J. Austral. Math. Soc. (Series A) 49 (1990), 111-128

MATCHING THEOREMS, FIXED POINT THEOREMS AND MINIMAX INEQUALITIES WITHOUT CONVEXITY

XIE-PING DING and KOK-KEONG TAN

(Received 20 April 1989)

Communicated by J. H. Rubinstein

Abstract

Matching theorems, fixed point theorems and minimax inequalities are obtained in H-spaces which generalize the corresponding results of Bae-Kim-Tan, Browder, Fan, Horvath, Kim, Ko-Tan, Shih-Tan, Takahashi, Tan and Tarafdar to non-compact and/or non-convex settings.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*) (1985 *Revision*): primary 47 H 10, 49 A 40; secondary 54 H 25, 54 C 60.
Keywords and phrases: matching theorem, fixed point theorem, minimax inequality, H-space, contractible, H-convex, weakly H-convex, compactly open (closed), H-compact, H-KKM, topological vector space, convex hull, lower semi-continuous.

1. Introduction

In 1972, by applying his infinite dimensional generalization [11, Lemma 1] of the classical Knaster-Kuratowski-Mazurkiewicz Theorem [18], Fan obtained a minimax inequality [12, Theorem 1] which has numerous applications to various and diverse branches of mathematics. Since then there are many generalizations in topological vector space setting, for example, [1], [2], [4], [5], [13], [20], [23], [24], [25], [26], [27] and [28]. In [14, 15, 16], Horvath obtained minimax inequalities by replacing convexity with pseudo-convexity or contractibility in a topological space but only in compact setting. In [3], using Horvath's approach in [16], Bardaro and Ceppitelli obtained some minimax inequalities in non-compact setting for mappings taking values in an ordered vector space.

This work was partially supported by NSERC of Canada under grant number A-8096.

In this paper, we shall use Bardaro and Ceppitelli's notions of " H-space", " H-convex", "weak- H-convex" and " H-compact" in [3] to first obtain some generalizations of Fan's matching theorems [13, Theorems 2 and 3] and some results of Horvath [16, Theorem 2], Kim [17, Theorem 2], Ko and Tan [19, Theorem 7B] and Tarafdar [27, Lemma 2.1] to non-convex setting. Next by applying our earlier results, some fixed point theorems are obtained generalizing those of Browder [6, Theorem 2], Horvath [16, Theorem 2′], Kim [17, Theorem 3] and Tarafdar [27, Theorems 2.2 and 2.3 and Corollaries 2.1 and 2.2] to non-convex and non-compact setting. Several very general minimax inequalities are also presented which improve those of Bae, Kim and Tan [2, Theorem 1], Fan [12, Corollary 1], [13, Theorem 6], Horvath [16, Propositions 1, 2 and 3], Shih and Tan [21, Theorem 1], Takahashi [25, Theorem 3] and Tan [26, Theorem 1].

For further and related works and applications and for mappings taking values in an ordered vector space, we refer to Ding, Kim and Tan [7] and Ding and Tan [8, 9, 10].

2. Matching theorems

Let X and Y be non-empty sets; we shall denote by 2^Y the family of all non-empty subsets of Y and $\mathscr{F}(X)$ the family of all non-empty finite subsets of X. If $F: X \to 2^Y$, define F^{-1}, $F^*: Y \to 2^X \cup \{\emptyset\}$ and $F^c: X \to 2^Y \cup \{\emptyset\}$ by

$$F^{-1}(y) = \{x \in X : y \in F(x)\}, \quad F^*(y) = \{x \in X : y \notin F(x)\} \quad \text{and}$$
$$F^c(x) = \{y \in Y : y \notin F(x)\}.$$

We shall denote by Δ_n the standard n dimensional simplex with the vertices $e_0, \dots, e_n$. If J is a non-empty subset of $\{0, \dots, n\}$, Δ_J will denote the convex hull of the vertices $\{e_j : j \in J\}$. If E is a vector space and $A \subset E$, we shall denote by $\mathrm{co}(A)$ the convex hull of A.

The following notions which were introduced by Bardaro and Ceppitelli in [3] were motivated by an earlier work of Horvath [16] in generalizing Ky Fan's infinite dimensional generalization of the Knaster-Kuratowski-Mazurkiewicz theorem [18] and Fan's minimax inequality [12] to topological spaces without convexity.

A pair $(X, \{F_A\})$ is said to be an *H-space* if X is a topological space and $\{F_A\}$ is a given family of non-empty contractible subsets of X, indexed by $A \in \mathscr{F}(X)$ such that $F_A \subset F_{A'}$ whenever $A \subset A'$. Let $(X, \{F_A\})$ be an H-space. A non-empty subset D of X is called (i) *H-convex* if $F_A \subset D$ for each $A \in \mathscr{F}(D)$; (ii) *weakly H-convex* if $F_A \cap D$ is non-empty and

https://doi.org/10.1017/S1446788700030275 Published online by Cambridge University Press

contractible for each $A \in \mathcal{F}(D)$ (this is equivalent to say that $(D, \{F_A \cap D\})$ is an H-space); (iii) *compactly open* (*closed*) *in* X if $D \cap C$ is open (closed) in C for each non-empty compact subset C of X.

Let $(Y, \{F_A\})$ be an H-space and X be a non-empty subset of Y. A non-empty subset X_0 of X is said to be *H-compact in* X if, for each $A \in \mathcal{F}(X)$, there exists a compact, weakly H-convex subset C_A of Y such that $X_0 \cup A \subset C_A$. A map $F: X \to 2^Y$ is called *H-KKM* if $F_A \subset \bigcup_{x \in A} F(x)$ for each $A \in \mathcal{F}(X)$. We remark here that our definition of "H-compact in X" is slightly more general than that of "H-compact" in [3]; however, the two notions coincide when $X = Y$.

The proof of the following useful result is contained in the proof of Theorem 1 of Horvath in [16] and is thus omitted.

LEMMA 1. *Let* X *be a topological space. For each non-empty subset* J *of* $\{0, \dots, n\}$, *let* F_J *be a non-empty contractible subset of* X. *If* $J \subset J'$ *implies* $F_J \subset F_{J'}$, *then there exists a continuous map* $f: \Delta_n \to X$ *such that* $f(\Delta_J) \subset F_J$ *for each non-empty subset* J *of* $\{0, \dots, n\}$.

The following result is a variation of Theorem 1 of Horvath [16]:

LEMMA 2. *Let* X *be a topological space and* $\{R_i\}_{i=0}^n$ *be a family of subsets of* X. *Suppose*

(i) *for each non-empty subset* J *of* $\{0, \dots, n\}$, *there exists a non-empty contractible subset* F_J *of* X *such that* $F_J \subset \bigcup_{j \in J} R_j$ *and* $F_J \subset F_{J'}$, *whenever* $J \subset J'$;

(ii) *for each* $i \in \{0, \dots, n\}$, $F_{\{0, \dots, n\}} \cap R_i$ *is closed in* $F_{\{0, \dots, n\}}$.

Then $\bigcap_{i=0}^n R_i \neq \varnothing$.

