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Fixed point theorems for condensing multivalued mappings on a locally convex topological space

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A general definition for a measure of nonprecompactness for bounded subsets of a locally convex linear topological space is given. Fixed point theorems for condensing multivalued mappings have been proved. These fixed point theorems are further generalizations of Kakutani's fixed point theorems.

1. Introduction

Using the concept of condensing mapping Sadovskiĭ [11] and Lifšic and Sadovskiĭ [9] have obtained respectively the generalizations of Schauder [12] and Tychonoff [14] fixed point theorems. Daneš [2] has obtained the generalization of Kakutani's fixed point theorem [7] by using the concept of multivalued condensing mapping. Reinermann [10] has also used condensing mapping defined in terms of a measure of noncompactness (nonprecompactness) of bounded sets to obtain generalizations of Schauder Theorem [12]. Using the multivalued condensing mapping defined in terms of a measure of precompactness Himmelberg, Porter and van Vleck [6] have proved a fixed point theorem which includes the fixed point theorems of Sadovskiĭ [11], Tychonoff [14], Glicksberg [5], Fan [4] and a part of a theorem of Browder [1].

The aim of this note is to obtain a fixed point theorem which will contain the above fixed point theorems of [2, 4, 5, 6, 7, 9, 10, 11, 12, 14].

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In §2 we have introduced a general definition of a measure of nonprecompactness of bounded sets in a locally convex linear topological space. In §3 we have given various definitions of condensing multivalued mappings and have unified them in a single definition. In §4 we have proved our main fixed point theorem and also obtained corollaries and a theorem which are similar but more general than the corresponding corollaries and theorem of [6].

We follow the notation and terminology as in [6]. A multivalued mapping $F : X \to Y$ is a mapping which assigns to each point $x \in X$ a nonempty set $F(x)$ of Y. F is a subset of $X \times Y$ whose domain is $F^{-1}(Y) = X$. The set $\{(x, y) : x \in X \text{ and } y \in F(x)\}$ is called the graph of F or simply F. For $\Omega \subset X$, a multivalued mapping $G : \Omega \to Y$ having the property that $G(x) \subset F(x)$ for each $x \in \Omega$ is called a submultivalued mapping of F. For $\Omega \subset X$, $F(\Omega) = \bigcup_{x\in\Omega} F(x)$.

A point $x \in X$ is a fixed point of a multivalued mapping $F : X \to X$ if $x \in F(x)$. It is obvious that a fixed point of a submultivalued mapping of F is also a fixed point of F.

A multivalued mapping $F : X \to Y$ of a topological space X into a topological space Y is called upper semicontinuous if for each closed subset A of Y, $F^{-1}(A)$ is closed. F has closed graph if Y is regular and F is upper semicontinuous and has closed values (see [8], p. 175). A multivalued mapping $F : X \to Y$ is lower semicontinuous if for each open subset A of Y, $F^{-1}(A)$ is open.

In the sequel (E, τ) will always denote a locally convex linear topological space, and $[p_\alpha : \alpha \in I]$ will denote the family of seminorms which generates the topology τ. Any topological concept, such as closedness, precompactness, compactness, boundedness, and so on, will be understood as 'with respect to the topology τ'. In all other cases, that is, when a topological concept is not meant with respect to τ, the corresponding topology will precede the concept; for example, p_α-precompact to mean that certain subset is precompact with respect to p_α-topology.

2. Measure of precompactness and nonprecompactness

We denote by $\mathcal{C}$ the class of all bounded subsets of (E, τ) .

DEFINITION 2.1. $\mu = [\mu_\alpha : \alpha \in I]$ will be said to define a measure of precompactness on $\mathcal{C}$, where for each $\alpha \in I$, μ_α is a set (interval) valued mapping of $\mathcal{C}$ into R^+ , the set of non-negative real numbers, having properties:

(i) $\mu_\alpha(\Omega) = [a, \infty)$ or (a, ∞) , $a \geq 0$ for each $\Omega \in \mathcal{C}$;

(ii) $\Omega_1 \subset \Omega_2 \in \mathcal{C}$ implies $\mu_\alpha(\Omega_1) \supset \mu_\alpha(\Omega_2)$ for every $\alpha \in I$;

(iii) $\mu_\alpha(\Omega) = \mu_\alpha(\text{co}\Omega)$ for each $\Omega \in \mathcal{C}$ where $\text{co}\Omega$ stands for the convex hull of Ω ;

(iv) $\mu_\alpha(\Omega_1 \cup \Omega_2) = \mu_\alpha(\Omega_1) \cap \mu_\alpha(\Omega_2)$ for $\Omega_1, \Omega_2 \in \mathcal{C}$;

(v) $\mu_\alpha(\Omega) = R^+$ if Ω is precompact and Ω is precompact if $\mu_\alpha(\Omega) \supset (0, \infty)$, for each $\alpha \in I$.

