A property of Cesàro-Perron integrals

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1. It is well known[1] that *if* $f(t)$ *is* (*a*) *integrable in the Lebesgue sense, or more generally* (*b*) *integrable in the Perron sense, over every interval* (α, β) *interior to* (a, b), *and if*

$$\lim_{\substack{\alpha \to a+0 \\ \beta \to b-0}} \int_\alpha^\beta f(t)\, dt \tag{1·1}$$

exists, then $f(t)$ *is integrable in the Perron sense over* (a, b) *to the value* (1·1).

The first result expresses the fact that a function integrable in the Cauchy-Lebesgue sense is also integrable in the Perron sense, while the second is fundamental in the proof[2] of the equivalence of the Special Denjoy and Perron integrals. These results have been extended by Miss M. E. Grimshaw[3] to the case when Perron integrability is replaced by Cesàro-Perron integrability of *unit order*[4] and the limits by Cesàro limits. Miss Grimshaw's theorem and its generalisation for Cesàro-Perron integrals of *any positive order*[5] have important applications. The object of this note is to prove the general case[6].

2. *Definition. If* $\lambda > 0$ *we say that* $f(t) \to l\ (C, \lambda)$ *as* $t \to a$, *or that* $C_\lambda\text{-}\lim_{t \to a} f(t) = l$, *if* (i) $f(t)$ *is integrable* $C_\mu P$ *in an interval* $(a - \eta, a + \eta)$, *where* $\mu = \max(\lambda - 1, 0)$, *and* (ii)

$$C_\lambda(f, a, a+h) = \frac{\lambda}{h} \underset{(C_\mu P)}{\int_0^h} \left(1 - \frac{u}{h}\right)^{\lambda - 1} f(a+u)\, du \tag{2·1}$$

tends to l *as* $h \to 0$. *The usual related notation is defined in the obvious way.*

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[1] See Saks, **8**, 247.

[2] See Saks, **8**, 247-252.

[3] Grimshaw, **7**.

[4] Burkill, **3**.

[5] Burkill, **3**, **4**, **6**.

[6] I am indebted to Dr John Todd who, after I had proved Theorem A, drew my attention to Miss Grimshaw's paper.

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THEOREM A. *If* $\lambda \geqq 0$ *and* (i) $f(t)$ *is integrable* $C_\lambda P$ *in* (α, β) *for every* α, β *such that* $a < \alpha < \beta < b$, (ii)

$$\underset{\substack{\alpha \to a+0 \\ \beta \to b-0 \\ (C_\lambda P)}}{C_\lambda\text{-lim}} \int_\alpha^\beta f(t)\, dt \tag{2·2}$$

exists, then $f(t)$ *is integrable* $C_\lambda P$ *in* (a, b) *to the value* (2·2).

PROOF. We need only consider the case $\lambda > 0$. Let

$$F(x) = \underset{\substack{\alpha \to a+0 \\ (C_\lambda P)}}{C_\lambda\text{-lim}} \int_\alpha^x f(t)\, dt \quad (a < x < b)$$

$$F(a) = 0$$

$$F(b) = \underset{x \to b-0}{C_\lambda\text{-lim}}\ F(x).$$

We shall show that, if ϵ is a given positive number, there is a function $M(x)$ such that

(1) $M(a) = 0$,

and, for[1] $a \leqq x \leqq b$,

(2) $M(x)$ is C_λ-continuous[2],

(3) $0 \leqq M(x) - F(x) < \epsilon$,

(4) $C_\lambda D_* M(x) \geqq f(x)$,

(5) $C_\lambda D_* M(x) > -\infty$,

where

$$C_\lambda D_* M(x) = \varliminf_{h \to 0} \frac{C_\lambda(M, x, x+h) - M(x)}{h/(\lambda+1)}.$$

We shall call $M(x)$ a $C_\lambda P$ *ϵ-major function for* $f(x)$ *in* (a, b), *approximating to* $F(x)$.

A similar argument will show the existence of a $C_\lambda P$ ϵ-minor function $m(x)$ for $f(x)$ in (a, b), approximating to $F(x)$.

This will show that $f(t)$ is integrable $C_\lambda P$ in (a, b), and that, for $a \leqq x \leqq b$,

$$(C_\lambda P) \int_a^x f(t)\, dt = F(x).$$

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[1] It is to be understood that conditions (2), (4) and (5) need only hold in a *one-sided* sense at $x = a$ and $x = b$.

[2] *i.e.* $C_\lambda(M, x, x+h) \to M(x)$ as $h \to 0$.