PROOF. By Lemma 1, there exists a continuous function $f: \Delta_n \to X$ such that $f(\Delta_J) \subset F_J$ for each non-empty subset J of $\{0, \dots, n\}$. For each $i = 0, \dots, n$, let $S_i = f^{-1}(F_{\{0, \dots, n\}} \cap R_i)$, then S_i is a closed subset of the simplex Δ_n. For each non-empty subset J of $\{0, \dots, n\}$, we have

$$\bigcup_{j \in J} S_j = f^{-1}\left(F_{\{0, \dots, n\}} \cap \left(\bigcup_{j \in J} R_j\right)\right) \supset f^{-1}(F_{\{0, \dots, n\}} \cap F_J)$$
$$= f^{-1}(F_J) \supset \Delta_J.$$

Therefore

$$\mathrm{co}\{e_j : j \in J\} \subset \bigcup_{j \in J} S_j.$$

By the Knaster-Kuratowski-Mazurkiewicz theorem [18], $\bigcap_{i=0}^n S_i \neq \varnothing$. Take any $p \in \bigcap_{i=0}^n S_i$, then $f(p) \in \bigcap_{i=0}^n (F_{\{0, \dots, n\}} \cap R_i)$ so that $\bigcap_{i=0}^n R_i \neq \varnothing$.

https://doi.org/10.1017/S1446788700030275 Published online by Cambridge University Press

The following result is the dual of Lemma 2 and generalizes Theorem 2 of Kim in [17] to non-convex setting.

LEMMA 3. *Let X be a topological space and $\{R_i\}_{i=0}^n$ be a family of subsets of X. Suppose*

(i) *for each non-empty subset J of $\{0, \dots, n\}$, there exists a non-empty contractible subset F_J of X such that $F_J \subset \bigcup_{j\in J} R_j$ and $F_J \subset F_{J'}$ whenever $J \subset J'$;*

(ii) *for each $i \in \{0, \dots, n\}$, $F_{\{0,\dots,n\}} \cap R_i$ is open in $F_{\{0,\dots,n\}}$.*

Then $\bigcap_{i=0}^n R_i \neq \varnothing$.

PROOF. By Lemma 1, there exists a continuous function $f: \Delta_n \to X$ such that $f(\Delta_J) \subset F_J$ for each non-empty subset J of $\{0, \dots, n\}$. For each $i = 0, \dots, n$, let $S_i = f^{-1}(F_{\{0,\dots,n\}} \cap R_i)$, then S_i is an open subset of the simplex Δ_n and for each non-empty subset J of $\{0, \dots, n\}$.

$$\bigcup_{j\in J} S_j = f^{-1}\left(F_{\{0,\dots,n\}} \cap \left(\bigcup_{j\in J} R_j\right)\right) \supset f^{-1}(F_{\{0,\dots,n\}} \cap F_J)$$
$$= f^{-1}(F_J) \supset \Delta_J.$$

Therefore $\mathrm{co}\{e_j : j \in J\} \subset \bigcup_{j\in J} S_j$. It follows from Corollary 1 of Shih and Tan [22] (also Theorem 1 of Kim [17]) that $\bigcap_{i=0}^n S_i \neq \varnothing$. Take any $p \in \bigcap_{i=0}^n S_i$, then $f(p) \in \bigcap_{i=0}^n (F_{\{0,\dots,n\}} \cap R_i)$ so that $\bigcap_{i=0}^n R_i \neq \varnothing$.

As applications of Lemmas 2 and 3, we have the following matching theorems.

THEOREM 1. *Let X be a topological space and $A_1, \dots, A_n$ be n closed subsets of X such that $\bigcup_{i=1}^n A_i = X$. For each non-empty subset J of $\{1, \dots, n\}$, let F_J be a non-empty contractible subset of X such that $F_J \subset F_{J'}$ whenever $J \subset J'$. Then there exists a non-empty subset J_0 of $\{1, \dots, n\}$ such that $F_{J_0} \cap \bigcap_{j\in J_0} A_j \neq \varnothing$.*

PROOF. Suppose the conclusion were not true, then $F_J \cap \bigcap_{j\in J} A_j = \varnothing$ for each non-empty subset J of $\{1, \dots, n\}$. For each $j = 1, \dots, n$, let $G_j = S \backslash A_j$, then G_j is open in X. It follows that $F_J \subset \bigcup_{j\in J} G_j$ for each non-empty subset J of $\{1, \dots, n\}$. By Lemma 3, $\bigcap_{j=1}^n G_j \neq \varnothing$, which contradicts the assumption $\bigcup_{i=1}^n A_i = X$. This completes the proof.

If X is a convex subset of a topological vector space and $x_1, \dots, x_n \in X$, let F_J be the convex hull of $\{x_j : j \in J\}$ for each non-empty subset J of $\{1, \dots, n\}$, we see then Theorem 1 generalizes Theorem 2 of Fan in [13].

Theorem 2. *Let X be a topological space and $B_1, \ldots, B_n$ be n open subsets of X such that $\bigcup_{i=1}^n B_i = X$. For each non-empty subset J of $\{1, \ldots, n\}$, let F_J be a non-empty contractible subset of X such that $F_J \subset F_{J'}$ whenever $J \subset J'$. Then there exists a non-empty subset J_0 of $\{1, \ldots, n\}$ such that $F_{J_0} \cap \bigcap_{j \in J_0} B_j \neq \varnothing$.*

Proof. Suppose the conclusion were not true, then $F_J \cap \bigcap_{j\in J} B_j = \varnothing$ for each non-empty subset J of $\{1, \ldots, n\}$. For each $j = 1, \ldots, n$, let $G_j = X \backslash B_j$, then G_j is closed in X. It follows that $F_J \subset \bigcup_{j \in J} G_j$ for each non-empty subset J of $\{1, \ldots, n\}$. By Lemma 2, $\bigcap_{j=1}^n G_j \neq \varnothing$, which contradicts the assumption $\bigcup_{j=1}^n B_j = X$. This completes the proof.

The above result generalizes Theorem 7B of Ko and Tan in [19] to a non-convex setting.

Theorem 3. *Let $(X, \{F_A\})$ be an H-space and $S\colon X \to 2^X$ be such that*
(a) $\bigcup_{x \in X} S(x) = X$;
(b) *for some $x_0 \in X$, $S^c(x_0)$ is compact and for each $x \in X$, $S^c(x_0) \cap S^c(x)$ is closed in $S^c(x_0)$;*
(c) *for each $x \in X$ and for each $A \in \mathscr{F}(X)$, $F_A \cap S^c(x)$ is closed in F_A.*
Then there exists $A \in \mathscr{F}(X)$ such that $F_A \cap \bigcup_{x \in A} S(x) \neq \varnothing$.

Proof. Suppose the assertion is false; then for each $A \in \mathscr{F}(X)$, $F_A \cap \bigcap_{x \in A} S(x) = \varnothing$ so $F_A \subset X \backslash \bigcap_{x\in A} S(x) = \bigcup_{x \in A} S^c(x)$. Define $G\colon X \to 2^X$ by $G(x) = S^c(x)$ for each $x \in X$; then G is an H-KKM map. By (c), for each $x \in X$ and for each $A \in \mathscr{F}(X)$, $F_A \cap G(x)$ is closed in F_A. Thus by Lemma 2 the family $\{G(x)\colon x \in X\}$ has the finite intersection property. By (b), $G(x_0)$ is compact and for each $x \in X$, $G(x_0) \cap G(x)$ is closed in $G(x_0)$. It follows that $\bigcap_{x \in X} G(x) \neq \varnothing$ which contradicts (a). Hence the assertion must hold.