For $\Omega \in \mathcal{C}$, $\hat{\mu}(\Omega) = [\hat{\mu}_\alpha(\Omega) : \alpha \in I]$ where $\hat{\mu}_\alpha(\Omega) = \inf \mu_\alpha(\Omega)$ may then be regarded as a measure of nonprecompactness of Ω . Thus all the entries in the parenthesis of $\hat{\mu}(\Omega)$ are zeros if and only if Ω is precompact.

EXAMPLE 2.1 (Kuratowski). For each $\Omega \in \mathcal{C}$, we define $\lambda(\Omega) = [\lambda_\alpha(\Omega) : \alpha \in I]$ where

$$\lambda_\alpha(\Omega) = \{\varepsilon > 0 : \Omega \text{ can be covered by a finite number of sets of } p_\alpha\text{-diameter} \leq \varepsilon\} .$$

Then λ is indeed a measure of precompactness on $\mathcal{C}$.

(i) $\lambda_\alpha(\Omega) = [\hat{\lambda}_\alpha(\Omega), \infty)$ or $(\hat{\lambda}_\alpha(\Omega), \infty)$. (ii), (iv), and (v) follow easily. For proof of (iii) we refer to Darbo [3] (the proof given by Darbo for a normed space applies also for a seminormed space).

EXAMPLE 2.2. Let $U_\alpha = \{x \in E : p_\alpha(x) \leq 1\}$.

For each $\Omega \in \mathcal{C}$, we define $\gamma(\Omega) = [\gamma_\alpha(\Omega) : \alpha \in I]$ where

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$\gamma_\alpha(\Omega) = \{\varepsilon > 0 :$ there exists a p_α-precompact subset S with

$$S + \varepsilon U_\alpha \supset \Omega\} .$$

(i) As before we take $a = \hat{\gamma}_\alpha(\Omega)$. The proof of (ii) and (iv) is trivial.

(iii) In view of (ii) it suffices to show that $\gamma_\alpha(\Omega) \subset \gamma_\alpha(\mathrm{co}\Omega)$. Let $t \in \gamma_\alpha(\Omega)$. Then there exists a p_α-precompact subset S such that $S + tU_\alpha \supset \Omega$. Since $\mathrm{co}S + tU_\alpha \supset \Omega$ and $\mathrm{co}S + tU_\alpha$ is convex, $\mathrm{co}S + tU_\alpha \supset \mathrm{co}\Omega$. Noting that $\mathrm{co}S$ is p_α-precompact, we conclude that $t \in \gamma_\alpha(\mathrm{co}\Omega)$.

(v) Let Ω be τ-precompact. Then Ω is p_α-precompact for each $\alpha \in I$. Since $\Omega + tU_\alpha \supset \Omega$ for all $t \geq 0$ and $\alpha \in I$, $\mu_\alpha(\Omega) = R^+$ for all $\alpha \in I$.

Next, let $\alpha \in I$ be arbitrary and $\mu_\alpha(\Omega) \supset (0, \infty)$.

Let $r > 0$ be any real number. Since $\frac{r}{2} \in \mu_\alpha(\Omega)$, there exists a p_α-precompact set S such that $S + \frac{r}{2} U_\alpha \supset \Omega$. Since S is p_α-compact, there exists a finite set F such that $F + \frac{r}{2} U_\alpha \supset S$. Now $F + rU_\alpha \supset S + \frac{r}{2} U_\alpha \supset \Omega$. Thus Ω is p_α-precompact. Since α is arbitrary, Ω is τ-precompact.