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Let numbers x_n be chosen so that

(i) $a < \ldots . < x_{-2} < x_{-1} < x_0 < x_1 < x_2 < \ldots . < b$,

(ii) $x_n \to a$ as $n \to -\infty$,

(iii) $x_n \to b$ as $n \to \infty$.

Let $M_n(x)$ be a $C_\lambda P$ $2^{-|n|-4}\epsilon$-major function for $f(x)$ in (x_{n-1}, x_n), approximating to $F(x) - F(x_{n-1})$, and write

$$D_n(x) = M_n(x) - F(x) + F(x_{n-1}).$$

Let

$$M_0(x) = F(x) + \sum_{\nu=-\infty}^{n-1} D_\nu(x_\nu) + D_n(x), \quad \begin{pmatrix} x_{n-1} \leqq x < x_n, \\ -\infty < n < \infty \end{pmatrix},$$

$$M_0(a) = 0,$$

$$M_0(b) = F(b) + \sum_{\nu=-\infty}^{\infty} D_\nu(x_\nu).$$

Then $M_0(x)$ is C_λ-continuous in $a < x < b$. Also, if $n \leqq 0$ and $x_{n-1} \leqq x < x_n$,

$$0 \leqq M_0(x) - F(x) < \sum_{\nu=-\infty}^{n} 2^{-|\nu|-4}\epsilon = 2^{-|n|-3}\epsilon,$$

which tends to zero as $n \to -\infty$ (*i.e.* as $x \to a+0$), while, if $n > 0$ and $x_{n-1} \leqq x < x_n$,

$$0 \leqq M_0(b) - F(b) - M_0(x) + F(x)$$
$$< \sum_{\nu=n}^{\infty} 2^{-|n|-4}\epsilon = 2^{-|n|-3}\epsilon,$$

which tends to zero as $n \to \infty$ (*i.e.* as $x \to b-0$). Thus $M_0(x) - F(x)$ is continuous, relative to (a, b), at $x = a$ and $x = b$, and hence it follows from (2·2) that $M_0(x)$ is C_λ-continuous in $a \leqq x \leqq b$. So $M_0(x)$ satisfies (1) and (2), and it is easily seen that (3) is satisfied with $\frac{1}{4}\epsilon$ in place of ϵ, while (4) and (5) are satisfied except possibly at $x = a$ and $x = b$[1].

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[1] If in the definition of a $C_\lambda P$ integral an exceptional null set S is allowed in condition (4) and an exceptional enumerable set E in condition (5), the rest of the proof, which shows explicitly how to "remove" an exceptional point, may be omitted. Burkill, **3**, 317, **5**, 46, **6**, 223, showed, in the case $0 < \lambda \leqq 1$, that such a definition is possible, but did not show explicitly (in the case of the set E) that the scope of the integral is not increased thereby. In the general case the set S is easily reduced to the set E by the addition to $M(x)$ of an appropriate function with infinite derivates at the points of S [Burkill, **2**, 274, **3**, 317], while the set E may be removed by repetition of the argument given here [or by repetition of Miss Grimshaw's argument in the case $\lambda = 1$]. Thus it may be shown that the scope of the $C_\lambda P$ integral ($\lambda > 0$) is not increased by the general definition. I am indebted to Miss W. L. C. Sargent for helping me to clear up this point.

Let $z_1(x)$, $z_2(x)$ be defined so that (i) $z_1(a) = 0$, (ii) $z_1(x)$ is continuous and steadily increasing for $a \leqq x \leqq b$, (iii) $z_1(b) < \frac{1}{4}\epsilon$, (iv) $z_1(x) \geqq (\lambda+1)\,|\,C_\lambda(M_0, a, x)\,|$ in some interval $(a, a+\eta_1)$, (v) $z_1(x)/(x-a)$ is steadily decreasing in $(a, a+\eta_1)$, while (vi) $z_2(a) = 0$, (vii) $z_2(x)$ is continuous and steadily increasing in $a \leqq x \leqq b$, (viii) $0 < z_2(b) < \frac{1}{4}\epsilon$, (ix) $z_2(x) \leqq z_2(b) - (\lambda+1)\,|\,C_\lambda(M_0, b, x) - M_0(x)\,|$ in some interval $(b-\eta_2, b)$, (x) $\{z_2(x) - z_2(b)\}/(x-b)$ is steadily increasing in $(b-\eta_2, b)$[1].

