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ON HAUSDORFF DIMENSION FOR ATTRACTORS OF ITERATED FUNCTION SYSTEMS

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Abstract

A conjecture on the Hausdorff dimension for Markov attractors of disjoint hyperbolic iterated function systems was given by Ellis and Branton. This paper proves the conjecture and generalizes the result to more general cases.

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1. Introduction

In [2] Ellis and Branton have discussed the Hausdorff dimension of attractors of disjoint hyperbolic iterated function systems. The main results of [2] are

THEOREM A. (Ellis and Branton) *Let A be the attractor of a disjoint hyperbolic iterated function system $(X; T_1, \dots, T_n)$. Suppose that*

$$s_i d(x, y) \le d(T_i x, T_i y) \le \bar{s}_i d(x, y) \qquad \forall x, y \in X, \quad 1 \le i \le n$$

for some constants $0 < s_i \le \bar{s}_i < 1$. Then

$$l \le \dim(A) \le u,$$

where $\sum_{i=1}^n s_i^l = 1$ and $\sum_{i=1}^n \bar{s}_i^u = 1$.

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THEOREM B. (Ellis and Branton) *Assume that* $(X; T_1, \ldots, T_n)$ *is the same as in Theorem A, and* M *is a primitive Markov transition matrix,* A_M *is the Markov attractor of the iterated function system associated with* M. *Then*

$$\dim(A_M) \leq u,$$

where

$$\|M\bar{S}^u\| = 1,$$

and

$$\bar{S} = \begin{pmatrix} \bar{s}_1 & & 0 \\ & \ddots & \\ 0 & & \bar{s}_n \end{pmatrix}.$$

For the lower bound of $\dim(A_M)$, they gave the following conjecture:

CONJECTURE. *Under the same assumptions as* Theorem B, *we have*

$$\dim(A_M) \geq l,$$

where

$$\|MS^l\| = 1,$$

and

$$S = \begin{pmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_n \end{pmatrix}.$$

Ellis and Branton showed that the conjecture is true in some special cases. In this paper we shall prove that the conjecture is true in the general case and extend the result in the case when M is not primitive and when not all T_i's are contractions. The main result of this paper is

THEOREM. *Suppose that* M *is a Markov transition matrix with at least one non-zero eigenvalue, and that* $(X, T_1, \ldots, T_n)$ *is a disjoint iterated function system such that*

$$s_i d(x, y) \leq d(T_i x, T_i y) \leq \bar{s}_i d(x, y)$$

and

$$0 < s_i \leq \bar{s}_i < 1 \quad .$$

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Then we have

$$l \le \dim(A_M) \le u,$$

where

$$\|MS^l\| = 1 \quad \text{and} \quad \|M\bar{S}^u\| = 1,$$

and A_M, S, $\bar{S}$ are as before.

In Section 2 we review the notions of iterated function systems and some results we will use. In Section 3 we will give the proof of the conjecture in the case M is irreducible. In the Section 4 we will generalize the result to the case M is reducible. In the last section we consider the situation when not all the T_i's are contractions.

This paper is a part of my Ph. D work under the supervision of Professor G. Brown in the University of New South Wales. I should like also to thank Professor C. Sutherland for his kind help.

2. Preliminaries

DEFINITION 1. An *iterated function system* $(X; T_1, \ldots, T_n)$ is a compact metric space, X, together with continuous maps T_i $X \mapsto X$.

We say $(X; T_1, \ldots, T_n)$ is *hyperbolic* if there exists a constant $0 < s < 1$ such that

$$d(T_i x, T_i y) \le s d(x, y) \qquad \forall x, y \in X, \quad 1 \le i \le n.$$

For a hyperbolic iterated function system $(X; T_1, \ldots, T_n)$, a subset A of X is called the *attractor* of the system if

(i) $\emptyset \ne A$ is closed;

(ii) $T_i(A) \subset A$, for $1 \le i \le n$;

(iii) A is minimal with respect to (i) and (ii).

Hutchinson proved that the attractor A exists for every hyperbolic iterated function system and $A = \bigcup_{i=1}^n T_i(A)$. In addition $\forall a \in A$, there exists a sequence $i_1, i_2, \ldots$ such that

$$\lim_{m\to\infty} T_{i_1} \circ T_{i_2} \circ \ldots \circ T_{i_m} x = a.$$

for all $x \in X$ (see [5]).

If the attractor A of a hyperbolic iterated function system $(X; T_1, \ldots, T_n)$ satisfies $T_i(A) \bigcap T_j(A) = \emptyset$ when $i \ne j$, then the system is called *disjoint*.

