Can. J. Math., Vol. XXVII, No. 2, 1975, pp. 261–270

INDUCED REPRESENTATIONS OF RINGS

MARC A. RIEFFEL

At the beginning of the chapter on induced representations in the treatise of Curtis and Reiner [**8**] on representation theory, they write "Most of the results have not yet found suitable generalization to rings with minimum condition or finite dimensional algebras, ...". The purpose of this paper is to indicate how some of the more basic theorems concerning induced representations can, in fact, be generalized to rings and algebras. In most cases we can do this by bringing together known results, so that in this sense this paper does not contain substantially new results. Perhaps of greatest interest is the connection we indicate in § 2 between the imprimitivity theorem and what have come to be called "the Morita theorems". In § 1 we discuss the Frobenius reciprocity theorem and related matters, while in § 3 we consider the intertwining number theorem of Mackey.

It may be of interest that almost all of the theorems which we consider are seen to be true (in an appropriate form) for representations of (possibly infinite) groups on modules over arbitrary commutative ground rings with unit.

1. The Frobenius Reciprocity Theorem. Although Levitzki [**15**] made an early attempt to generalize Frobenius' definition of induced representations [**10**] to the setting of algebras (by using idempotents), and a similar formulation is implicit in a treatment for groups given by Weyl [**26**, p. 335], the first general, functorial, definition of induced representations for rings and algebras which we have found in the literature was given in 1955 by D. G. Higman [**11**]. His definition can be stated in slightly more current terminology as follows. Let A be a ring and let B be a subring of A. (Throughout this paper rings are always assumed to have an identity element, and subrings are assumed to contain the identity element of the containing ring. All modules are unital.) Then the obvious restriction functor, $W \mapsto W_B$, from the category of left A-modules to the category of left B-modules has both an adjoint, $V \mapsto {}^A V$, and a coadjoint, $V \mapsto V^A$, (for the definition of adjoint functors see [**17**, p. 79]). These are defined on a left B-module, V, by

(1.1) $\quad {}^A V = A \otimes_B V$

and

(1.2) $\quad V^A = \mathrm{Hom}_B\,(A, V)$

Received August 14, 1973 and in revised form, March 11, 1974. This research was partially supported by the National Science Foundation grant GP-25082.

respectively, where in the first case A is viewed as a left-A-right-B-bimodule, and in the second case as a left-B-right-A-bimodule. These are just the change of rings operations familiar from other areas of algebra (see for example [**5**, p. 28]). We will call ${}^A V$ and V^A the adjoint and coadjoint induced modules obtained by inducing V from B to A.

Higman indicated that if the constructions (1.1) and (1.2) are applied to the case of a group and a subgroup of finite index (by using their group algebras), they both lead to Frobenius' original definition of induced representations. Since Higman's proof of this fact is buried in his discussion of self-dual S-rings, we will begin by sketching a direct proof here. In the process we will indicate what happens if the subgroup does not have finite index.

Let G be a group, and let R be a commutative ring, which will remain fixed throughout this paper. We will denote the group algebra of G over R by $R(G)$. The representations of G which we will consider are those for which G acts as a group of automorphisms of some R-module. We will refer to the space of such a representation of G as a G-module. (The term G-R-module would be more appropriate if there were any chance of confusion about which ground ring is being used.) Then it is easily verified that G-modules correspond to $R(G)$-modules (see [**5**, p. 149]), and, even more, that the category of G-modules is isomorphic to the category of $R(G)$-modules.

Let H be a subgroup of G, and let V be an H-module. Frobenius gave his definition of the induced representation of G obtained from V only for the case in which G is finite and R is a field, but we make neither of these assumptions here. His definition, in more modern guise, and in this slightly greater generality, is as follows. The space of the induced G-module is the R-module, $R_H(G, V)$, of all functions, F, from G to V which satisfy the relation

$$(1.3) \quad F(xs) = s^{-1}(F(x))$$

for all $x \in G$ and $s \in H$, with the action of G on $R_H(G, V)$ given by

$$(1.4) \quad (yF)(x) = F(y^{-1}x)$$

for $x, y \in G$. We will now see how this definition is related to Higman's definitions in terms of the change of ring operations.

