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A CONVERSE OF THE LOEWNER–HEINZ INEQUALITY, GEOMETRIC MEAN AND SPECTRAL ORDER

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Abstract Let A, B be non-negative bounded self-adjoint operators, and let a be a real number such that $0 < a < 1$. The Loewner–Heinz inequality means that $A \leqq B$ implies that $A^a \leqq B^a$. We show that $A \leqq B$ if and only if $(A+\lambda)^a \leqq (B+\lambda)^a$ for every $\lambda > 0$. We then apply this to the geometric mean and spectral order.

Keywords: Loewner–Heinz inequality; geometric mean; spectral order

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1. Introduction

Let A, B be bounded self-adjoint operators on a Hilbert space $\mathbb{H}$. $A \leqq B$ means that $(Ax, x) \leqq (Bx, x)$ for every $x \in \mathbb{H}$. A real continuous function $f(t)$ defined on a real interval is said to be *operator monotonic*, provided that $A \leqq B$ implies that $f(A) \leqq f(B)$ for any two operators A and B whose spectra are in the interval. The Loewner–Heinz inequality means that the power function t^a is operator monotonic on $[0, \infty)$ for $0 < a < 1$. $\log t$ is also operator monotonic on $(0, \infty)$. $f(t)$ is operator monotonic if and only if $f(t)$ has a holomorphic extension $f(z)$ to the open upper half plane such that $f(z)$ maps it into itself, i.e. $f(z)$ is a Pick function (see [**2**, **4**]).

Recall that for $A \geqq 0$, $B \geqq 0$ the geometric mean $A\#B$ is defined and represented as

$$A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$$

if A is invertible. This binary operation is monotone increasing with respect to each variable, that is to say, for $A \leqq B$, $C \leqq D$,

$$A\#C \leqq B\#D.$$

But, in general, the converse does not hold. For details, we refer the reader to [**3**] or [**1**, Chapter 4].

Throughout this paper A, B, C, D stand for bounded self-adjoint operators, and for a real number λ we write $A+\lambda$, for short, instead of $A+\lambda I$. The objective of this paper is to show that $A \leqq B$ if $(A+\lambda)^a \leqq (B+\lambda)^a$ for a real number $0<a<1$ and for every real number $\lambda>0$; we actually prove a more general result. We then apply this fact to the operator geometric mean: we show precisely that $A \leqq B$ and $C \leqq D$ if

$$(A+\lambda)\#(C+\mu) \leqq (B+\lambda)\#(D+\mu)$$

for every $\lambda>0$ and every $\mu>0$. We also give a necessary and sufficient condition to be $A^n \leqq B^n$ for every n.

2. A converse of the Loewner–Heinz inequality

We start with a general result.

Lemma 2.1. *Let $h(t)$ be a differentiable function defined in a neighbourhood of $t=a$ with $h'(a)>0$. Let A, B be bounded self-adjoint operators. If*

$$h(a+\lambda_n A) \leqq h(a+\lambda_n B) \tag{2.1}$$

for $\{\lambda_n\}_{n=1}^{\infty}$ such that $\lambda_n \downarrow 0$, then $A \leqq B$.

Proof. We note that for sufficiently small λ_n the functional calculus $h(a+\lambda_n A)$ is well defined. From (2.1) it follows that

$$\frac{h(a+\lambda_n A)-h(a)}{\lambda_n} \leqq \frac{h(a+\lambda_n B)-h(a)}{\lambda_n}. \tag{2.2}$$

Let $\{E_t\}$ be the spectral family of A. We then get

$$\frac{h(a+\lambda_n A)-h(a)}{\lambda_n} = \int_{-\|A\|}^{\|A\|} \frac{h(a+\lambda_n t)-h(a)}{\lambda_n}\,\mathrm{d}E_t.$$

For an arbitrary $\epsilon>0$ there exists n_0 such that

$$\left|\frac{h(a+\lambda_n t)-h(a)}{\lambda_n}\right| \leqq |(h'(a)+\epsilon)t|$$

for $n \geqq n_0$ and for $-\|A\| \leqq t \leqq \|A\|$. Since $|(h'(a)+\epsilon)t|$ is continuous, by Lebesgue's theorem,

$$\lim_{n\to\infty}\int_{-\|A\|}^{\|A\|} \frac{h(a+\lambda_n t)-h(a)}{\lambda_n}\,\mathrm{d}E_t = \int_{-\|A\|}^{\|A\|} h'(a)t\,\mathrm{d}E_t = h'(a)A.$$