EXAMPLE 2.3. Let (E, τ) , $[p_\alpha : \alpha \in I]$ and $\mathbf{C}$ be as before. For $\Omega \in \mathbf{C}$, we define $\nu(\Omega) = [\nu_\alpha(\Omega) : \alpha \in \Omega]$ where

$\nu_\alpha(\Omega) = \{\varepsilon > 0 :$ there exists a precompact set S such that

$$S + \varepsilon U_\alpha \supset \Omega\} .$$

The proof that ν is a measure of nonprecompactness on $\mathbf{C}$ is similar to that of Example 2.2. We note that for each $\Omega \in \mathbf{C}$, $\nu_\alpha(\Omega) \subset \gamma_\alpha(\Omega)$ for each $\alpha \in I$.

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3. Condensing mappings

Himmelberg, Porter and van Vleck [6] have defined a measure of precompactness for any subset of (E, τ) in the following way.

Let $\mathcal{B}$ be a base of convex neighbourhoods of 0. Then for $\Omega \subset E$, $Q(\Omega)$, the measure of precompactness of Ω, is defined to be the collection of all $B \in \mathcal{B}$ such that $S + B \supset \Omega$ for some precompact subset S of E. With this notion of measure of precompactness they have introducted a definition of condensing mapping.

Let X be a nonempty subset of a locally convex linear topological space (E, τ). Let $[p_\alpha : \alpha \in I]$ and $\mathcal{C}$ be as before. Let $F : X \to X$ be a multivalued mapping.

DEFINITION 3.1. F is condensing with respect to Q if for each τ-bounded but not τ-precompact set $\Omega \subset X$ with $F(\Omega) \subset \Omega$ we have $Q(F(\Omega)) \supsetneq Q(\Omega)$.

DEFINITION 3.2. F is condensing with respect to μ if for each bounded but not precompact set $\Omega \subset X$ with $F(\Omega) \subset \Omega$, there exists a $\alpha \in I$ such that $\hat{\mu}_\alpha(F(\Omega)) < \hat{\mu}_\alpha(\Omega)$ where $\mu = [\mu_\alpha : \alpha \in I]$ is a measure of precompactness on $\mathcal{C}$.

DEFINITION 3.3. F is condensing with respect to μ if for each bounded but not precompact set $\Omega \subset X$ with $F(\Omega) \subset \Omega$, there exists $\alpha \in I$ such that $\mu_\alpha(F(\Omega)) \not\supseteq \mu_\alpha(\Omega)$.

DEFINITION 3.4. F is condensing if for each $\Omega \subset X$ with $F(\Omega) \subset \Omega$,

(a) the condition that $\Omega - \mathrm{clco}F(\Omega)$ is compact implies the compactness of $\mathrm{cl}\Omega$; or

(b) the condition that $\Omega - \mathrm{co}F(\Omega)$ is empty or single point implies the compactness of $\mathrm{cl}\Omega$.

DEFINITION 3.5. F is condensing if for each $\Omega \subset X$ with $F(\Omega) \subset \Omega$, the condition that $\Omega - \mathrm{co}F(\Omega)$ is empty or single point implies that Ω is precompact.

Definition 3.1 is due to Himmelberg, Porter and van Vleck [6]. For a single valued mapping, Definition 3.2 has been used by Reinermann [10] and Stallbohm [13] with $\mu = \lambda$, and Definition 3.4 is due to Lifšic and

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Sadovskiĭ [9]. Definition 3.5 is a slight variant of the one given by Daneš [2].

(A) It is easy to see that Definition 3.2 implies Definition 3.3 for each measure μ.

(B) Definition 3.1 implies Definition 3.3 for suitable measure μ. Let Definition 3.1 hold. We index the base $\mathcal{B}$ by $\mathcal{B} = \{B_\alpha : \alpha \in I\}$. Let p_α be the Minkowski functional on B_α. Let $U_\alpha = \{x \in E : p_\alpha(x) \le 1\}$. Clearly $B_\alpha = U_\alpha$. We now consider the measure ν as defined in Example 2.3. We now show that Definition 3.3 holds with respect to ν. Let Ω be any bounded but not precompact subset of X with $F(\Omega) \subset \Omega$. Then we have $Q(F(\Omega)) \supsetneqq Q(\Omega)$; that is, there exists a $B_\alpha \in \mathcal{B}$ such that $B_\alpha \in Q(F(\Omega))$ but $B_\alpha \notin Q(\Omega)$. Hence it follows that $1 \in \nu_\alpha(F(\Omega))$ but $1 \notin \nu_\alpha(\Omega)$. Also since $F(\Omega) \subset \Omega$, it follows from (ii) of Definition 2.1 that $\nu_\alpha(F(\Omega)) \supsetneqq \nu_\alpha(\Omega)$.