We consider coadjoint induced modules first. We do not assume that G is finite or that H has finite index in G. Now $R(H)$ is a subalgebra of $R(G)$ and so we can apply (1.2) to construct the coadjoint induced representation $V^{R(G)}$. Let us write V^G instead of $V^{R(G)}$, and similarly for Hom_H or $\otimes_H$. Now any element, T, of $V^G = \mathrm{Hom}_H(R(G), V)$ is determined by its values on the R-basis for $R(G)$ consisting of the elements of G, that is, by a V-valued function on G. To obtain agreement with the conventions we used above in describing Frobenius' definition of induced representations, we associate to T the function, F_T, defined on G by

$$(1.5) \quad F_T(x) = T(x^{-1}).$$

We thus obtain a mapping from V^G to $R_H(G, V)$ which is easily seen to be a G-module (or $R(G)$-module) isomorphism. Thus the functor $V \mapsto V^G$ is naturally equivalent to the functor $V \mapsto R_H(G, V)$. (Here and elsewhere we leave details concerning morphisms to the reader.)

We turn now to the adjoint induced modules, which we write as ${}^G V$ instead of ${}^{R(G)}V$. Again we wish to express $R(G) \otimes_H V$ as a collection of V-valued functions on G. A calculation which suggests how to do this is the following. Let $f \in R(G)$, $v \in V$, and let $\{x_c : c \in G/H\}$ be a collection of coset representatives for H in G. Then, writing f as the formal linear combination $\sum_{x \in G} f(x)x$, we have

$$\begin{aligned} f \otimes v &= \sum_{x} f(x)x \otimes v \\ &= \sum_{c \in G/H} \sum_{t \in H} f(x_c t)x_c t \otimes v \qquad (1.6) \\ &= \sum_{c \in G/H} x_c \otimes \Big(\sum_{t \in H} f(x_c t)tv \Big). \end{aligned}$$

This suggests that we associate to f and v the V-valued function, $b(f, v)$, on G defined by

$$(1.7) \quad b(f, v)(x) = \sum_{t \in H} f(xt)tv.$$

It is easily verified that b extends to an injection from $R(G) \otimes_H V$ into $R_H(G, V)$. However, b is not in general surjective, for the range of b will be the subspace, $R_H{}^\circ(G, V)$, of $R_H(G, V)$ consisting of functions which have value zero off of the union of a finite number of cosets of H in G. Thus the functor $V \mapsto {}^G V$ is naturally equivalent to the functor $V \mapsto R_H{}^\circ(G, V)$.

In particular, if H is of finite index in G, then $R_H{}^\circ(G, V) = R_H(G, V)$, and the functors $V \mapsto {}^G V$ and $V \mapsto V^G$ are naturally equivalent. In other words, in this case we have the interesting phenomenon that the adjoint and coadjoint of the restriction functor from the category of $R(G)$-modules to the category of $R(H)$-modules are naturally equivalent. This phenomenon was investigated for general extensions of rings by Higman [**11**], and later by Morita [**19**], who showed that the ring extensions exhibiting this phenomenon are exactly the "Frobenius extensions" introduced earlier by Kasch [**13**]. This phenomenon also holds for unitary representations of compact groups [**22**]. We remark that a description of induced representations in terms of homogeneous bundles can be found in [**9**].

The statement that the constructions (1.1) and (1.2) provide the adjoint and coadjoint of the restriction functor should be considered to be the appropriate generalization of the Frobenius reciprocity theorem to the setting of rings.

THE FROBENIUS RECIPROCITY THEOREM FOR RINGS. *Let A be a ring and* *let B be a subring of A. Then*

(1.8) $\mathrm{Hom}_A({}^A V, W) \cong \mathrm{Hom}_B(V, W_B)$

and

(1.9) $\mathrm{Hom}_A(W, V^A) \cong \mathrm{Hom}_B(W_B, V)$

for all A-modules W and B-modules V.

These natural equivalences are well-known and follow from the basic properties of tensor products (see [**5**, p. 28] or [**4**, p. 133]). The Frobenius reciprocity theorem for groups in the form given on [**12**, p. 556] is an immediate consequence of the above theorem. More classical versions can be derived from this form as in [**12**].

The theorem on induction in stages for adjoint induced representations (see [**8**, p. 267] for the case of groups) follows immediately from the associativity of tensor products, while for coadjoint representations it follows from similar considerations.

THEOREM ON INDUCTION IN STAGES. *Let A be a ring, B a subring of A, and C a subring of B. Let U be a C-module. Then there are natural equivalences of functors*

$${}^A({}^B U) \cong {}^A U \quad \textit{and} \quad (U^B)^A \cong U^A.$$

We remark in conclusion that "Shapiro's lemma" (see [**14**, p. 69] or [**25**, p. 131]), which states that

$$H^n(G, V^G) \cong H^n(H, V),$$

where the H^n are cohomology groups (and R is the ring of integers), is an almost immediate consequence of (1.9) together with Corollary 1 of [**1**]. (I thank C. C. Moore for bringing "Shapiro's Lemma "to my attention.)