Then, for $0 < h < \eta_1$, it follows from (v) that

$$\frac{C_\lambda(z_1, a, a+h)}{h/(\lambda+1)} = \frac{\lambda(\lambda+1)}{h^2}\int_0^h \left(1 - \frac{u}{h}\right)^{\lambda-1} z_1(a+u)\,du$$

$$\geqq \frac{\lambda(\lambda+1)\,z_1(a+h)}{h^3}\int_0^h \left(1 - \frac{u}{h}\right)^{\lambda-1} u\,du$$

$$= \frac{z_1(a+h)}{h}.$$

and hence from (iv) that

$$\varliminf_{h \to +0} \frac{C_\lambda(M_0 + z_1, a, a+h)}{h/(\lambda+1)} \geqq \varliminf_{h \to +0} \left\{\frac{z_1(a+h)}{h} + \frac{C_\lambda(M_0, a, a+h)}{h/(\lambda+1)}\right\}$$

$$\geqq \varliminf_{h \to +0} \frac{(\lambda+1)}{h}\{\,|\,C_\lambda(M_0, a, a+h)\,| + C_\lambda(M_0, a, a+h)\}$$

$$\geqq 0,$$

while, for $-\eta_2 < h < 0$, it follows from (x) that

$$\frac{C_\lambda(z_2, b, b+h) - z_2(b)}{h/(\lambda+1)} = \frac{\lambda(\lambda+1)}{h^2}\int_0^h \left(1 - \frac{u}{h}\right)^{\lambda-1} \{z_2(b+u) - z_2(b)\}\,du$$

$$\geqq \frac{z_2(b+h) - z_2(b)}{h},$$

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[1] For instance, if η_1 is chosen so that $(\lambda+1)\,|\,C_\lambda(M_0, a, x)\,| < \frac{1}{4}\epsilon$ in $(a, a+\eta_1)$, we may take

$$z_1(x) = (\lambda+1)(x-a) \overline{\text{bound}}_{x \leqq t \leqq a+\eta_1} \left\{\frac{1}{t-a} \overline{\text{bound}}_{a \leqq u \leqq t} \,|\, C_\lambda(M_0, a, u)\,|\right\}$$

in $(a, a+\eta_1)$, and $z_1(x)$ constant in $(a+\eta_1, b)$, while if η_2 is chosen so that

$$(\lambda+1)\,|\,C_\lambda(M_0, b, x) - M_0(b)\,| < z_2(b)$$

in $(b-\eta_2, b)$, where $z_2(b)$ satisfies (viii), we may take

$$z_2(x) = z_2(b) - (\lambda+1)(b-x) \overline{\text{bound}}_{b-\eta_2 \leqq t \leqq x} \frac{1}{b-t}\left\{\overline{\text{bound}}_{t \leqq u \leqq b} \,|\, C_\lambda(M_0, b, u) - M_0(b)\,|\right\}$$

in $(b-\eta_2, b)$, and $z_2(x)$ linear in $(a, b-\eta_2)$.

and hence from (ix) that

$$\underline{\lim}_{h \to -0} \frac{C_\lambda(M_0 + z_2, b, b+h) - M_0(b) - z_2(b)}{h/(\lambda+1)}$$

$$\geqq \underline{\lim}_{h \to -0} \left\{ \frac{C_\lambda(M_0, b, b+h) - M_0(b)}{h/(\lambda+1)} + \frac{z_2(b+h) - z_2(b)}{h} \right\}$$

$$\geqq \underline{\lim}_{h \to -0} \frac{(\lambda+1)}{h} \{C_\lambda M_0, b, b+h) - M_0(b) - |C_\lambda(M_0, b, b+h) - M_0(b)|\}$$

$$\geqq 0.$$

If we now define

$$M(x) = M_0(x) + z_1(x) + z_2(x) + \tfrac{1}{8}\epsilon\left(\frac{x-a}{b-a}\right)^{\frac{1}{2}} + \tfrac{1}{8}\epsilon\left\{1 - \left(\frac{b-x}{b-a}\right)^{\frac{1}{2}}\right\}$$

for $a \leqq x \leqq b$, we see that $M(x)$ satisfies conditions (1)—(5).

3. *Applications.*

I. Theorem A has been used elsewhere[1] to prove that a function integrable in the $C_\lambda L$ sense (Cesàro-Lebesgue, $\lambda \geqq 0$) is also integrable in the $C_\lambda P$ sense, *i.e.* that the $C_\lambda L$ integral is included in the $C_\lambda P$ integral.

II. Miss Sargent has recently defined[2] a $C_\lambda D$ integral (Special Cesàro-Denjoy, $\lambda =$ integer), and used Theorem A in proving that it is equivalent to the $C_\lambda P$ integral.

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[1] Bosanquet, 1.

[2] Sargent, **9**.

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