Theorem 3 can be restated in its contrapositive form and in terms of the complement $G(x)$ of $S(x)$ in X as follows.

Theorem 4. *Let $(X, \{F_A\})$ be an H-space and $G\colon X \to 2^X$ be such that*
(a) *G is an H-KKM map;*
(b) *for some $x_0 \in X$, $G(x_0)$ is compact and for each $x \in X$, $G(x_0) \cap G(x)$ is closed in $G(x_0)$;*
(c) *for each $x \in X$ and for each $A \in \mathscr{F}(X)$, $F_A \cap G(x)$ is closed in F_A.*
Then $\bigcap_{x \in X} G(x) \neq \varnothing$.

As an immediate consequence of Theorem 4, we have the following.

THEOREM 5. *Let* $(X, \{F_A\})$ *be an* H*-space and* $F, G: X \to 2^X$ *be such that*

(a) *for each* $x \in X$, $F(x) \subset G(x)$ *and* $x \in F(x)$;

(b) *for each* $x \in X$, $F^*(x)$ *is* H*-convex;*

(c) *for some* $x_0 \in X$, $G(x_0)$ *is compact and for each* $x \in X$, $G(x_0) \cap G(x)$ *is closed in* $G(x_0)$;

(d) *for each* $x \in X$ *and for each* $A \in \mathscr{F}(X)$, $F_A \cap G(x)$ *is closed in* F_A.

Then $\bigcap_{x\in X} G(x) \neq \varnothing$.

PROOF. By Theorem 4, we need only to show that G is an H-KKM map. If G were not H-KKM, then there exists $A \in \mathscr{F}(X)$ such that F_A is not contained in $\bigcup_{x\in A} G(x)$; let $y \in F_A$ be such that $y \notin \bigcup_{x\in A} G(x)$. It follows that $A \subset G^*(y) \subset F^*(y)$ by (a) so that $F_A \subset F^*(y)$ by (b). As $y \in F_A$, we must have $y \in F^*(y)$ so that $y \notin F(y)$ which contradicts (a). This completes the proof.

Theorem 5 generalizes Lemma 2.1 of Tarafdar in [27] to non-convex setting and to a pair of maps and Theorem 2 of Horvath in [16] in several aspects. As another immediate consequence of Theorem 4, we have the following.

COROLLARY 1. *Let* $(X, \{F_A\})$ *be an* H*-space and let* $G: X \to 2^X$ *be such that*

(a) G *is* H*-KKM;*

(b) *for some* $x_0 \in X$, $G(x_0)$ *is compact and for each* $x \in X$, $G(x)$ *is closed in* X.

Then $\bigcap_{x\in X} G(x) \neq \varnothing$.

As another application of Lemma 2, we have the following

THEOREM 6. *Let* $(X, \{F_A\})$ *be an* H*-space and* $S: X \to 2^X$ *be such that*

(a) *for some* $x_0 \in X$, $S^*(x_0)$ *is compact and for each* $x \in X$, $S^*(x_0) \cap S^*(x)$ *is closed in* $S^*(x_0)$;

(b) *for each* $x \in X$ *and for each* $A \in \mathscr{F}(X)$, $F_A \cap S^*(x)$, *is closed in* F_A.

Then there exists $A \in \mathscr{F}(X)$ *such that* $F_A \cap \bigcap_{x\in A} S^{-1}(x) \neq \varnothing$.

PROOF. Suppose the assertion were false; then for each $A \in \mathscr{F}(X)$, $F_A \cap \bigcap_{x\in A} S^{-1}(x) = \varnothing$ so that $F_A \subset X \setminus \bigcap_{x\in A} S^{-1}(x) = \bigcup_{x\in A} X \setminus S^{-1}(x) = \bigcup_{x\in A} S^*(x)$. Define $G: X \to 2^X$ by $G(x) = S^*(x)$ for each $x \in X$;

https://doi.org/10.1017/S1446788700030275 Published online by Cambridge University Press

then G is an H-KKM map. It follows from (b) and Lemma 2 that the family $\{G(x): x \in X\}$ has the finite intersection property so that by (a) $\bigcap_{x \in X} G(x) \neq \varnothing$. Take any $y \in \bigcap_{x \in X} G(x)$, then for each $x \in X$, $y \in G(x) = S^*(x)$ and hence $x \notin S(y)$. Thus $S(y) = \varnothing$, which is a contradiction. Hence the assertion must hold.

Lemma 4. *Let $(Y, \{F_A\})$ be an H-space and X be a non-empty subset of Y. Let $B: X \to 2^Y$ be such that*

(a) *for each $x \in X$, $B(x)$ is compactly open in Y;*

(b) $\bigcup_{x \in X} B(x) = Y$;

(c) *there exists a non-empty compact weakly H-convex subset C of Y such that $X \subset C$.*

Then there exists $A \in \mathscr{F}(X)$ such that $F_A \cap \bigcap_{x \in A} B(x) \neq \varnothing$.

Proof. By (a) and (c), $B(x) \cap C$ is open in C for each $x \in S$. By (b), $C = \bigcup_{x \in X}(B(x) \cap C)$. Thus there exists $\{x_0, \dots, x_n\} \in \mathscr{F}(X)$ such that $C = \bigcup_{i=0}^n (B(x_i) \cap C)$. For each $i \in \{0, \dots, n\}$, let $G(x_i) = C \backslash (B(x_i) \cap C)$; then $G(x_i)$ is closed in C. By (c), for each non-empty $J \subset \{x_0, \dots, , x_n\}$ $(\subset X \subset C)$, $F_J \cap C$ is a non-empty contractible subset of C such that $F_J \subset F_{J'}$ whenever $J \subset J'$. Now suppose that the assertion were false, then for each non-empty subset A of $\{x_0, \dots, x_n\}$, $F_A \cap \bigcap_{x \in A} B(x) = \varnothing$ so that $(F_A \cap C) \cap \bigcap_{x \in A}(B(x) \cap C) = \varnothing$ and hence

$$F_A \cap C \subset C \backslash \left(\bigcap_{x \in A} (B(x) \cap C) \right) = \bigcup_{x \in A} G(x).$$

By Lemma 2, $\bigcap_{i=0}^n G(x_i) \neq \varnothing$; but

$$\bigcap_{i=0}^n G(x_i) = \bigcap_{i=0}^n (C \backslash (B(x_i) \cap C)) = C \backslash \bigcup_{i=0}^n (B(x_i) \cap C)$$

which contradicts the fact that $C = \bigcup_{i=0}^n (B(x_i) \cap C)$. Hence the conclusion of Lemma 4 must hold.

Theorem 7. *Let $(Y, \{F_A\})$ be an H-space and X be a non-empty subset of Y. Let $B: X \to 2^Y$ be such that*

(a) *for each $x \in X$, $B(x)$ is compactly open in Y;*

(b) $\bigcup_{x \in X} B(x) = Y$;

(c) *there exists a non-empty subset X_0 of X which is H-compact in X such that $Y \backslash \bigcup_{x \in X_0} B(x)$ is empty or compact.*

Then there exists $A \in \mathscr{F}(X)$ such that $F_A \cap \bigcap_{x \in A} B(x) \neq \varnothing$.

PROOF. *Case* 1. Suppose $Y = \bigcup_{x\in X_0} B(x)$, then the conclusion follows from Lemma 4.