Since the right-hand side of (2.2) also converges to $h'(a)B$, we get that $A \leqq B$. □

Theorem 2.2. *Let $f(t)$ be a non-constant operator monotonic function in a neighbourhood of $t=a$. Then, $A \leqq B$ if and only if there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that $t_n \downarrow 0$ and*

$$f(a+t_n A) \leqq f(a+t_n B).$$

Proof. $A \leqq B$ clearly yields that $a + t_n A \leqq a + t_n B$. Hence, we get that $f(a + t_n A) \leqq f(a + t_n B)$. Since a non-constant operator monotonic function is increasing, by Lemma 2.1, $A \leqq B$ follows from $f(a + t_n A) \leqq f(a + t_n B)$. □

Theorem 2.3. *Let $A \geqq 0$, let $B \geqq 0$, and let $0 < a < 1$. The following are then equivalent:*

(i) $A \leqq B$;

(ii) $A + \lambda \leqq B + \lambda$ *for every* $\lambda \geqq 0$;

(iii) $(A + \lambda)^a \leqq (B + \lambda)^a$ *for every* $\lambda \geqq 0$;

(iv) $(A + \lambda_n)^a \leqq (B + \lambda_n)^a$ *for a sequence* $\{\lambda_n\}_{n=1}^{\infty}$ *such that* $\lambda_n > 0$ *and* $\lambda_n \to \infty$ *as* $n \to \infty$;

(v) $(t_n A + 1)^a \leqq (t_n B + 1)^a$ *for a sequence* $\{t_n\}_{n=1}^{\infty}$ *such that* $t_n > 0$ *and* $t_n \to 0$ *as* $n \to \infty$.

Proof. (i) $\Longrightarrow$ (ii) and (iii) $\Longrightarrow$ (iv) are trivial; (ii) $\Longrightarrow$ (iii) is the Loewner–Heinz inequality. From (iv) it follows that

$$\left(1 + \frac{A}{\lambda_n}\right)^a \leqq \left(1 + \frac{B}{\lambda_n}\right)^a,$$

which ensures (v). By Theorem 2.2 we get (v) $\Longrightarrow$ (i). □

It is not difficult to see the following in the same way as above.

Theorem 2.4. *Let $A \geqq 0$ and let $B \geqq 0$. The following are then equivalent:*

(i) $A \leqq B$;

(ii) $\log(A + \lambda) \leqq \log(B + \lambda)$ *for every* $\lambda > 0$;

(iii) $\log(A + \lambda_n) \leqq \log(B + \lambda_n)$ *for a sequence* $\{\lambda_n\}_{n=1}^{\infty}$ *such that* $\lambda_n > 0$ *and* $\lambda_n \to \infty$ *as* $n \to \infty$;

(iv) $\log(t_n A + 1) \leqq \log(t_n B + 1)$ *for a sequence* $\{t_n\}_{n=1}^{\infty}$ *such that* $t_n > 0$ *and* $t_n \to 0$ *as* $n \to \infty$.

Theorem 2.3 indicates that (iv) $\Longrightarrow$ (iii). The following gives a direct proof of this in the case of $a = \frac{1}{2}$.

Proposition 2.5. *Let $A \geqq 0$, let $B \geqq 0$ and let $\lambda > 0$. The following then hold.*

(i) *If* $(A + \lambda)^{1/2} \leqq (B + \lambda)^{1/2}$, *then* $(A + \mu)^{1/2} \leqq (B + \mu)^{1/2}$ *for every* $\mu \colon 0 < \mu \leqq \lambda$.

(ii) *If* $(\lambda A + 1)^{1/2} \leqq (\lambda B + 1)^{1/2}$, *then* $(\mu A + 1)^{1/2} \leqq (\mu B + 1)^{1/2}$ *for every* $\mu \colon \mu \geqq \lambda$.