(C) Definition 3.3 with each measure μ implies Definition 3.5 if F has bounded range. Let Definition 3.3 hold with a measure μ. Let $\Omega \subset X$, $F(\Omega) \subset \Omega$, and $\Omega - \text{co}F(\Omega) = Z$ where $Z = \emptyset$ or a single point. Obviously $\mu_\alpha(Z) = R^+$ for each $\alpha \in I$.

Since $\Omega \subset Z \cup \text{co}F(\Omega)$, it follows that Ω is bounded and we have for each $\alpha \in I$, $\mu_\alpha(\Omega) \supset \mu_\alpha(Z \cup \text{co}F(\Omega))$ by (ii) of Definition 2.1 equal to $\mu_\alpha(Z) \cap \mu_\alpha(F(\Omega))$ by (iv) and (iii) of Definition 2.1. Again since $\Omega \supset Z \cup F(\Omega)$, we have for each $\alpha \in I$, $\mu_\alpha(\Omega) \subset \mu_\alpha(Z) \cap \mu_\alpha(F(\Omega))$ by (ii) and (iv) of Definition 2.1. Thus for each $\alpha \in I$, $\mu_\alpha(\Omega) = \mu_\alpha(Z) \cap \mu_\alpha(F(\Omega))$. From this and the fact that $\mu_\alpha(Z) = R^+$ for each $\alpha \in I$, it follows that $\mu_\alpha(\Omega) = \mu_\alpha(F(\Omega))$ for each $\alpha \in I$, which in view of Definition 3.3 implies that Ω is precompact.

(D) Obviously Definition 3.4 implies Definition 3.5.

4. Fixed point theorems

The proof of the following lemma can be found in [6].

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LEMMA 4.1. *Let X be a topological space. Let $F : X \to X$ be a multivalued mapping with closed graph. If there exists a nonempty subset A of X such that $F(A) \subset A$ and $\mathrm{cl}A$ is compact, then there exists a nonempty, closed and compact subset K of X such that $K \subset F(K)$.*

THEOREM 4.1. *Let X be a nonempty complete convex subset of a Hausdorff locally convex linear topological space E. Let $F : X \to X$ be a condensing multivalued mapping in the sense of Definition* 3.5 *with convex values and closed graph. Then F has a fixed point.*

Proof. Unlike [11], [9], and [6], we will not use ordinals. Let $x \in X$. Set $A = \{x\} \cup \left\{\bigcup_{n=1}^{\infty} F^n(x)\right\}$. Then clearly $F(A) \subset A$ and $A - \mathrm{co}F(A) \subset \{x\}$. Since F is condensing, A is precompact. Also $\mathrm{cl}A \subset X$ and $\mathrm{cl}A$ is compact as X is complete. Hence by Lemma 4.1, there exists a nonempty compact subset K of X such that $F(K) \supset K$.

Let $S = \{Y \subset X : K \subset Y,\ F(Y) \subset Y \text{ and } Y \text{ is convex}\}$. S is nonempty as $X \in S$. S is a partially ordered set with respect to the relation $\le$ where $Y_1 \le Y_2$ if and only if $Y_1 \supset Y_2$ with $Y_1, Y_2 \in S$.

We first prove that every chain in S has an upper bound in S. Let T be a chain in S. Then $Z = \bigcap_{Y \in T} Y$ is an upper bound. Clearly $Z \subset X$, $K \subset Z$, $F(Z) \subset Z$, and Z is convex. Hence $Z \in S$. Thus by Zorn's Lemma there is a maximal element $Z_0 \in S$.

We next prove that for each $Y \in S$, $\mathrm{co}F(Y) \in S$.

(a) $\mathrm{co}F(Y) \subset X$ as $F(Y) \subset Y \subset X$ and X is convex.

(b) $K \subset \mathrm{co}F(Y)$.

Since $K \subset Y$ and $K \subset F(K)$, we have $K \subset F(K) \subset F(Y)$. Hence $K \subset \mathrm{co}F(Y)$.