Case 2. Suppose $Y\setminus\bigcup_{x\in X_0} B(x)$ is non-empty and compact, then by (b), $Y = \bigcup_{x\in X} B(x) \supset Y\setminus\bigcup_{x\in X_0} B(x)$ so that we can find $A = \{x_1, \dots, x_n\} \subset X\setminus X_0$ such that $\bigcup_{x\in A} B(x) \supset Y\setminus\bigcup_{x\in X_0} B(x)$. Thus $\bigcup_{x\in X_0\cup A} B(x) = Y$. Since X_0 is H-compact in X, by Lemma 4 again, we obtain the desired result.

Theorem 7 may be restated in its contrapositive form and in terms of the complement $F(x)$ of $B(x)$ in Y as follows.

THEOREM 8. *Let $(Y, \{F_A\})$ be an H-space and X be a non-empty subset of Y. Let $F: X \to 2^Y$ be an H-KKM map such that*

(a) *for each $x \in X$, $F(x)$ is compactly closed in Y;*

(b) *there exists a non-empty subset X_0 of X which is H-compact in X such that $\bigcap_{x\in X_0} F(x)$ is empty or compact.*

Then $\bigcap_{x\in X} F(x) \neq \varnothing$.

Lemma 4, Theorem 7 and Theorem 8 generalize Lemma 1, Theorem 3 and Theorem 4 of Fan in [13], respectively, to a non-convex setting. We emphasize that our Theorem 8 is a true generalization of Theorem 4 of Fan in [13] while Theorem 1 of Bradaro-Ceppitelli in [3] only generalizes a special case (namely, when $X = Y$) of the corresponding result.

3. Fixed point theorems

We first shall apply Lemma 3 to obtain the following fixed point theorem which generalizes Theorem 3 of Kim in [17] to a non-convex setting and to a pair of maps.

THEOREM 9. *Let X be a topological space, $x_0, \dots, x_n \in X$ and $S, T: X \to 2^X$ be such that*

(a) *for each $i = 0, \dots, n$, $S(x_i) \subset T(x_i)$;*

(b) *for each non-empty subset A of $\{x_0, \dots, x_n\}$, there exists a non-empty contractible subset F_A of X such that $F_A \subset F_{A'}$ whenever $A \subset A'$;*

(c) *for each $i = 0, \dots, n$, $F_{\{x_0,\dots,x_n\}} \cap S(x_i)$ is closed in $F_{\{x_0,\dots,x_n\}}$;*

(d) *for each non-empty subset A of $\{x_0, \dots, x_n\}$ with $A \subset T^{-1}(y)$ for some $y \in X$, $F_A \subset T^{-1}(y)$;*

(e) $\bigcup_{i=0}^n S(x_i) = X$.

Then there exists $\hat{x} \in X$ such that $\hat{x} \in T(\hat{x})$.

Proof. For each $x \in X$, let $F(x) = T^c(x)$ and $G(x) = S^c(x)$. Suppose $F_A \subset \bigcup_{x\in A} G(x)$ for each non-empty subset A of $\{x_0, \dots, x_n\}$. By (c), for each $i = 0, \dots, n$, $F_{\{x_0,\dots,x_n\}} \cap G(x_i)$ is open in $F_{\{x_0,\dots,x_n\}}$. By Lemma 3, $\bigcap_{i=0}^n G(x_i) \neq \varnothing$, which contradicts (e). Thus there must exist a non-empty subset A of $\{x_0, \dots, x_n\}$ such that F_A is not contained in $\bigcup_{x\in A} G(x)$. Take any $\hat{x} \in F_A$ with $\hat{x} \notin \bigcup_{x\in A} G(x)$. It follows that for each $x \in A$, $\hat{x} \in S(x) \subset T(x)$ by (a) so that $x \in T^{-1}(\hat{x})$. Therefore $A \subset T^{-1}(\hat{x})$ and hence $F_A \subset T^{-1}(\hat{x})$ by (d). As $\hat{x} \in F_A$, we have $\hat{x} \in T^{-1}(\hat{x})$ so that $\hat{x} \in T(\hat{x})$.

Theorem 10. *Let $(X, \{F_A\})$ be an H-space and $S, T: X \to 2^X$ be such that*

(a) *for each $x \in X$, $S(x) \subset T(x)$;*

(b) $\bigcup_{x\in X} S(x) = X$;

(c) *for some $x_0 \in X$, $S^c(x_0)$ is compact and for each $x \in X$, $S^c(x_0) \cap S^c(x)$ is closed in $S^c(x_0)$;*

(d) *for each $x \in X$ and for each $A \in \mathcal{F}(X)$, $F_A \cap S^c(x)$ is closed in F_A;*

(e) *for each $x \in X$, $T^{-1}(x)$ is H-convex.*

Then there exists $\hat{x} \in X$ such that $\hat{x} \in T(\hat{x})$.

Proof. By Theorem 3, there exists $A \in \mathcal{F}(X)$ such that $F_A \cap \bigcap_{x\in A} S(x) \neq \varnothing$. Take any $\hat{x} \in F_A \cap \bigcap_{x\in A} S(x)$; then $\hat{x} \in F_A$ and $A \subset S^{-1}(\hat{x}) \subset T^{-1}(\hat{x})$ by (a). By (e), $F_A \subset T^{-1}(\hat{x})$; but then $\hat{x} \in T^{-1}(\hat{x})$ so that $\hat{x} \in T(\hat{x})$.

The following is an immediate consequence of Theorem 10.

Corollary 2. *Let $(X, \{F_A\})$ be an H-space and $S, T: X \to 2^X$ be such that*

(a) *for each $x \in X$, $S(x) \subset T(x)$;*

(b) $\bigcup_{x\in X} S(x) = X$;

(c) *for some $x_0 \in S$, $S^c(x_0)$ is compact and for each $x \in X$, $S(x)$ is open in X;*

(d) *for each $x \in X$, $T^{-1}(x)$ is H-convex.*

Then there exists $\hat{x} \in X$ such that $\hat{x} \in T(\hat{x})$.

Theorem 10 and Corollary 2 generalize Theorem 2.3 and Corollary 2.2 of Tarafdar in [27] respectively to a non-convex setting and to a pair of maps.

Theorem 11. *Let $(X, \{F_A\})$ be an H-space and $S, T: X \to 2^X$ be such that*

(i) *for each* $x \in X$, $S(x) \subset T(x)$;

(ii) *for each* $y \in X$, $S^{-1}(y)$ *is open in* X;

(iii) *for each* $x \in X$, $T(x)$ *is H-convex;*

(iv) *there exist a non-empty compact subset* L *of* X *and a point* $y_0 \in X$ *such that* $y_0 \in S(x)$ *for all* $x \in X \backslash L$.

Then there exists a point $\hat{x} \in X$ *such that* $\hat{x} \in T(\hat{x})$.

Proof. Suppose the assertion is false, that is, $x \notin T(x)$ for all $x \in X$. For each $x \in X$, let $G(x) = S^*(x)$ and $F(x) = T^*(x)$. Then we have the following properties:

(a) by (i), for each $y \in X$, $F(y) \subset G(y)$;

(b) for each $x \in X$, since $x \notin T(x)$, we must have $x \in F(x)$;

(c) by (ii), $G(y)$ is closed in X for each $y \in X$; by (iv), $G(y_0)$ is a subset of L so that $g(y_0)$ is compact;

(d) since $F^*(x) = T(x)$ for each $x \in X$, by (iii) $F^*(x)$ is H-convex for each $x \in X$.