Extensions of a number of the ideas of this section to representations of locally compact groups and Banach algebras can be found in [**21**; **22**; **23**].

2. The Imprimitivity theorem. Just as basic to the theory of induced representations of groups as the Frobenius reciprocity theorem, is the imprimitivity theorem, which describes which representations of a group G are induced from representations of a subgroup H. Since we have seen that for rings there are two types of induced representations, the adjoint and coadjoint induced representations, there should be two imprimitivity theorems for induced representations of rings. In this section we will only discuss the imprimitivity theorem for adjoint induced representations of rings (construction (1.1)), but an entirely parallel discussion can be given for coadjoint induced representations.

What we must do is discover what special properties are possessed by A-modules which happen to be of the form $A \otimes_B V$ for some B-module V. Appropriate special properties are suggested by what have come to be known as "the Morita theorems" [**18**; **2**, Chapter 2; **7**]. Let A_B denote A viewed as a right B-module, and let $E = \mathrm{End}_B(A_B)$, the ring of B-linear maps of A into itself. To any element $a \in A$ we can associate the map of A into itself consisting of left multiplication by a, and in this way we can identify A with a subring of E. Now A_B is in an obvious way a left-E-right-B-bimodule, and from this it is clear that if V is any B-module, then $A \otimes_B V$ will in fact be a left E-module, with the action of E being an extension of the original action of its subring A on $A \otimes_B V$. Thus a necessary condition that a left A-module W be induced from a left B-module is that the action of A on W be extendable to an action of all of E on W.

To obtain a sufficient condition we must make hypotheses on A and B which will ensure that E is large enough to see well how B acts on A. For this purpose the Morita theorems suggest that we require that A_B be finitely generated, projective, and a generator (see [**2**, p. 68]). Then from the Morita theorems (see especially [**2**, p. 65]) we immediately obtain the following theorem.

THE EQUIVALENCE THEOREM. *Let A be a ring and B a subring of A. Assume that as a right B-module A is a finitely generated projective generator. Let $E =$* $\mathrm{End}_B(A_B)$. *Then the category of left E-modules is equivalent to the category of left B-modules, the equivalence being given in one direction by $V \mapsto A \otimes_B V$ for any left B-module V.*

From this one obtains almost immediately:

THE IMPRIMITIVITY THEOREM FOR RINGS. *Let A be a ring and B a subring of A. Assume that as a right B-module A is a finitely generated projective generator. Let $E =$* $\mathrm{End}_B(A_B)$, *and identify A with the subring of E consisting of left multiplication by elements of A. Then a left A-module W is (adjoint) induced from a left B-module if and only if the action of A on W can be extended to an action of all of E on W.*

To see that the imprimitivity theorem for groups is a special case of the above theorem, we must first identify the algebra E in this case. As before let G be a group and H a subgroup of G, so that $E = \mathrm{End}_{(RH)}\,(R(G)_{(RG)})$. Then $R(G)$ is free as a right $R(H)$-module, with basis consisting of a set of coset representatives for H in G. In particular, as a right $R(H)$-module $R(G)$ is a projective generator. Furthermore it is finitely generated if H has finite index in G, which we assume from now on. But, from the fact that a set of coset representatives, $\{x_c : c \in G/H\}$, is a basis for $R(G)$ as a right $R(H)$-module, it is easily seen that the collection of operators $P_c \otimes x$ forms a basis for E over R as c ranges over G/H and x ranges over G, where P_c is the natural projection of $R(G)$ onto its subspace of functions which have value zero off the coset c, and $P_c \otimes x$ acts by applying first x and then P_c.

https://doi.org/10.4153/CJM-1975-032-0 Published online by Cambridge University Press

Now if W is an $R(G)$-module and if we wish to extend the action of $R(G)$ on W to an action of all of E on W, it suffices to specify how each basis element $P_c \otimes x$ is to act on W, as long as this is done in a way consistent with the way in which these basis elements are added and multiplied in E. Examination of this consistency requirement leads to the imprimitivity theorem in its usual form for groups (see [**8**, p. 346]).

