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Optimal Solutions of Restricted Subadditive Inequalities

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Consider the following question:

"Let k be a given non-negative integer, and let g denote a real valued function of one variable, defined on a known finite open interval, and possessing a continuous k-th derivative $g^{(k)}$. How might simple real zeros of $g^{(k)}$ be efficiently approximated, by using only values of g and points in the domain of g?"

A standard approach to this question is to choose successively a (prescribed) total of $n(> k)$ points to be the abscissae for sequences of k-th divided differences. The *signs* of the differences are then used to locate the zeros; see Wallace [4] and the references therein to the pioneering work of R.S. Booth, S.M. Johnson, J. Kiefer and others.

Of central importance is the particular rule (or *strategy*) $S_k = S_k(n)$ by which these n points are selected. Some strategies estimate the zeros of $g^{(k)}$ more *efficiently* than others, so workers have sought the *most* efficient strategy $\overline{S}_k$, for given k. To date, $\overline{S}_k$ has been exhibited for $k = 0, 1, 2, 3, 4, 5, 6, 14$ only.

The main result of the first part of this thesis is the establishment of $\overline{S}_{2(2^{A+1}-1)}$, A a fixed non-negative integer. Also determined is an extension of a result of Booth on $\overline{S}_4$. Both results are found by analysis of a particular *maximal* solution of the *restricted subadditive inequality*

$$\psi_{2p}(n+2p+1) \leqslant \psi_{2p}(n+\ell) + \psi_{2p}(n+2p-\ell),\ n \geqslant 0,\ 0 \leqslant \ell \leqslant p,$$

(p a fixed non-negative integer); namely, that sequence $U_{2p} = \{U_{2p}(n)\}$, $n \geqslant 0$, which is defined, for fixed non-negative integer p, by the initial conditions

$$U_{2p}(0) = U_{2p}(1) = U_{2p}(2) = \ldots = U_{2p}(2p) = 1$$

and by the restricted subadditive recursion

$$U_{2p}(n+2p+1) = \min_{0 \leqslant \ell \leqslant p} (U_{2p}(n+\ell) + U_{2p}(n+2p-\ell)),\ n \geqslant 0\,.$$

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Various number theoretic and algebraic properties of U_{2p} are also exhibited.

The problem of multi-stage allocation processes is one of the many classes that can be solved by dynamic programming; see Bellman [**1**], Iwamoto [**2, 3**]. The main aim of the second part of the thesis is to introduce a new class of optimal decision-making problems arising in a certain (discrete) multi-stage allocation process in manufacturing. It is also shown how these problems can be solved by a certain (discrete) dynamic programming approach.

Specifically, it is first established that the solutions to the problems in question can be modelled by a particular *minimal* solution of the weighted generalised restricted *superadditive* inequality

$$(1)\quad \chi_{\alpha,\beta}(n) \geqslant \begin{cases} \alpha(1-\beta^3)\chi_{\alpha,\beta}(n-1) + (1-\alpha)\beta^3\chi_{\alpha,\beta}(n-3) + \alpha\beta^3\chi_{\alpha,\beta}(n-4), \\ (\alpha^2(1-\beta^2) + \beta^2(1-\alpha^2))\chi_{\alpha,\beta}(n-2) + \alpha^2\beta^2\chi_{\alpha,\beta}(n-4), \ n \geqslant 4 \\ \beta(1-\alpha^3)\chi_{\alpha,\beta}(n-1) + (1-\beta)\alpha^3\chi_{\alpha,\beta}(n-3) + \alpha^3\beta\chi_{\alpha,\beta}(n-4), \end{cases}$$

(α, β fixed in $0 < \alpha, \beta < 1$); namely, by that sequence $P_{\alpha,\beta} = \{P_{\alpha,\beta}(n)\}$, $n \geqslant 0$, which is defined, for fixed α, β in $0 < \alpha, \beta < 1$, by the initial conditions

$$(2a)\quad \begin{cases} P_{\alpha,\beta}(0) = 1; \qquad P_{\alpha,\beta}(1) = \max\begin{cases} \alpha(1-\beta^3) \\ \beta(1-\alpha^3) \end{cases}; \\ P_{\alpha,\beta}(2) = \max\begin{cases} \alpha(1-\beta^3)P_{\alpha,\beta}(1) \\ (\alpha^2(1-\beta^2) + \beta^2(1-\alpha^2)) \\ \beta(1-\alpha^3)P_{\alpha,\beta}(1) \end{cases}; \\ P_{\alpha,\beta}(3) = \max\begin{cases} \alpha(1-\beta^3)P_{\alpha,\beta}(2) + (1-\alpha)\beta^3 \\ (\alpha^2(1-\beta^2) + \beta^2(1-\alpha^2))P_{\alpha,\beta}(1) \\ \beta(1-\alpha^3)P_{\alpha,\beta}(2) + (1-\beta)\alpha^3 \end{cases}; \end{cases}$$

and by the generalised restricted superadditive recursion

$$(2b)\quad P_{\alpha,\beta}(n) = \max\begin{cases} \alpha(1-\beta^3)P_{\alpha,\beta}(n-1) + (1-\alpha)\beta^3P_{\alpha,\beta}(n-3) + \alpha\beta^3P_{\alpha,\beta}(n-4), \\ (\alpha^2(1-\beta^2) + \beta^2(1-\alpha^2))P_{\alpha,\beta}(n-2) + \alpha^2\beta^2P_{\alpha,\beta}(n-4), \ n \geqslant 4 \\ \beta(1-\alpha^3)P_{\alpha,\beta}(n-1) + (1-\beta)\alpha^3P_{\alpha,\beta}(n-3) + \alpha^3\beta P_{\alpha,\beta}(n-4), \end{cases}$$

It is then shown how the values of $P_{\alpha,\beta}$ generated by recursions (2a,b) can be utilised to solve the aforementioned problems. (Note that determining a minimal solution $\chi_{\alpha,\beta}$ of the superadditive inequality (1) is tantamount to finding a maximal solution of a subadditive inequality ((1) with $\chi_{\alpha,\beta}$ replaced by $-\chi_{\alpha,\beta}$).)

The analysis of U_{2p} and $P_{\alpha,\beta}$ summarised above suggests several areas for future investigation. Three of these are discussed briefly in the final chapter.

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