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EXAMPLE. Let

$$\Sigma_n^+ = \{(i_1, i_2, \ldots) \mid 1 \le i \le n\}$$

and define maps $\sigma_i : \ \Sigma_n^+ \mapsto \Sigma_n^+$ by

$$\sigma_i(i_1, i_2, \ldots) = (i, i_1, i_2, \ldots), \qquad 1 \le i \le n.$$

If we define a metric, d, on Σ_n^+ by

$$d(\mathbf{i}, \mathbf{j}) = 2^{-k} \quad \text{when } i_1 = j_1, \ldots, i_k = j_k;\ i_{k+1} \ne j_{k+1},$$

where $\mathbf{i} = (i_1,\ i_2,\ \ldots)$, $\mathbf{j} = (j_1,\ j_2,\ \ldots)$. Then $((\Sigma_n^+, d); \sigma_1, \ldots, \sigma_n)$ is a disjoint hyperbolic iterated function system satisfying

$$d(\sigma_i(\mathbf{i}), \sigma_i(\mathbf{j})) = \frac{1}{2} d(\mathbf{i}, \mathbf{j}) \qquad 1 \le i \le n,$$

with $A = (\Sigma_n^+, d)$ as its attractor. By Theorem A we know that

$$\dim((\Sigma_n^+, d)) = \frac{\log n}{\log 2}\ .$$

DEFINITION 2. An $n \times n$ matrix M is called a *Markov transition matrix* if all of its entries are 1 or 0.

We say a sequence (finite or infinite) $i_1, i_2, \ldots$ is *M-admissible*, if

$$M_{i_j i_{j+1}} = 1$$

for all $j = 1, 2, \ldots,$ and where $i_j \in \{1, 2, \ldots, n\}$.

Let

$$\Sigma_M^+ = \{(i_1, i_2, \ldots) \mid (i_1, i_2, \ldots) \ \text{ is } M\text{-admissible}\}.$$

Then, under the metric d defined above, Σ_M^+ is a closed, therefore compact, subspace of (Σ_n^+, d).

DEFINITION 3. A non-negative square matrix M (all entries of M are non-negative, written $M \ge 0$) is called *primitive* if $M^k > 0$ (all entries > 0) for some positive integer k.

DEFINITION 4. An $n \times n$ matrix M is called *reducible* if there is a permutation that puts it into the form

$$\widetilde{M} = \begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix}$$

where M_{11} and M_{22} are square matrices. Otherwise M is called *irreducible*.

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Obviously, a primitive matrix is irreducible. For an irreducible non-negative matrix M, we have the following Frobenius Theorem (see [4]).

THEOREM. (Frobenius) *An irreducible non-negative matrix M always has a positive eigenvalue* λ. *The moduli of all the other eigenvalues do not exceed* λ. *And there is an eigenvector associated to* λ *with all positive coordinates.*

In this paper we use $\|M\|$ to denote the maximal modulus of eigenvalues of M.

DEFINITION 5. Let $(X; T_1, \dots, T_n)$ be a hyperbolic iterated function system with attractor A, and let M be a Markov transition matrix. We say that a point $a \in A$ is *M-attractive*, if there exists an M-admissible sequence $i_1, i_2, \dots$ such that

$$a = \lim_{m\to\infty} T_{i_1} \circ T_{i_2} \circ \dots \circ T_{i_m} x$$

for all $x \in X$. The set of all M-attractive points of A, denoted as A_M, is called the Markov attractor of the system associated with M.

Let $B_i = A_M \bigcap T_i(A) = \{a | a = \lim_{m\to\infty} T_i \circ T_{i_2} \circ \dots \circ T_{i_m} x, (i, i_2, \dots, i_m, \dots) \in \Sigma_M^+\}$. Then we have

$$B_i = \bigcup_{M_{ij}=1} T_i(B_j) \ .$$

For $((\Sigma_n^+, d); \sigma_1, \dots, \sigma_n)$, (Σ_M^+, d) is the Markov attractor. If M is a primitive Markov transition matrix, it is shown in [2] that

$$\dim((\Sigma_M^+, d)) = \frac{\log \|M\|}{\log 2} \quad .$$

3. Proof of the conjecture

In this section we give a proof of the conjecture when M is irreducible. That is, we will prove

