where N is the radical of A. Since imposing conditions on all of the A/N^i often imposes very strong conditions on A itself, it might be conjectured that requiring that each A/N^i be dominant might be sufficient to imply that A is generalised uniserial. However, this is false, as is shown by the example (19). The radical of A in this case, is the set of all matrices of the form in (20)

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \alpha_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_6 & 0 & 0 \\ 0 & 0 & \alpha_7 & 0 & 0 \end{bmatrix} \tag{20}$$

and $N^2 = 0$. But A/N and A/N^2 are both dominant but A is not generalised uniserial; the primitive ideal composed of the matrices with all columns zero but the third has more than one composition series.

Thus, in particular, the result (3, Lemma) that a QF-3 algebra A with $N^2 = 0$ is generalised uniserial cannot be generalised by replacing the QF-3 condition by one assuming that A is dominant.

REFERENCES

(1) Emil Artin, C. J. Nesbitt and R. M. Thrall, Rings with minimum condition, *Univ. of Michigan Publications in Math.*, No. 1, 1944.

(2) Nathan Jacobson, *Structure of Rings* (Amer. Math. Soc. Colloquium Publication No. 37, New York, 1956).

(3) Yutaka Kawada, A generalisation of Morita's theorem concerning generalised uniserial algebras, *Proc. Japan Acad.* **34** (1958), 404-406.

(4) Kiiti Morita, Duality for modules, *Sci. Rep. Tokyo Kyoiku Daigaku*, **6**, no. 150 (1958), 83-142.

(5) R. M. Thrall, Some generalisations of quasi-Frobenius algebras, *Trans. Amer. Math. Soc.*, **64** (1948), 173-183.

(6) Drury W. Wall, Characterisations of generalised uniserial algebras. I, *Trans. Amer. Math. Soc.*, **90** (1959), 161-170.

(7) Drury W. Wall, Characterisations of generalised uniserial algebras. II, *Proc. Amer. Math. Soc.*, **9** (1958), 915-919.

(8) Drury W. Wall, Algebras with unique minimal faithful representations, *Duke Math., J.* **25** (1958), 321-330.

(9) Drury W. Wall, Cartan invariants of algebras with unique minimal faithful representations, *Illinois J. Math.*, **4** (1960), 133-142.

The University of Iowa
Iowa City, Iowa