Proof. To show (i) we may assume that $\mu = 0$ without of loss of generality. $f(t) := t^{1/2}(t + 2\lambda^{1/2})^{1/2}$ is operator monotonic on $[0, \infty)$; indeed, $f(t)$ has a holomorphic extension $f(z) = z^{1/2}(z + 2\lambda^{1/2})^{1/2}$ to the open upper half-plane. Since $0 < \arg f(z) \leqq \arg z$ for $0 < \arg z < \pi$, $f(z)$ is a Pick function, and hence $f(t)$ is operator monotonic. Since $f(t - \lambda^{1/2}) = (t^2 - \lambda)^{1/2}$ for $t \geqq \lambda^{1/2}$, we have $f((A + \lambda)^{1/2} - \lambda^{1/2}) = A^{1/2}$, and hence $A^{1/2} \leqq B^{1/2}$. Part (ii) follows from part (i). □

Unfortunately, we do not know whether Proposition 2.5 holds in other cases $a \neq \frac{1}{2}$. According to the notation introduced in [**7**] we can rewrite (i) in the above proposition as follows: for $0 < \mu \leqq \lambda$, $(t + \mu)^{1/2} \preceq (t + \lambda)^{1/2}$ on $0 \leqq t < \infty$.

Theorem 2.3 also states that if $A^a \leqq B^a$ but $A \nleqq B$, then the set of λ such that $(A + \lambda)^a \leqq (B + \lambda)^a$ is bounded above. On the contrary, in the case where $a \geqq 1$ we have the following.

Proposition 2.6. *Let $A \geqq 0$, let $B \geqq 0$ and let $a \geqq 1$. Then, $A^a \leqq B^a$ implies that $(A + \mu)^a \leqq (B + \mu)^a$ for every $\mu > 0$, namely, $(t + \mu)^a \preceq t^a$ on $0 \leqq t < \infty$.*

Proof. If a is a natural number, we can see this by the binomial expansion of $(A + \mu)^a$. In general, consider a function $f(t) = (t^{1/a} + \mu)^a$. Since its holomorphic extension $f(z) = (z^{1/a} + \mu)^a$ is a Pick function, $f(t^a) = (t + \mu)^a$ yields the required result. □

3. Application

We apply Theorem 2.3 to the operator geometric mean.

Theorem 3.1. *For $A, B, C, D \geqq 0$ the following are equivalent:*

(i) *$A \leqq B$ and $C \leqq D$;*

(ii) *$(A + \lambda)\#(C + \mu) \leqq (B + \lambda)\#(D + \mu)$ for every $\lambda; \mu > 0$;*

(iii) *$(sA + 1)\#(tC + 1) \leqq (sB + 1)\#(tD + 1)$ for every $s, t > 0$.*

Proof. (i) $\Longrightarrow$ (ii) and (i) $\Longrightarrow$ (iii) are clear. To show that (ii) $\Longrightarrow$ (i) we may assume that A and B are invertible, because (ii) holds for $A + \epsilon$, $B + \epsilon$ in place of A, B. Part (ii) then yields that

$$(A + \lambda)^{1/2}((A + \lambda)^{-1/2}(C + \mu)(A + \lambda)^{-1/2})^{1/2}(A + \lambda)^{1/2} \leqq (B + \lambda)^{1/2}((B + \lambda)^{-1/2}(D + \mu)(B + \lambda)^{-1/2})^{1/2}(B + \lambda)^{1/2}. \quad (3.1)$$

Divide both sides of (3.1) by $\lambda^{1/2}$ and let $\lambda \to \infty$. We then get that

$$(C + \mu)^{1/2} \leqq (D + \mu)^{1/2}$$

for every $\mu > 0$. By Theorem 2.3 we obtain that $C \leqq D$. Divide both sides of (3.1) by $\mu^{1/2}$ and let $\mu \to \infty$ to get that $(A + \lambda)^{1/2} \leqq (B + \lambda)^{1/2}$, which ensures that $A \leqq B$. One can see analogously that (iii) $\Longrightarrow$ (i). □

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It is not difficult to see that the above theorem also holds for the harmonic mean, i.e. $A!B := (\frac{1}{2}(A^{-1}+B^{-1}))^{-1}$, but does not hold for the arithmetic mean.