THEOREM 1. *Suppose that* $(X; T_1, \dots, T_n)$ *is a disjoint hyperbolic iterated function system satisfying*

$$s_i d(x, y) \le d(T_i x, T_i y) \le \bar{s}_i d(x, y) \qquad \forall x, y \in X, \quad 1 \le i \le n,$$

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where $0 < s_i \le \bar{s}_i < 1$, M *is an irreducible Markov transition matrix, and* A_M *is the Markov attractor associated with* M. *Then*

$$l \le \dim(A_M) \le u$$

where

$$\|MS^l\| = 1 \qquad \textit{and} \qquad \|M\bar{S}^u\| = 1$$

and

$$S = \begin{pmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_n \end{pmatrix}, \qquad \bar{S} = \begin{pmatrix} \bar{s}_1 & & 0 \\ & \ddots & \\ 0 & & \bar{s}_n \end{pmatrix}.$$

In order to prove Theorem 1, we need to consider Σ_n^+. Now we define another metric d' on Σ_n^+ by

$$d'(\mathbf{i}, \mathbf{j}) = \begin{cases} (s_{i_1} \dots s_{i_k})^l, & i_1 = j_1, \dots, i_k = j_k; \quad i_{k+1} \ne j_{k+1} \\ 0, & i_1 = j_1, i_2 = j_2, \dots \\ 1, & i_1 \ne j_1. \end{cases}$$

where $\mathbf{i} = (i_1, i_2, \dots)$, $\mathbf{j} = (j_1, j_2, \dots)$.

It is easy to see that d' is a metric on Σ_n^+. Clearly, $((\Sigma_n^+, d'); \sigma_1, \dots, \sigma_n)$ is also a disjoint hyperbolic iterated function system with attractor (Σ_n^+, d'). In fact, (Σ_n^+, d') is a self-similar set with $\dim(\Sigma_n^+, d') = k$, where k satisfies

$$s_1^{lk} + s_2^{lk} + \dots + s_n^{lk} = 1.$$

For a Markov transition matrix M, (Σ_M^+, d') is also the Markov attractor of $((\Sigma_n^+, d'); \sigma_1, \dots, \sigma_n)$. We want to prove

PROPOSITION 1. *Suppose that* M *is an irreducible Markov transition matrix. Then*

$$\dim((\Sigma_M^+, d')) = 1.$$

PROOF. For any cover of Σ_M^+, say $\beta = \{B_j\}$, use $\beta(p)$ to denote the sum $\sum_j (d'(B_j))^p$, where p is a positive number. Let $\mathbf{v} = (v_1, v_2, \dots, v_n)'$ with $\sum_{i=1}^n v_i = 1$ be the eigenvector of MS^l associated with $\|MS^l\| = 1$ in Frobenius' Theorem.

Let $[i_1, \dots, i_k]$ be the set of all sequences of $\sum_n^+$ with $(i_1, \dots, i_k)$ as their first k entries and call it a *block*. Using β_k to represent the cover of Σ_M^+ consisting of all the M-admissible blocks $[i_1, i_2, \dots, i_k]$, we have

$$d'([i_1, i_2, \dots, i_k]) = (s_{i_1} s_{i_2} \dots s_{i_k})^l,$$

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where $d'[i_1, i_2, \dots, i_k]$ is the diameter of the set $[i_1, i_2, \dots, i_k]$. Clearly,

$$d'([i_1, i_2, \dots, i_k]) \to 0 \qquad \text{as} \quad k \to \infty.$$

So, if $\beta_k(1) < \infty$ for any k, then $\dim(\Sigma_M^+, d') \le 1$. Now

$$\begin{aligned}\beta_k(1) &= \sum_{\substack{i_1 \dots i_k \\ \text{admissible}}} d'([i_1, \dots, i_k]) = \sum_{\substack{i_1 \dots i_k \\ \text{admissible}}} (s_{i_1} \dots s_{i_k})^l \\ &\le c \sum_{\substack{i_1 \dots i_k \\ \text{admissible}}} (s_{i_1} \dots s_{i_k})^l v_{i_k} = c \sum_{i_1 \dots i_k} s_{i_1}^l (MS^l)_{i_1 i_2} \dots (MS^l)_{i_{k-1} i_k} v_{i_k} \\ &= c \sum_{i_1} s_{i_1}^l \sum_{i_k} (MS^l)_{i_1 i_k}^{k-1} v_{i_k} = c \sum_{i_1} s_{i_1}^l v_{i_1} \le c \sum_{i=1}^n s_i^l < \infty \quad ,\end{aligned}$$

where $c = 1/\min_i\{v_i\}$.

