A Problem of Robert Simson's.

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The following problem appears in Robert Simson's "Opera Quaedam Reliqua," pp. 472–504:

"Si a duobus punctis datis A, B ad circulum positione datum CDE inflectantur utcumque duae rectae AC, BC circumferentiae rursus in D, E occurrentes; juncta DE vel continebit datum angulum cum recta ad datum punctum vergente; vel parallela erit rectae positionae datae; vel verget ad datum punctum:" *i.e.* if from two given points A and B any two straight lines AC, BC are drawn to a circle CDE given in position, and they meet the circumference again in D and E, then the straight line DE (I.) will make a constant angle with a straight line passing through a fixed point, or (II.) will be parallel to a straight line given in position, or (III.) will pass through a given point. This final form of the result was only arrived at by Simson after he obtained the aid of Matthew Stewart.

As indicated, the discussion of the problem by Simson occupies over thirty pages quarto, but the proof may be given as follows:

Case I.(a). When AB *does not pass through the centre of the given circle.* This is the most general case. Find K (Fig. 1) on AB, such that AB . AK = AC . AD, then K is a definite point. Join S, the centre of the circle, to K, and find F the harmonic conjugate of K with respect to the circle; then SF . SK = ST^2. Let AF meet the circle in L and M, then KL and KM cut off equal arcs MP and QL.

If BL be joined and cut the circle again in N, NP will be parallel to AB. (For AK . AB = AM . AL, and therefore LMKB is a cyclic quadrilateral. Hence $\angle$ AKM = $\angle$ MLB = $\angle$ MPN). Similarly if CB cuts the circle in E, EH is parallel to AB. Hence PN $\|$ HE, and therefore arc PH = arc EN. Again, if

DF meet the circle in G, KG and KD cut off equal arcs, *i.e.* arc HD = arc GR.

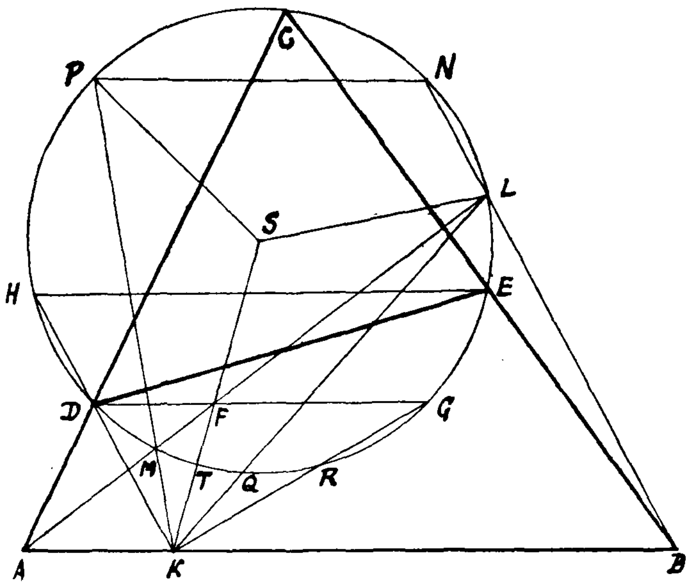

Fig. 1.

Now KP = KL (Pappus' Lemma), and therefore $\angle$ PSK = $\angle$ LSK, and arc TP = arc TL. Similarly, since arc HD = arc GR, KH = KG and arc TH = arc TG. Hence arc GL = arc PH = arc EN, and arc GE = arc NL. Thus

$$\angle \text{EDF or } \angle \text{EDG} = \angle \text{NML} = \text{constant}.$$

Case I.(b). When AB *passes through the centre.* Find K (Fig. 2) in AB such that AB . BK = CB . BE. Then since $\angle$ ABC = $\angle$ KBE, and the triangles ABC, KBE are thus similar, $\angle$ ACE = $\angle$ BKE. Then find F in AB such that SF . SK = ST^2. Let ES cut the circle in G, then SF . SK = SG^2 = SG . SE and $\angle$ GSF = $\angle$ KSE. Therefore the triangles GSF, KSE are similar, and $\angle$ FGS = $\angle$ SKE = $\angle$ ACE. If FG cuts the circle in D′ and AC in D, we have $\angle$ D′GE = $\angle$ DCE, wherefore arc ED′ = arc ED and D and D′ coincide. Hence FDG are collinear, and $\angle$ FDE is a right angle, and therefore constant.

Case II. When A *and* B *are inverse points with respect to the circle.* Let AB cut the circle in Y. Then $\angle$ DCY = $\angle$ YCE, and therefore Y is the middle point of the arc DE, and AB is perpendicular to DE. Thus DE is parallel to a fixed line.

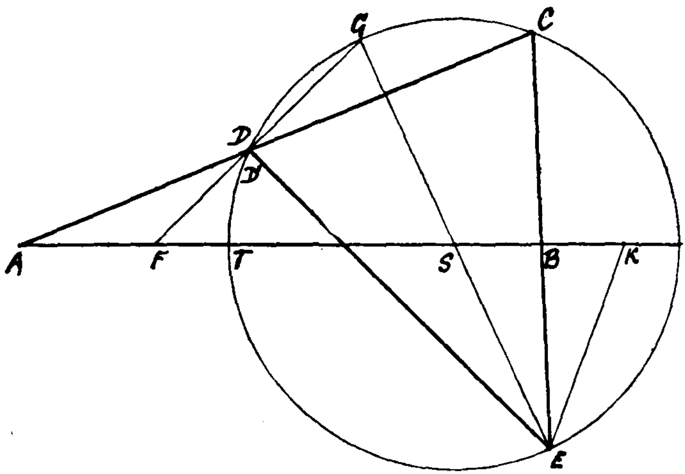

Fig. 2.

Case III. When A *and* B *are diagonal points of any cyclic quadrilateral* DCUE. Then DE passes through F the pole of AB with respect to the given circle, *i.e.* through a fixed point.

By reciprocation of Simson's result with respect to a circle whose centre is F, we obtain the following proposition: Let S be a conic with focus F, and a, b be two straight lines in the plane of the conic. If any tangent meet these lines in U and V, and if from these points tangents d, e be drawn to the conic and intersect at a point x, then the angle that Fx makes with either tangent d or e is constant.

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