Let a be a real number. The power (non-operator) mean is then defined by

$$M_a(s,t) = \left(\frac{s^a+t^a}{2}\right)^{1/a}, \qquad M_\infty(s,t) = \max(t,s), \qquad M_{-\infty}(s,t) = \min(t,s)$$

for $s,t>0$. M_1 and M_{-1} are the arithmetic mean and harmonic mean, respectively. We now write, for $A \geqq 0$ and for $\lambda \geqq 0$,

$$M_a(A,\lambda) = \left(\frac{A^a+\lambda^a}{2}\right)^{1/a}.$$

We then have the following.

Proposition 3.2. *Let $a>1$ and let $A,B \geqq 0$. The following are then equivalent:*

(i) $A^a \leqq B^a$;

(ii) $M_a(A,\lambda) \leqq M_a(B,\lambda)$ *for every* $\lambda > 0$.

Moreover, if A, B are invertible, then the following is equivalent to the above:

(iii) $M_{-a}(A,\lambda) \leqq M_{-a}(B,\lambda)$ *for every* $\lambda > 0$.

Proof. Part (ii) is rewritten as $(A^a+\lambda^a)^{1/a} \leqq (B^a+\lambda^a)^{1/a}$. By Theorem 2.3 this is equivalent to (i). If A, B are invertible, (i) means that $A^{-a} \geqq B^{-a}$, which is equivalent to (iii) because of Theorem 2.3. □

Although the above proof is very easy, the implication that (ii) $\implies$ (i) is not an obvious fact; for instance, if we slightly change the condition (ii) in the case of $a=2$ to be $(A^2+2\lambda A+\lambda^2)^{1/2} \leqq (B^2+2\lambda B+\lambda^2)^{1/2}$ for every $\lambda>0$, then we gain not that $A^2 \leqq B^2$ but that $A \leqq B$.

Incidentally, in the case $0<a<1$, by Proposition 2.6 we easily obtain the following.

Remark 3.3. Let $0<a<1$ and let $A,B \geqq 0$. Then, $A \leqq B$ if and only if $M_a(A,\lambda) \leqq M_a(B,\lambda)$ for every $\lambda>0$.

Proposition 3.2 leads us to the following theorem.

Theorem 3.4. *Let $\{E_t\}$ and $\{F_t\}$ be spectral families of $A \geqq 0$ and $B \geqq 0$, respectively. The following are then equivalent:*

(i) $A^a \leqq B^a$ *for every* $a>0$;

(ii) $M_\infty(A;\lambda) \leqq M_\infty(B;\lambda)$ *for every* $\lambda>0$;

(iii) $M_{-\infty}(A;\lambda) \leqq M_{-\infty}(B;\lambda)$ *for every* $\lambda>0$;

(iv) $E_t \geqq F_t$ *for every* t.

Before proceeding to the proof, we set out some concepts. For A, B with spectral families $\{E_t\}$, $\{F_t\}$, respectively, we write $A \prec B$ if $E_t \geqq F_t$ for every t; this order is called the *spectral order* and the equivalence of (i) and (iv) in Theorem 3.4 was shown by Olson [**5**]. In the case where A, B are not necessarily non-negative, in [**6**] we have seen that $A \prec B$ if and only if $\mathrm{e}^{tA} \leqq \mathrm{e}^{tB}$ for all $t > 0$.

Proof. To get (i) $\Longrightarrow$ (ii) note that by Proposition 3.2 $M_a(A, \lambda) \leqq M_a(B, \lambda)$ for every $a > 1$ and for every $\lambda > 0$. Since, for a fixed λ, $M_a(t, \lambda)$ converges uniformly to $M_\infty(t, \lambda)$ on $-\|A\| \leqq t \leqq \|A\|$ as $a \to \infty$, $M_a(A, \lambda)$ converges to $M_\infty(A, \lambda)$ in the normed sense. $M_a(B, \lambda)$ also converges to $M_\infty(B, \lambda)$. We therefore get (ii). Assume (i) again. By Proposition 2.6 we have that $(A+\epsilon)^a \leqq (B+\epsilon)^a$ for $\epsilon > 0$. By Proposition 3.2 we deduce that $M_{-a}(A + \epsilon, \lambda) \leqq M_{-a}(B + \epsilon, \lambda)$. Letting $a \to \infty$ entails that $M_{-\infty}(A + \epsilon, \lambda) \leqq M_{-\infty}(B + \epsilon, \lambda)$, and then letting $\epsilon \to 0$ yields (iii), because $M_{-\infty}(t, \lambda)$ is continuous for t. We next show that (ii) $\Longrightarrow$ (iv). To do so, we claim that