Now we show that the set

$$\{\ \beta(1) \mid \beta = \{B_j\} \ \text{ is a cover of } \ \Sigma_M^+\ \}$$

has a positive lower bound. Hence the 1-dimensional Hausdorff measure of (Σ_M^+, d') is positive, therefore $\dim((\Sigma_M^+)) \ge 1$.

Since (Σ_M^+, d') is compact, we need only consider finite covers. Suppose that $\beta = \{B_j, 1 \le j \le m\}$ is a cover of Σ_M^+. For any B_j, there exists $x, y \in B_j$, such that

$$d'(B_j) = d'(x, y) = (s_{i_1} s_{i_2} \dots s_{i_{k_j}})^l.$$

Hence for any $z \in B_j$, we have

$$d'(x, z) \le d'(x, y) \quad ,$$

so $z \in [i_1, \dots, i_k]$, and we get

$$B_j \subset [i_1, \dots, i_{k_j}].$$

It is reasonable to assume that each B is a block $[i_1, \dots, i_k]$ for some $i_1, \dots, i_k$.

Let t be the maximal length of a block in the cover $\beta = \{B_1, \dots, B_m\}$. For each $B_j = [i_1, \dots, i_{k_j}]$, consider the cover $\alpha_t(B_j)$ of B_j by blocks of length t. Then

$$\begin{aligned}\alpha_t(B_j)(1) &= \sum_{\nu_{k_j+1}, \dots, \nu_t}{}' d'([i_1, \dots, i_{k_j}, \nu_{k_j+1}, \dots, \nu_t]) \\ &= \sum_{\nu_{k_j+1}, \dots, \nu_t}{}' (s_{i_1} \dots s_{i_{k_j}} s_{\nu_{k_j+1}} \dots s_{\nu_t})^l \quad ,\end{aligned}$$

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where the sum $\sum'$ is over admissible sequences with $M_{i_{k_j}\nu_{k+1}} = 1$. Thus

$$\begin{aligned}
\alpha_t(B_j)(1) &= (s_{i_1}\dots s_{i_{k_j}})^l \sum_{\nu_{k_j+1},\dots,\nu_t} (M_{i_{k_j}\nu_{k_j+1}} s^l_{\nu_{k_j+1}}) \cdots (M_{\nu_{t-1}\nu_t} s^l_{\nu_t}) \\
&\le c(s_{i_1}\dots s_{i_{k_j}})^l \sum_{\nu_{k_j+1},\dots,\nu_t} (M_{i_{k_j}\nu_{k_j+1}} s^l_{\nu_{k_j+1}}) \cdots (M_{\nu_{t-1}\nu_t} s^l_{\nu_t}) v_{\nu_t} \\
&= c(s_{i_1}\dots .s_{i_{k_j}})^l \sum_{\nu} (MS^l)^{t-k}_{i_{k_j}\nu} v_\nu \\
&= c(s_{i_1}\dots s_{i_{k_j}})^l v_{i_{k_j}} \le c d'(B_i) \quad .
\end{aligned}$$

Hence

$$\begin{aligned}
\beta(1) &= \sum_j d'(B_j) \ge c^{-1} \sum_j \alpha_t(B_j)(1) \\
&= c^{-1} \sum_j \sum_{\nu_{k_j+1},\dots,\nu_t}{}' (s_{i_1}\dots s_{i_{k_j}} s_{\nu_{k_j+1}} \dots s_{\nu_t})^l \\
&\ge c^{-1} \sum_j \sum_{\nu_{k_j+1},\dots,\nu_t}{}' (s_{i_1}\dots s_{i_{k_j}} s_{\nu_{k_j+1}} \dots s_{\nu_t})^l v_{\nu_t} \\
&\ge c^{-1} \sum_{\substack{i_1\dots i_t \\ \text{admissible}}} (s_{i_1}\dots s_{i_t})^l v_{i_t} \\
&= c^{-1} \sum_{i,j} (MS^l)^t_{ij} v_j \\
&= c^{-1} \sum_i v_i = c^{-1} \quad .
\end{aligned}$$

Now we have proved Proposition 1.

If we use $\bar{s}_i$ instead of s_i and u instead of l, we obtain the same result.
For the proof of Theorem 1, we also need the following lemma (see [6]).