The above result, as well as the description of E, can be reformulated in a way which is algebraically more attractive. Let $C(G/H)$ denote the algebra under pointwise multiplication of all R-valued functions on G/H, which whenever convenient we will tacitly identify with the R-valued functions on G which are constant on cosets of H. Then the operator consisting of pointwise multiplication of elements of $R(G)$ by a fixed element of $C(G/H)$ is easily seen to commute with the right action of $R(H)$ on $R(G)$, and so $C(G/H)$ can be identified with a subalgebra of E.

Let $x \in G$ and $\phi \in C(G/H)$, with both x and ϕ viewed as elements of E. Then it is easily seen that

$$x(\phi f) = (x\phi)(xf)$$

for all $f \in R(G)$, where by definition $x\phi(y) = \phi(x^{-1}y)$ for all $y \in G$. Furthermore, it is easily verified that with the action $\phi \mapsto x\phi$, G acts as a group of automorphisms of the algebra $C(G/H)$.

Now given any algebra, C, and any group, G, acting as automorphisms of C, there is a construction (see [**24**]) of an algebra which can be considered to be the "semidirect product" of G and C, and which has the property that the modules over this algebra are in bijective correspondence with the spaces W which are simultaneously G and C modules in such a way that

$$x(cw) = (xc)(xw)$$

for all $x \in G$, $c \in C$, $w \in W$, where xc denotes the result of applying the automorphism corresponding to x to the element c of C.

Returning to the situation of a group G and subgroup H, it is easily seen that the algebra E is isomorphic to the "semidirect product"

$$C(G/H) \overset{s}{\otimes} R(G).$$

The statement of the imprimitivity theorem then becomes the statement that an $R(G)$-module W is induced from some representation of H if and only if it can be made into a $C(G/H)$-module in such a way that

$$x(\phi w) = (x\phi)(xw)$$

for all $x \in G$, $\phi \in C(G/H)$, $w \in W$. This form is very close to the form of the imprimitivity theorem for induced representations of locally compact groups given by Blattner [**3**].

3. Mackey's Intertwining Number Theorem. The generalization of Mackey's intertwining number theorem [**8**, p. 327] to the context of rings should concern the relations between representations induced from two subrings of a ring. The first proof which Mackey gave for his intertwining number theorem [**16**, p. 580] (which is the proof presented on [**8**, p. 325]) uses his tensor product theorem, which in turn involves being able to take inner tensor products of representations. But the possibility of taking inner tensor products seems to involve the presence of a Hopf algebra structure. (See [**20**]. It would be interesting to have a theorem to this effect.) And so one would not expect that this proof would generalize to give a proof of an intertwining number theorem in the context of induced representations of arbitrary rings. Mackey then gave a second proof (in [**16**, § II]) of an intertwining number theorem which, in fact, applied to open subgroups of separable locally compact groups. The proof of this version is fairly long. We will sketch here a proof, for the case of discrete groups and an arbitrary commutative ground ring, which involves little more than two applications of the Frobenius reciprocity theorem and one application of the imprimitivity theorem. In the process we will see how the theorem can at least to some extent be generalized to rings. As in the previous section we will treat only adjoint induced representations, but a quite parallel treatment can be given for coadjoint induced representations.

Let A be a ring, and let B and C be subrings of A. Let U be a left B-module and V be a left C-module. The intertwining number theorem concerns the space

(3.1) $\quad \mathrm{Hom}_A({}^A U, {}^A V)$

(or, more specifically, the dimension of this space if the rings are finite dimensional semisimple algebras over a splitting field). If we apply the Frobenius reciprocity theorem in form (1.8) to (3.1) we find that (3.1) is naturally equivalent to

(3.2) $\quad \mathrm{Hom}_B\,(U, ({}^A V)_B)$.

Now $({}^A V)_B$ is just $A \otimes_C V$ where A is now viewed as a left-C-right-B-bimodule. To proceed further we must make the assumption that A is decomposable as a B-C-bimodule (which will usually be the case if B and C are semisimple). Thus assume that

$$A = \bigoplus D_i$$

where the D_i are B-C-sub-bimodules of A. (We allow this direct sum to have an infinite number of summands.) Then

(3.3) $\quad A \otimes_C V \cong \bigoplus D_i \otimes_C V$

as B-modules. Substituting (3.3) in (3.2) we obtain the following relation, which seems to be about the best one can do in obtaining a generalization of

Mackey's intertwining number theorem to rings if one does not have further information about the structure of the D_i's.