$$M_\infty(A, \lambda) = \lambda E_\lambda + A(I - E_\lambda).$$

In fact,

$$\begin{aligned}
M_\infty(A, \lambda) &= \int_{-\infty}^{\infty} M_\infty(t, \lambda)\,\mathrm{d}E_t \\
&= \int_{-\infty}^{\lambda} M_\infty(t, \lambda)\,\mathrm{d}E_t + \int_{\lambda+0}^{\infty} M_\infty(t, \lambda)\,\mathrm{d}E_t \\
&= \int_{-\infty}^{\lambda} \lambda\,\mathrm{d}E_t + \int_{\lambda+0}^{\infty} t\,\mathrm{d}E_t \\
&= \lambda E_\lambda + \int_{\lambda+0}^{\infty} t\,\mathrm{d}E_t(I - E_\lambda) \\
&= \lambda E_\lambda + \int_{-\infty}^{\infty} t\,\mathrm{d}E_t(I - E_\lambda) \\
&= \lambda E_\lambda + A(I - E_\lambda).
\end{aligned}$$

We therefore get that $M_\infty(A, \lambda) - \lambda = (A - \lambda)(I - E_\lambda)$, whose null space is $E_\lambda \mathbb{H}$. Since

$$0 \leqq M_\infty(A, \lambda) - \lambda \leqq M_\infty(B, \lambda) - \lambda,$$

by comparing the null spaces we get that $F_\lambda \mathbb{H} \subseteqq E_\lambda \mathbb{H}$. This yields (iv). We next show that (iii) $\Longrightarrow$ (iv). Since $M_{-\infty}(A, \lambda) = AE_\lambda + \lambda(I - E_\lambda)$,

$$0 \leqq (\lambda - B)F_\lambda = \lambda - M_{-\infty}(B, \lambda) \leqq \lambda - M_{-\infty}(A, \lambda) = (\lambda - A)E_\lambda.$$

One can see that the null space of $(\lambda - A)E_\lambda$ is $(E_{\lambda-0}\mathbb{H})^\perp$. By comparing the null spaces of operators in the above inequalities, we obtain

$$(E_{\lambda-0}\mathbb{H})^\perp \subseteqq (F_{\lambda-0}\mathbb{H})^\perp.$$

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This implies that $F_{\lambda-0} \leqq E_{\lambda-0}$ for every $\lambda > 0$. We therefore get

$$F_\lambda \leqq F_{\lambda+\epsilon-0} \leqq E_{\lambda+\epsilon-0} \leqq E_{\lambda+\epsilon}$$

for an arbitrary $\epsilon > 0$. Since E_λ is continuous from the right, this yields (iv). $\square$

We note that in the above proofs of (ii) $\Longrightarrow$ (iv) and (iii) $\Longrightarrow$ (iv), the non-negativity of A, B is not used. So, (ii)–(iv) in Theorem 3.4 are equivalent even if A, B are not non-negative; indeed, since $M_\infty(t, \lambda)$ and $M_{-\infty}(t, \lambda)$ are non-decreasing continuous functions with respect to t, by [**5**, Corollary 1], (ii) and (iii) follow from (iv).

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References

1. R. Bhatia, *Positive definite matrices*, Princeton Series in Applied Mathematics (Princeton University Press, 2007).
2. A. Koranyi, On a theorem of Löwner and its connections with resolvents of self-adjoint transformations, *Acta Sci. Math. (Szeged)* **17** (1956), 63–70.
3. F. Kubo and T. Ando, Means of positive linear operators, *Math. Annalen* **246** (1980), 205–224.
4. K. Löwner, Über monotone Matrix functionen, *Math. Z.* **38** (1934), 177–216.
5. M. P. Olson, The self-adjoint operators of a von Neumann algebra form a conditionally complete lattice, *Proc. Am. Math. Soc.* **28** (1971), 537–544.
6. M. Uchiyama, Commutativity of self-adjoint operators, *Pac. J. Math.* **161** (1993), 385–392.
7. M. Uchiyama, A new majorization between functions, polynomials, and operator inequalities, *J. Funct. Analysis* **231** (2006), 221–244.