LEMMA. *Let X,Y be two metric space with a map $f: \quad X \mapsto Y$, and δ, c be positive constants. Then we have*

(a) *if $d(f(x), f(y)) \ge c d(x, y)^\delta$, then*

$$\dim(Y) \ge \frac{1}{\delta}\dim(X) \ ;$$

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(b) *if* $f(X) = Y$ *and* $d(f(x), f(y)) \leq cd(x, y)^{\delta}$, *then*

$$\dim(Y) \leq \frac{1}{\delta}\dim(X) \ .$$

Now we can prove Theorem 1.

PROOF OF THEOREM 1. Define $f: \ \Sigma_M^+ \mapsto A_M$ by

$$f(i_1, i_2, \dots) = \lim_{m\to\infty} T_{i_1} \circ T_{i_2} \circ \dots \circ T_{i_m} x \qquad \forall x \in X \ .$$

Since the limit on the right hand side is independent of x, f is well defined. By disjointness we have

$$c = \inf\{\, d(x, y) | x \in T_i(A), y \in T_j(A), i \neq j \,\} > 0.$$

For any $\mathbf{i} = (i_1, \ i_2, \ \dots)$, $\mathbf{j} = (j_1, \ j_2, \ \dots) \in \Sigma_M^+$, we estimate the distance between $a = \lim_{m\to\infty} T_{i_1} \circ T_{i_2} \circ \dots \circ T_{i_m} x$ and $b = \lim_{m\to\infty} T_{j_1} \circ T_{j_2} \circ \dots \circ T_{j_m} x$.

Suppose

$$d'(\mathbf{i}, \mathbf{j}) = (s_{i_1} s_{i_2} \dots s_{i_k})^l \quad ,$$

that is

$$i_t = j_t, \qquad 1 \leq t \leq k; \qquad i_{k+1} \neq j_{k+1}.$$

Let

$$a' = \lim_{m\to\infty} T_{i_{k+1}} \circ \dots \circ T_{i_m} x \quad \text{and} \qquad b' = \lim_{m\to\infty} T_{j_{k+1}} \circ \dots \circ T_{j_m} x.$$

We have $a' \in T_{i_{k+1}}(A)$, $b' \in T_{j_{k+1}}(A)$, so that

$$d(a', b') \geq c \ .$$

Moreover,

$$a = T_{i_1} \circ T_{i_2} \circ \dots \circ T_{i_k} a', \qquad b = T_{i_1} \circ T_{i_2} \circ \dots \circ T_{i_k} b' \ .$$

Hence

$$d(a, b) \geq s_{i_1} s_{i_2} \dots s_{i_k} \, d(a', b') \geq s_{i_1} s_{i_2} \dots s_{i_k} c = cd'(\mathbf{i}, \ \mathbf{j})^{l^{-1}}.$$

Applying the first part of the lemma ($\delta = l^{-1}$) we know

$$\dim(A_M) \geq \frac{1}{l^{-1}} \dim(\Sigma_M^+, d') = l \ .$$

On the other hand, using the second part of the lemma ($\delta = u^{-1}$) and the result of Proposition 1 when $\bar{s}_i$ is instead of s_i and u instead of l, we obtain

$$\dim(A_M) \leq u \ .$$

The proof is completed.

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4. Generalization

Now we assume that M is reducible. At first we suppose that

$$M = \begin{pmatrix} M_1 & M_{12} \\ 0 & M_2 \end{pmatrix}$$

where M_1 and M_2 are $m \times m$ and $(n-m) \times (n-m)$ irreducible matrices respectively.

If $M_{12} = 0$, we consider $(X; T_1, \dots, T_m)$ and $(X; T_{m+1}, \dots, T_n)$, and get

$$l_1 \leq \dim(A_{M_1}) \leq u_1 \quad \text{and} \quad l_2 \leq \dim(A_{M2}) \leq u_2$$

where

$$\|M_1 S_1^{l_1}\| = \|M_1 \bar{S}_1^{u_1}\| = 1 \quad \text{and} \quad \|M_2 S_2^{l_2}\| = \|M_2 \bar{S}_2^{u_2}\| = 1$$

and

$$S_1 = \begin{pmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_m \end{pmatrix}, \qquad S_2 = \begin{pmatrix} s_{m+1} & & 0 \\ & \ddots & \\ 0 & & s_n \end{pmatrix},$$

with similar definitions for $\bar{S}_1$ and $\bar{S}_2$.