$$\mathrm{Hom}_A\,({}^AU,{}^AV) \cong \bigoplus \mathrm{Hom}_B\,(U, D_i \otimes_C V).$$

To see that Mackey's intertwining number theorem is in part a special case of this relation, consider a group G with subgroups H and K. If $R(G)$ is viewed as an $R(H)$-$R(K)$-bimodule, then it has an obvious bimodule decomposition in terms of the H-K-double cosets of G, which can be expressed as

$$(3.4)\quad R(G) \cong \bigoplus_{D\in H\backslash G/K} R(D),$$

where $H\backslash G/K$ denotes the collection of H-K-double cosets of G, and $R(D)$ denotes the subspace of $R(G)$ spanned by the elements of D. Suppose that U is an H-module and that V is a K-module. Then, applying the above result, we see that

$$(3.5)\quad \mathrm{Hom}_G({}^GU,{}^GV) \cong \bigoplus_{D\in H\backslash G/K} \mathrm{Hom}_H(U, R(D) \otimes_K V).$$

This result somewhat resembles Mackey's intertwining number theorem.

To obtain the additional details contained in Mackey's intertwining number theorem we must analyze further the structure of the H-modules $R(D) \otimes_K V$, and express them as induced representations. This analysis is the main part of the proof of Mackey's subgroup theorem [**8**, p. 324], and follows readily from the imprimitivity theorem. (The proofs in [**16**] and [**8**] are slightly ad hoc.) If we view $R(D)$ as a left H-module, then there is an obvious transitive system of imprimitivity, namely the one associated with the cosets of K contained in D. If $\{x_D\}_{D\in H\backslash G/K}$ is a family of double coset representatives, so that $D = Hx_DK$, then this system of imprimitivity is based (see [**8**, 50.2]) on the quotient of G by the stability subgroup of $R(x_DK)$, which is easily seen to be $H \cap x_DKx_D^{-1}$. Let us denote $H \cap x_DKx_D^{-1}$ by H_{x_D}. Then, according to the imprimitivity theorem, the H-module $R(D)$ is equivalent to the module obtained by inducing up to H the H_{x_D}-module $R(x_DK)$ if H_{x_D} has finite index in H. That is,

$$(3.6)\quad R(D) \cong R(H) \otimes_{H_{x_D}} R(x_DK).$$

We note that $R(x_DK)$ is a right K-module, and that the equivalence (3.6) is, in fact, an equivalence of H-K-bimodules. Using the associativity of tensor products [**8**, p. 67], we find that

$$R(D) \otimes_K V \cong R(H) \otimes_{H_{x_D}} (R(x_DK) \otimes_K V),$$

and we recognize the right hand side as being the module obtained by inducing the H_{x_D}-module $R(x_DK) \otimes_K V$ up to H. Now it is easily verified that the H_{x_D}-module $R(x_DK) \otimes_K V$ is isomorphic to the conjugate by x_D of the restriction of V to H_{x_D}, that is, the module $V_{(x_D)}$ whose elements are those of V

but for which the action of H_{x_D} is defined by

$$y(\dot{v}) = (x_D^{-1}yx_D)v$$

for $y \in H_{x_D}$, $v \in V$, where $\dot{v}$ denotes v viewed as an element of $V_{(x_D)}$. Thus we see that

$$R(D) \otimes_K V \cong {}^H(V_{(x_D)}).$$

If we apply this in (3.5), we find that

$$\mathrm{Hom}_G({}^GU, {}^GV) \cong \bigoplus_{D \in H\backslash G/K} \mathrm{Hom}_H(U, {}^H(V_{(x_D)})).$$

If we assume that H_{x_D} is of finite index in H for all x_D, so that, as we saw in the first section, ${}^H(V_{(x_D)}) \cong (V_{(x_D)})^H$, then we can apply the Frobenius reciprocity theorem for coadjoint induced representations (1.9) to the right hand side. This completes the proof of the following slight generalization of Mackey's intertwining number theorem (in that it applies when working over any commutative ground ring with unit).

INTERTWINING SPACE THEOREM FOR GROUPS. *Let H and K be subgroups of the group G, let U be an H-module and let V be a K-module. Let H_{x_D} and $V_{(x_D)}$ be defined as above for $D \in H\backslash G/K$. Assume that H_{x_D} is of finite index in H for all $D \in H\backslash G/K$. Then*

$$\mathrm{Hom}_G({}^GU, {}^GV) \cong \bigoplus_{D \in H\backslash G/K} \mathrm{Hom}_{H_{x_D}}(U_{H_{x_D}}, V_{(x_D)}).$$

To see that the theorem is false if we do not assume that the H_{x_D} are of finite index in H it suffices to consider the example in which G is an infinite group, $H = G$, K consists of only the identity element of G, and in which U and V are the trivial representations of H and K respectively.