(c) $F(\mathrm{co}F(Y)) \subset \mathrm{co}F(Y)$.

Since $F(Y) \subset Y$ and Y is convex, $\mathrm{co}F(Y) \subset Y$. Hence $F(\mathrm{co}F(Y)) \subset F(Y) \subset \mathrm{co}F(Y)$.

(d) $\mathrm{co}F(Y)$ is convex.

Now since for each $Y \in S$, $F(Y) \subset Y$ and Y is convex, we have

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$\mathrm{co}F(Y) \subset Y$. Thus $Y \leq \mathrm{co}F(Y)$ for each $Y \in S$. In particular $Z_0 \leq \mathrm{co}F(Z_0)$. But since Z_0 is a maximal element in S , it follows that $Z_0 = \mathrm{co}F(Z_0)$; that is, $Z_0 - \mathrm{co}F(Z_0) = \emptyset$. Hence by condensing of F , Z_0 is precompact. Therefore, $\mathrm{cl}Z_0 \subset X$ and $\mathrm{cl}Z_0$ is compact. The rest of the argument is as given in [6]. Let $G = F \cap (\mathrm{cl}Z_0 \times \mathrm{cl}Z_0)$. Then G is closed and compact subset of $X \times X$. Also $G^{-1}(\mathrm{cl}Z_0)$ is a closed subset of $\mathrm{cl}Z_0$ containing Z_0 . Thus domain $G = G^{-1}(\mathrm{cl}Z_0) = \mathrm{cl}Z_0$. Hence G is a multivalued mapping of $\mathrm{cl}Z_0$ into $\mathrm{cl}Z_0$, with convex values and compact graph. (G is also upper semicontinuous.) Hence by the theorem of Glicksberg [5] or of Fan [4], G has a fixed point in $\mathrm{cl}Z_0$. This fixed point is also a fixed point of F .

REMARKS 4.1. The same remark as given in ([6], p. 637) applies in the present situation; that is, the above theorem remains true for non Hausdorff (nonseparated) E if the further assumption that X is closed is assumed. For details see [6] as quoted above.

REMARK 4.2. If F is assumed to be condensing with respect to Definition 3.4, then the above theorem remains true with the completeness condition on X replaced by the condition that X is closed. The same proof applies, because in this case $\mathrm{cl}A$ and $\mathrm{cl}Z_0$ appeared in the proof would be compact directly due to the condensing of F . By Remark 4.1 we can then remove the Hausdorff condition on E as the condition that X is closed is already assumed. The resulting version of the theorem will include fixed point theorems of Lifšic and Sadovskiĭ [9].

COROLLARY 4.1. *Let X be a nonempty complete convex subset of a Hausdorff locally convex linear topological space E . Let $F : X \to X$ be a multivalued mapping with convex values, closed graph and bounded range. If F is condensing in the sense of Definition* 3.3*, then F has a fixed point.*

Proof. This follows from Theorem 4.1 and (C) of §3.

REMARK 4.3. In view of (B), §3, it follows that the fixed point theorem of Himmelberg, Porter and van Vleck ([6], Theorem 1) is a special

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case of our Corollary 4.1.

The following theorem includes the corresponding theorem of ([6], Theorem 3).

THEOREM 4.2. *Let* X *be a nonempty complete convex subset of a locally convex linear topological space* E. *Let* $F : X \to X$ *be a lower semicontinuous multivalued mapping with closed convex values. Then* F *has a fixed point if either of the following conditions hold:*

(a) X *is compact and metrizable;*

(b) the subspace uniformity on X *is metrizable and* F *is condensing in the sense of Definition* 3.5.

Proof. *(a)* Same proof as in [6] applies.

(b) We proceed as in the proof of Theorem 4.1 until the set Z_0 with $\mathrm{co}F(Z_0) = Z_0$ is obtained. By Corollary 2a, p. 176 of [8], $F(\mathrm{cl}Z_0) \subset \mathrm{cl}F(Z_0) \subset \mathrm{cl}Z_0$. We then apply case *(a)* to $F \cap (\mathrm{cl}Z_0 \times \mathrm{cl}Z_0) : \mathrm{cl}Z_0 \to \mathrm{cl}Z_0$. For details see [6].

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