Thus all hypotheses of Theorem 5 are satisfied. By Theorem 5, $\bigcap_{y \in X} G(y) \neq \varnothing$. Take any $u \in \bigcap_{y \in X} G(y)$, then $u \notin \bigcup_{y \in X} S^{-1}(y) = X$, which is impossible. Therefore there must exist $\hat{x} \in X$ such that $\hat{x} \in T(\hat{x})$.

As an immediate consequence of Theorem 11, we have

Corollary 3. *Let* X *be a convex subset of a topological vector space* E *and* $S, T: X \to 2^X$ *be such that*

(i) *for each* $x \in X$, $S(x) \subset T(x)$;

(ii) *for each* $y \in X$, $S^{-1}(y)$ *is open in* X;

(iii) *for each* $x \in X$, $T(x)$ *is convex;*

(iv) *there exist a non-empty compact subset* L *of* X *and a point* $y_0 \in X$ *such that* $y_0 \in S(x)$ *for all* $x \in X \backslash L$.

Then there exists a point $\hat{x} \in X$ *such that* $\hat{x} \in T(\hat{x})$.

Proof. For each $A \in \mathscr{F}(X)$, let $F_A = \mathrm{co}(A)$; then all hypotheses of Theorem 11 are satisfied; the conclusion follows from Theorem 11.

Even when $S = T$, Corollary 3 improves Theorem 2 of Browder in [6] where X is also assumed to be closed.

Theorem 12. *Let* $(X, \{F_A\})$ *be an* H*-space and* $S, T: X \to 2^X$ *be such that*

(a) *for each* $x \in X$, $S(x) \subset T(x)$;

(b) *for some* $x_0 \in X$, $S^*(x_0)$ *is compact and for each* $x \in X$, $S^*(x_0) \cap S^*(x)$ *is closed in* $S^*(x_0)$;

(c) *for each* $x \in X$ *and for each* $A \in \mathscr{F}(x)$, $F_A \cap S^*(x)$ *is closed in* F_A;

(d) *for each* $x \in X$, $T(x)$ *is* H*-convex.*

Then there exists $\hat{x} \in X$ *such that* $\hat{x} \in T(\hat{x})$.

PROOF. By Theorem 6, there exists $A \in \mathscr{F}(X)$ such that $F_A \cap \bigcap_{x \in A} S^{-1}(x) \neq \varnothing$. Take any $\hat{x} \in F_A \cap \bigcap_{x \in A} S^{-1}(x)$; then $\hat{x} \in F_A$ and $A \subset S(\hat{x}) \subset T(\hat{x})$ by (a). By (d), $F_A \subset T(\hat{x})$. Therefore $\hat{x} \in T(\hat{x})$.

Theorem 12 generalizes Theorem 2.2 of Tarafdar in [27] to a non-convex setting and to a pair of mappings. The following result is an immediate consequence of Theorem 12.

COROLLARY 4. *Let* $(X, \{F_A\})$ *be an* H*-space and* $S, T: X \to 2^X$ *be such that*

(a) *for each* $x \in X$, $S(x) \subset T(x)$;

(b) *for some* $x_0 \in X$, $S^*(x_0)$ *is compact and for each* $x \in X$, $S^*(x)$ *is closed in* X;

(c) *for each* $x \in X$, $T(x)$ *is* H*-convex.*

Then there exists $\hat{x} \in X$ *such that* $\hat{x} \in T(\hat{x})$.

Corollary 4 generalizes Corollary 2.1 of Tarafdar in [27] to a non-convex setting and Theorem $2'$ of Horvath in [16] to a non-compact setting.

4. Minimax inequalities

Throughout this section, X denotes a topological space and $h: X \times X \to \mathbb{R}$ denotes a fixed real-valued function. For each $(x, r) \in X \times \mathbb{R}$, let $H(x, r) = \{y \in X: h(y, x) \leq r\}$. We shall assume that the function h has the following property: For each $A \in \mathscr{F}(X)$, the set $F_A = \bigcap\{H(x, r): A \subset H(x, r)$ and $(x, r) \in X \times \mathbb{R}\}$ is contractible. Clearly, we have $F_A \subset F_{A'}$, whenever $A \subset A'$. Hence $(X, \{F_A\})$ becomes an H-space.

THEOREM 13. *Let* $f, g: X \times X \to \mathbb{R}$ *be such that*

(i) $g(x, y) \leq f(x, y)$ *for each* $(x, y) \in X \times X$;

(ii) *for each* $y, z \in X$ *and for each* $A \in \mathscr{F}(X)$, *if* $f(z, y) < f(x, y)$ *for each* $x \in A$, *then there exists* $w \in X$ *such that* $h(x, w) < h(z, w)$ *for each* $x \in A$;

(iii) *for each fixed* $x \in X$ *and for each* $A \in \mathscr{F}(X)$, $g(x, y)$ *is a lower semi-continuous function of* y *on* F_A.

For any $\lambda \in \mathbb{R}$, if there exist a non-empty compact subset L of X and $x_0 \in L$ such that

(iv) $g(x_0, y) > \lambda$ *for all* $y \in X \backslash L$,

(v) $g(x, y)$ *is also a lower semi-continuous function of* y *on* L,

then either there exists $\hat{y} \in L$ such that $g(x, \hat{y}) \leq \lambda$ for all $x \in X$ or there exists $\hat{x} \in X$ such that $f(\hat{x}, \hat{x}) > \lambda$.

PROOF. Suppose $f(x, x) \leq \lambda$ for all $x \in X$. For each $x \in X$, let

$$F(x) = \{y \in X : f(x, y) \leq \lambda\} \quad \text{and} \quad G(x) = \{y \in X : g(x, y) \leq \lambda\}.$$

(a) For each $x \in X$, $F(x) \subset G(x)$ by (i) and $x \in F(x)$ by the assumption.

(b) Suppose $A \subset F^*(y)$ for some $y \in X$, then $A \cap F^1(y) = \emptyset$ so that for any fixed $z \in F^{-1}(y)$,

$$f(z, y) \leq \lambda < f(a, y) \quad \text{for all } a \in A;$$

by (ii), there exists $w \in X$ such that

$$h(a, w) < h(z, w) \quad \text{for all } a \in A.$$

Choose $r_0 \in \mathbb{R}$ such that $h(a, w) \leq r_0 < h(z, w)$ for all $a \in A$; then $A \subset H(w, r_0)$ and $z \notin H(w, r_0)$ so that $z \in F_A$ for any $z \in F^{-1}(y)$. It follows that $F_A \subset F^*(y)$. Thus $F^*(y)$ is H-convex for each $y \in X$.

(c) By (iv), $G(x_0) \subset L$ and by (v) $G(x_0)$ is closed in L; thus $G(x_0)$ is compact. Moreover, for each $x \in X$, $G(x) \cap L$ is closed in L by (v) so that $G(x_0) \cap G(x) = G(x_0) \cap (G(x) \cap L)$ is closed in $G(x_0)$.

(d) By (iii), for each $x \in X$ and for each $A \in \mathcal{F}(X)$, $F_A \cap G(x)$ is closed in F_A.

Therefore all hypotheses of Theorem 5 are satisfied. By Theorem 5, $\bigcap_{x \in X} G(x) \neq \emptyset$. Let $\hat{y} \in \bigcap_{x \in X} G(x)$. Then $\hat{y} \in L$ as $G(x_0) \subset L$ and $g(x, \hat{y}) \leq \lambda$ for all $x \in X$.