We remark that another generalization of Mackey's intertwining number theorem can be found in [**9**].

It is not difficult to use the approach of this section to formulate a generalization to Hopf algebras [**20**] of Mackey's tensor product theorem ([**16**] or [**8**, p. 325]). However, the hypotheses become sufficiently cumbersome that we do not consider it worthwhile to include such a generalization here. On the other hand, some of the arguments employed above can be used to give additional motivation for the steps in the proof of this theorem in its original form. We also remark that in [**20**] we showed how to generalize Burnside's theorem concerning faithful representations of groups to the setting of Hopf algebras.

Finally, we remark that there is one very important part of the theory of induced representations of groups which we do not at present see how to generalize to rings, namely Clifford's theory of induced representations of group extensions ([**6**], or [**8**, §§ 49, 50 and 51]). The main difficulty is that we do not know an appropriate definition of a "normal" subring of a ring.

References

1. W. W. Adams and M. A. Rieffel, *Adjoint functors and derived functors with an application to the cohomology of semigroups*, J. Algebra *7* (1967), 25–34.
2. H. Bass, *Algebraic K-theory* (W. A. Benjamin, New York, 1968).
3. R. J. Blattner, *Positive definite measures*, Proc. Amer. Math. Soc. *14* (1963), 423–428.
4. N. Bourbaki, *Algèbre*, Ch. 3, Act. Sci. Ind. no. *1044* (Hermann, Paris, 1958).
5. H. Cartan and S. Eilenberg, *Homological algebra* (Princeton University Press, Princeton, 1956).
6. A. H. Clifford, *Representations induced in an invariant subgroup*, Ann. of Math. *38* (1937), 533–550.
7. P. M. Cohn, *Morita equivalence and duality*, Queen Mary College Mathematical Notes, London, n.d.
8. C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras* (Interscience Publ., New York, 1962).
9. A. Dress, *An intertwining number theorem for integral representations and applications*, Math. Z. *116* (1970), 153–165.
10. G. Frobenius, *über Relationen zwischen den Characteren einer Gruppe und denen ihrer Untergruppen*, Sitzber. Preuss. Akad. Wiss. (1898), 501–515.
11. D. G. Higman, *Induced and produced modules*, Can. J. Math. *7* (1955), 490–508.
12. B. Huppert, *Endliche Gruppen, I* (Springer-Verlag, Berlin, 1967).
13. F. Kasch, *Grundlagen einer Theorie der Frobeniuserweiterungen*, Math. Ann. *127* (1954), 453–474.
14. S. Lang, *Rapport sur la cohomologie des groupes* (W. A. Benjamin Inc., New York, 1966).
15. J. Levitzki, *über vollständigreduzible Ringe und Unterringe*, Math. Z. *33* (1931), 663–691.
16. G. W. Mackey, *On induced reparesentations of groups*, Amer. J. Math. *73* (1951), 576–592.
17. S. MacLane, *Categories for the working mathematician* (Springer-Verlag, New York, 1971).
18. K. Morita, *Duality for modules and its applications to the theory of rings with minimum conditions*, Tokyo Kyoiku Daigoku, Sect. A, *6* (1958), 83–142.
19. ——— *Adjoint pairs of functors and Frobenius extensions*, Tokyo Kyoiku Daigaku, Sect. A, *9* (1965), 40–71.
20. M. A. Rieffel, *Burnside's theorem for representations of Hopf Algebras*, J. Algebra *6* (1967), 123–130.
21. ——— *Induced Banach representations of Banach algebras and locally compact groups*, J. Functional Analysis *1* (1967), 443–491.
22. ——— *Unitary representations induced from compact subgroups*, Studia Math. *42* (1971), 145–175.
23. ——— *Induced representations of C*-algebras*, Bull. Amer. Math. Soc. *78* (1972), 606–609.
24. M. Takesaki, *Covariant representations of C*-algebras and their locally compact automorphism groups*, Acta Math. *119* (1967), 273–303.
25. E. Weiss, *Cohomology of groups* (Academic Press, New York, 1969).
26. H. Weyl, *Theory of groups and quantum mechanics* (Princeton University Press, Princeton, 1931).

The University of California,
Berkeley, California