Theorem 13 generalizes Proposition 1 of Horvath in [16] to non-compact topological spaces and hence also generalizes the corresponding results of Ben-El-Mechaiekh, Deguire and Granas in [4] and of Fan in [12].

COROLLARY 5. *Let $\phi, \psi : X \times X \to \mathbb{R}$ be such that*

(i) $\phi \leq \psi$ *on the diagonal* $\Delta = \{(x, x) : x \in X\}$ *and* $\phi \geq \psi$ *on* $(X \times X) \backslash \Delta$;

(ii) *for each fixed* $x \in X$, $y \to \phi(y, y) - \phi(x, y)$ *is lower semi-continuous on* X;

(iii) *for each* $y, z \in X$ *and for each* $A \in \mathcal{F}(X)$, *if* $\psi(a, y) < \psi(z, y)$ *for all* $a \in A$, *then there exists* $w \in X$ *such that* $h(a, w) < h(z, w)$ *for all* $a \in A$;

https://doi.org/10.1017/S1446788700030275 Published online by Cambridge University Press

(iv) *there exist a non-empty compact subset* L *of* X *and* $x_0 \in L$ *such that* $\phi(y, y) > \phi(x_0, y)$ *for all* $y \in X \setminus L$.

Then there exists $\hat{y} \in L$ *such that* $\phi(\hat{y}, \hat{y}) \leq \phi(x, \hat{y})$ *for all* $x \in X$.

Proof. Define $f, g: X \times X \to \mathbb{R}$ by

$$f(x, y) = \psi(y, y) - \psi(x, y), \qquad g(x, y) = \phi(y, y) - \phi(x, y).$$

Then f and g satisfy the hypotheses of Theorem 13 with $\lambda = 0$ and $f(x, x) = 0$ for all $x \in X$. By Theorem 13 there exists $\hat{y} \in L$ such that $g(x, \hat{y}) \leq 0$ for all $x \in X$; that is, $\phi(\hat{y}, \hat{y}) \leq \phi(x, \hat{y})$ for all $x \in X$.

The above result generalizes Proposition 2 of Horvath in [16] and Theorem 1 of Shih and Tan in [21] which in turn generalizes Corollary 1 of Fan in [12].

Corollary 6. *Let* $a: X \to \mathbb{R}$ *and* $f, g: X \times X \to \mathbb{R}$ *be such that*

(i) *for each* $r \in \mathbb{R}$, *the set* $\{y \in X: a(y) \leq r\}$ *is empty or contractible*;

(ii) $g(x, y) \leq f(x, y)$ *for all* $x, y \in X$;

(iii) *for* $x, y, z \in X$, *if* $f(z, y) < f(x, y)$, *then* $a(x) < a(z)$;

(iv) *for each fixed* $x \in X$ *and for any* $r \in \mathbb{R}$, $g(x, y)$ *is a lower semi-continuous function of* y *on* $\{y \in X: a(y) \leq r\}$.

For any $\lambda \in \mathbb{R}$, *if there exist a non-empty compact subset* L *of* X *and* $x_0 \in L$ *such that*

(v) $g(x_0, y) > \lambda$ *for all* $y \in X \setminus L$.

(vi) $g(x, y)$ *is also a lower semi-continuous function of* y *on* L,

then either there exists $\hat{y} \in L$ *such that* $g(x, \hat{y}) \leq \lambda$ *for all* $x \in X$ *or there exists* $\hat{x} \in X$ *such that* $f(\hat{x}, \hat{x}) > \lambda$.

Proof. Define $h: X \times X \to \mathbb{R}$ by $h(x, y) = a(x)$; then for each $A \in \mathcal{F}(X)$, $F_A = \bigcap\{H(x, r): A \subset H(x, r) \text{ and } (x, r) \in X \times \mathbb{R}\} = \bigcap\{\{y \in X: a(y) \leq r\}: A \subset \{y \in X: a(y) \leq r\} \text{ and } r \in \mathbb{R}\} = \{y \in X: a(y) \leq \bar{r}\}$ where $\bar{r} = \inf\{r \in \mathbb{R}: A \subset \{y \in X: a(y) \leq r\}$. Thus Theorem 13 can be applied to obtain the desired conclusion.

Corollary 6 generalizes Proposition 3 of Horvath in [16] to a non-compact setting.

Theorem 14. *Let* $f, g: X \times X \to \mathbb{R}$ *be such that*

(a) $g(x, y) \leq f(x, y)$ *for each* $x, y \in X$;

(b) *for each fixed* $x \in X$, $g(x, y)$ *is a lower semi-continuous function of* y *on* C *for each non-empty compact subset* C *of* X;

https://doi.org/10.1017/S1446788700030275 Published online by Cambridge University Press

(c) *for each* $y, z \in X$ *and for each* $A \in \mathscr{F}(X)$, *if* $f(z, y) < f(x, y)$ *for each* $x \in A$, *then there exists* $w \in X$ *such that* $h(x, w) < h(z, w)$ *for each* $x \in A$.

For any $\lambda \in \mathbb{R}$, *if there exist a non-empty subset* X_0 *of* X *and a non-empty compact subset* K *of* X *such that for each* $B \in \mathscr{F}(X)$, *there is a compact weakly* H*-convex subset* C_B *of* X *having the following properties*:

(d) $X_0 \cup B \subset C_B$;

(e) *for each* $y \in C_B \backslash K$, *there is* $x \in C_B$ *such that* $g(x, y) > \lambda$,

then either there exists $\hat{y} \in K$ *such that* $g(x, \hat{y}) \leq \lambda$ *for all* $x \in X$ *or there exists* $\hat{x} \in X$ *such that* $f(\hat{x}, \hat{x}) > \lambda$).

PROOF. Suppose $f(x, x) \leq \lambda$ for all $x \in X$. For each $x \in X$, let

$$K(x) = \{y \in K : g(x, y) \leq \lambda\};$$

then $K(x)$ is closed in K by (b). Let $B \in \mathscr{F}(X)$ be given. By hypotheses, there exists a compact weakly H-convex subset C_B of X satisfying (d) and (e).

Now for each $x \in C_B$, let

$$F(x) = \{y \in C_B : f(x, y) \leq \lambda\}, \quad G(x) = \{y \in C_B : g(x, y) \leq \lambda\}.$$

Then we have

(i) for each $x \in C_B$, $F(x) \subset G(x)$ by (a) and $x \in F(x)$ by assumption;

(ii) since C_B is weakly H-convex, $(C_B, \{F_A \cap C_B\})$ is also an H-space; let $A \in \mathscr{F}(C_B)$ be an arbitrarily given set such that $A \subset F^*(y)$; then $A \cap F^{-1}(y) = \emptyset$ so that for any fixed $z \in F^{-1}(y)$, $f(z, y) \leq \lambda < f(a, y)$ for all $a \in A$. By (c), there is $w \in X$ such that $h(a, w) < h(z, w)$ for all $a \in A$. Choose $r_0 \in \mathbb{R}$ such that $h(a, w) \leq r_0 < h(z, w)$ for all $a \in A$; then $A \subset H(w, r_0)$ and $z \notin H(w, r_0)$ so that $z \notin F_A$ for all $z \in F^{-1}(y)$. It follows that $F_A \cap C_B \subset F^*(y)$ and hence $F^*(y)$ is H-convex for each $y \in C_B$.

(iii) by (b), for each $x \in C_B$, $G(x)$ is closed in C_B and is therefore also compact.

By Theorem 5 with $X = C_B$, $\bigcap_{x \in C_B} G(x) \neq \emptyset$. In other words, there exists a point $y_0 \in C_B$ such that $g(x, y_0) \leq \lambda$ for all $x \in C_B$. By (e), we must have $y_0 \in K$ so that $y_0 \in \bigcap_{x \in B} K(x)$ by (d). This shows that $\{K(x) : x \in X\}$ has the finite intersection property. By the compactness of K, we have $\bigcap_{x \in X} K(x) \neq \emptyset$. Take any $\hat{y} \in \bigcap_{x \in X} K(x)$, then $\hat{y} \in K$ and $g(x, \hat{y}) \leq \lambda$ for all $x \in X$. This completes the proof.

As an immediate consequence of Theorem 14, we obtain the following very general minimax inequality in a topological vector space.

Theorem 15. *Let X be a non-empty convex subset in a topological vector space E. Let $f, g: X \times X \to \mathbb{R}$ be such that*

(a) $g(x, y) \leq f(x, y)$ *for each* $x, y \in X$;

(b) *for each fixed* $x \in X$, $g(x, y)$ *is a lower-semicontinuous function of* y *on* C *for each non-empty compact subset* C *of* X.

For any $\lambda \in \mathbb{R}$, *if*

(c) *for each fixed* $y \in X$, *the set* $\{x \in X: F(x, y) > \lambda\}$ *is convex,*

(d) *there exist a non-empty compact convex subset* X_0 *of* X *and a non-empty compact subset* K *of* X *such that for each* $y \in X \backslash K$, *there exists* $x \in \operatorname{co}(X_0 \cup \{y\})$ *such that* $g(x, y) > \lambda$,

then either there exists $\hat{y} \in K$ *such that* $g(x, \hat{y}) \leq \lambda$ *for all* $x \in X$ *or there exists* $\hat{x} \in X$ *such that* $f(\hat{x}, \hat{x}) > \lambda$.

Proof. For each $(x, y) \in X \times X$, let $h(x, y) = -f(x, y)$; then we have

$$H(y, r) = \{x \in X: h(x, y) \leq r\} = \{x \in X: f(x, y) \geq -r\}.$$

By (c), for each $y \in X$ and for each $r \in \mathbb{R}$, $H(y, r)$ is convex, so that for each $A \in \mathscr{F}(X)$, $F_A = \bigcap\{H(y, r): A \subset H(y, r) \text{ and } (y, r) \in X \times \mathbb{R}\}$ is convex and hence F_A is a non-empty contractible subset of X. Thus $(X, \{F_A\})$ is an H-space. For each $B \in \mathscr{F}(X)$ let $C_B = \operatorname{co}(X_0 \cup B)$. It is easy to see that all hypotheses of Theorem 14 are satisfied so that the conclusion follows.

Theorem 15 is equivalent to a minimax inequality of Bae, Kim and Tan [2, Theorem 1] which in turn generalizes minimax inequalities of Tan [26, Theorem 1], Allen [1, Theorem 2], Yen [28, Theorem 1] and Fan [13, Theorem 6]. For applications of Theorem 15 to variational inequalities and fixed point theorems, we refer to Bae, Kim and Tan [2].

We now observe the following.

Lemma 5. *Let $(Y, \{F_A\})$ be an H-space, X be a non-empty subset of Y, $\psi: X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ and $\alpha \in \mathbb{R}$.*

(1) *If* $\psi(x, x) \leq \alpha$ *for all* $x \in X$ *and for each* $y \in Y$, *the set* $\{x \in X: \psi(x, y) > \alpha\}$ *is H-convex, then for each* $A \in \mathscr{F}(X)$ *and for each* $y \in F_A$, $\min_{x\in A} \psi(x, y) \leq \alpha$.

(2) *If* $\psi(x, x) \leq \alpha$ *for all* $x \in X$, *define* $F: X \to 2^Y$ *by* $F(x) = \{y \in Y: \psi(x, y) \leq \alpha\}$ *for all* $x \in X$. *Then* F *is an H-KKM map if and only if for each* $A \in \mathscr{F}(x)$ *and for each* $y \in F_A$, $\min_{x\in A} \psi(x, y) \leq \alpha$.

Proof. (1) Let $A \in \mathscr{F}(X)$ and $y \in F_A$ be given. Suppose $\min_{x\in A} \psi(x, y) > \alpha$; then $A \subset \{x \in X: \psi(x, y) > \alpha\}$ so that by assumption $F_A \subset \{x \in$

https://doi.org/10.1017/S1446788700030275 Published online by Cambridge University Press

$X: \psi(x, y) > \alpha\}$. As $y \in F_A$, it follows that $\psi(y, y) > \alpha$ which is a contradiction. Hence we must have $\min_{x \in A} \psi(x, y) \leq \alpha$.

(2) Suppose F is H-KKM. Let $A \in \mathscr{F}(X)$ and $y \in F_A$; as $y \in F_A \subset \bigcup_{x \in A} F(x)$, we must have $\psi(x, y) \leq \alpha$ for some $x \in A$ and hence

$$\min_{x \in A} \psi(x, y) \leq \alpha.$$

Conversely, if F is not H-KKM, then there exists $A \in \mathscr{F}(X)$ such that $F_A \not\subset \bigcup_{x \in A} F(x)$. Let $y \in F_A$ be such that $y \notin \bigcup_{x \in A} F(x)$; it follows that $\psi(x, y) > \alpha$ for all $x \in A$ so that $\min_{x \in A} \psi(x, y) > \alpha$.

We remark here that the condition "for each $A \in \mathscr{F}(X)$ and for each $y \in F_A$, $\min_{x \in A} \psi(x, y) \leq \alpha$" is a generalization of the notion "*α-DQCV in* x" introduced by Zhou and Chen in [29].

As an application of Theorem 8, we present another very general minimax inequality:

THEOREM 16. *Let $(Y, \{F_A\})$ be an H-space, X be a non-empty subset of Y, $\phi: X \times Y \to \mathbb{R} \cup \{\pm, \infty\}$ and $\alpha \in \mathbb{R}$ be such that*

(a) *for each fixed $x \in X$, $\phi(x, y)$ is a lower semi-continuous function of y on C for each non-empty compact subset C of Y;*

(b) *for each $A \in \mathscr{F}(X)$ and for each $y \in F_A$, $\min_{x \in A} \phi(x, y) \leq \alpha$;*

(c) *there exists a non-empty subset X_0 of X which is H-compact in X such that the set $\{y \in X: \phi(x, y) \leq \alpha$ for all $x \in X_0\}$ is compact.*

Then either there exists a point $\hat{y} \in Y$ such that $\phi(x, \hat{y}) \leq \alpha$ for all $x \in X$ or there exists a point $\hat{x} \in X$ such that $\phi(\hat{x}, \hat{x}) > \alpha$.

PROOF. Suppose $\phi(x, x) \leq \alpha$ for all $x \in X$. Define $F: X \to 2^Y$ by $F(x) = \{y \in Y: \phi(x, y) \leq \alpha\}$ for each $x \in X$. Then by (b) and Lemma 5, F is an H-KKM map and by (a), for each $x \in X$, $F(x)$ is compactly closed in Y and by (c), $\bigcap_{x \in X_0} F(x)$ is compact. Thus by Theorem 8, $\bigcap_{x \in X} F(x) \neq \varnothing$. Take any $\hat{y} \in \bigcap_{x \in X} F(x)$; then $\phi(x, y) \leq \alpha$ for all $x \in X$.

As an application of Theorem 16, we have the following new minimax inequality:

THEOREM 17. *Let $(Y, \{F_A\})$ be an H-space, X be a non-empty subset of Y, ϕ, $\psi: X \times Y \to \mathbb{R} \cup \{\pm, \infty\}$ and $\alpha \in \mathbb{R}$ be such that*

(a) *$\phi(x, y) \leq \psi(x, y)$ for all $(x, y) \in X \times Y$;*

(b) *for each fixed $x \in X$, $\phi(x, y)$ is a lower semi-continuous function of y on C for each non-empty compact subset C of Y;*

(c) *for each fixed $y \in Y$, the set $\{x \in X: \psi(x, y) > \alpha\}$ is H-convex;*

(d) *there exists a non-empty subset* X_0 *of* X *which is* H*-compact in* X *such that the set* $\{y \in Y : \phi(x, y) \leq \alpha$ *for all* $x \in X_0\}$ *is compact.*

Then either there exists a point $\hat{y} \in Y$ *such that* $\phi(x, \hat{y}) \leq \alpha$ *for all* $x \in X$ *or there exists a point* $\hat{x} \in X$ *such that* $\psi(\hat{x}, \hat{x}) > \alpha$.

PROOF. Suppose $\psi(x, x) \leq \alpha$ for all $x \in X$. Then by (c) and Lemma 5, for each $A \in \mathscr{F}(X)$ and for each $y \in F_A$, $\min_{x \in A} \psi(x, y) \leq \alpha$, so that by (a), $\min_{x \in A} \phi(x, y) \leq \alpha$. Hence by (a) and Theorem 16, there exists $\hat{y} \in Y$ such that $\phi(x, \hat{y}) \leq \alpha$ for all $x \in X$.

Even when Y is a subset of a topological vector space, Theorem 17 generalizes a minimax inequality of Takahashi [25, Theorem 3] in several ways.

References

[1] G. Allen, 'Variational inequalities, complementarity problems, and duality theorems', *J. Math. Anal. Appl.* **58** (1977), 1–10.

[2] J. S. Bae, W. K. Kim and K. K. Tan, 'Another generalization of Ky Fan's minimax inequality and its applications', submitted.

[3] C. Bardaro and R. Ceppitelli, 'Some further generalizations of Knaster-Kuratowski-Mazurkiewicz theorem and minimax inequalities', *J. Math. Anal. Appl.* **132** (1988), 484–490.

[4] H. Ben-El-Mechaiekh, P. Deguire and A. Granas, 'Points fixes et coincidences pour les fonctions multivoques II. (Applications de type ϕ et ϕ^*)', *C. R. Acad. Sci. Paris Sér. I Math.* **295** (1982), 381–384.

[5] H. Brézis, L. Nirenberg and G. Stampacchia, 'A remark on Ky Fan's minimax principle', *Bull. Un. Mat. Ital.* (4) **6** (1972), 293–300.

[6] F. E. Browder, 'Coincidence theorems, minimax theorems, and variational inequalities', *Contemp. Math.* **26** (1984), 67–80.

[7] X. P. Ding, W. K. Kim and K. K. Tan, 'A new minimax inequality on H-spaces with applications', to appear in *Bull. Austral. Math. Soc.*

[8] X. P. Ding and K. K. Tan, 'Fixed point theorems and minimax inequalities without convexity', submitted.

[9] ——, 'Minimax inequality of von Neumann type and system of inequalities without convexity', submitted.

[10] ——, 'Non-convex generalizations of results on sets with convex sections', submitted.

[11] K. Fan, 'A generalization of Tychonoff's fixed point theorem', *Math. Ann.* **142** (1961), 305–310.

[12] ——, 'A minimax inequality and applications', *Inequalities*, Vol. III, (edited by O. Shisha), pp. 103–113, (Academic Press, New York, 1972).

[13] ——, 'Some properties of convex sets related to fixed point theorems', *Math. Ann.* **226** (1984), 519–537.

[14] C. Horvath, 'Point fixes et coincidences pur les applications multivoques sans convexité', *C. R. Acad. Sci. Paris* **296** (1983), 403–406.

[15] ——, 'Point fixes et coincidences dans les espaces topologiques compacts contractiles', *C. R. Acad. Sci. Paris* **299** (1984), 519–521.

[16] ——, 'Some results on multivalued mappings and inequalities without convexity', *Nonlinear and convex analysis*, edited by B. L. Lin and S. Simons, pp. 99–106, (Marcel Dekker, 1987).
[17] W. K. Kim, 'Some applications of the Kakutani fixed point theorem', *J. Math. Anal. Appl.* **121** (1987), 119–122.
[18] B. Knaster, C. Kuratowski and S. Mazurkiewicz, 'Ein Beweis des Fixpunktsatzes für n-dimensionale Simplexe', *Fund. Math.* **14** 91929), 132–137.
[19] H. M. Ko and K. K. Tan, 'Coincidence theorems and matching theorems', submitted.
[20] M. H. Shih and K. K. Tan, 'A further generalization of Ky Fan's minimax inequality and its applications', *Studia Math.* **78** (1984), 279–287.
[21] ——, 'The Ky Fan minimax principle, sets with convex sections, and variational inequalities', *Differential geometry, calculus of variations, and their applications*, edited by G. M. Rassias and T. M. Rassias, pp. 471–481, (Marcel Dekker, 1985).
[22] ——, 'Shapley selections and covering theorems of simplexes', *Nonlinear and convex analysis*, edited by B. L. Lin and S. Simons, pp. 245–251, (Marcel Dekker, 1987).
[23] ——, 'A minimax inequality and Browder-Hartman-Stampacchia variational inequalitiés for multi-valued monotone operators', *Proc. Fourth Franco-SEAMS Conference*, (Chiang Mai, Thailand, 1988).
[24] ——, 'A geometric property of convex sets with applications to minimax type inequalities and fixed point theorems', *J. Austral. Math. Soc. Ser. A* **45** (1988), 169–183.
[25] W. Takahashi, 'Fixed point, minimax, and Hahn-Banach theorems', *Nonlinear functional analysis and its applications*, Part 2, pp. 419–427, (Proc. Sympos. Pure Math., vol. 45, 1986).
[26] K. K. Tan, 'Comparison theorems on minimax inequalities, variational inequalities, and fixed point theorems', *J. London Math. Soc.* **28** (1983), 555–562.
[27] E. Tarafdar, 'On minimax principles and sets with convex sections', *Publ. Math. Debrecen* **29** (1982), 219–226.
[28] C. L. Yen, 'A minimax inequality and its applications to variational inequalities', *Pacific J. Math.* **97** (1981), 477–481.
[29] J. X. Zhou and G. Chen, 'Diagonal convexity conditions for problems in convex analysis and quasi-variational inequalities', *J. Math. Anal. Appl.* **132** (1988), 213–225.

Depatment of Mathematics
Sichuan Normal University
Chengdu, Sichuan
People's Republic of China

Department of Mathematics,
Statistics and Computing Science
Dalhousie University
Halifax, Nova Scotia
Canada, B3H 3J5