Clearly,

$$A_M = A_{M_1} \bigcup A_{M_2} \quad \text{and} \quad l \leq \dim(A_M) \leq u$$

where

$$l = \max\{l_1, l_2\}, \qquad u = \max\{u_1, u_2\}.$$

And we have

$$\|MS^l\| = \max\{\|M_1 S_1^l\}\|, \qquad \|M_2 S_2^l\|\} = 1,$$

and

$$\|M\bar{S}^{\bar{u}}\| = \max\{\|M_1 \bar{S}_1^u\|, \qquad \|M_2 \bar{S}_2^u\|\} = 1 \ .$$

If $M_{12} \neq 0$, the M-admissible sequence related to M_{12} is

$$i_1, i_2, \dots, i_k, j_{k+1}, j_{k+2}, \dots$$

where $(i_1, \dots, i_k)$ and $(j_{k+1}, j_{k+2}, \dots)$ are M_1 and M_2 admissible respectively with $M_{i_k j_{k+1}} = 1$. Since the set

$$\{(i_1, i_2, \dots, i_k) \mid \ (i_1, i_2, \dots, i_k) \quad \text{is M-admissible, } k = 1, 2, \dots\}$$

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is countable, we denote the related composite maps as $T(1), T(2), \dots$. Using $A_{M_{12}}$ to denote the set of all the M-attractive points related with M_{12} , we have

$$A_{M_{12}} = \bigcup_{i=1}^{\infty} T(i)(A_{M_2}).$$

So

$$\begin{aligned}\dim(A_{M_{12}}) &= \dim\left(\bigcup_{i=1}^{\infty} T(i)(A_{M_2})\right)\\ &= \sup_{1\le i<\infty}\left\{\dim\left(T(i)(A_{M_2})\right)\right\} = \dim(A_{M_2}).\end{aligned}$$

But

$$A_M = A_{M_1}\bigcup A_{M_2}\bigcup A_{M_{12}};$$

hence

$$\dim(A_M) = \max\{\dim(A_{M_1}), \dim(A_{M_2}\}$$

and

$$l \le \dim(A_M) \le u.$$

Certainly

$$\|MS^l\| = 1 \quad\text{and}\quad \|M\bar{S}^u\| = 1.$$

In general, a Markov transition matrix M with $\|M\| > 0$, can acquire the form

$$\widetilde{M} = \begin{pmatrix} M_1 & & & * \\ & \ddots & & \\ & & M_{k-1} & \\ 0 & & & M_k \end{pmatrix}$$

through permutation, where $M_1, \dots, M_{k-1}$ are irreducible , and M_k irreducible or of the form

$$M_k = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}.$$

Without loss of generality, we can assume

$$M = \begin{pmatrix} M_1 & & & * \\ & \ddots & & \\ & & M_{k-1} & \\ 0 & & & M_k \end{pmatrix}.$$

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If M_k is irreducible, from the above we know that

$$\dim(A_M) = \max\{\dim(A_{M_1}), \ldots, \dim(A_{M_k})\}.$$

Hence

$$l \le \dim(A_M) \le u$$

where

$$l = \max\{l_1, \ldots, l_k\} \quad \text{and} \quad u = \max\{u_1, \ldots, u_k\}$$

and l_i, u_i have the same meaning as in the case $k = 2$.

In the case

$$M_k = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix},$$

there is no infinite M-admissible related to M_k, so $A_{M_k} = \emptyset$. Since $\|M\| > 0$, we must have $k > 1$. Again we have

$$\dim(A_M) = \max\{\dim(A_{M_1}), \ldots, \dim(A_{M_{k-1}}), \dim(A_{M_k})\}$$

and

$$l \le \dim(A_M) \le u$$

where

$$l = \max\{l_1, \ldots, l_{k-1}\} \quad \text{and} \quad u = \max\{u_1, \ldots, u_{k-1}\}.$$

In both cases we can easily see

$$\|MS^l\| = 1 \quad \text{and} \quad \|M\bar{S}^u\| = 1.$$

Now we have proved

THEOREM 2. *Suppose that $(X; T_1, \ldots, T_n)$ is a disjoint hyperbolic iterated function system, and M is a Markov transition matrix with at least one non-zero eigenvalue. Then we have*

$$l \le \dim(A_M) \le u$$

where

$$\|MS^l\| = 1 \quad \textit{and} \quad \|M\bar{S}^u\| = 1.$$

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5. Not all T_i's need to be contractions

Feiste showed in [3] that for an iterated function system $(X; T_1, \dots, T_n)$, if it is cyclically contracting with respect to an irreducible Markov transition matrix M, the Markov attractor A_M exists. We will show that the result of Theorem 2 holds in this case.

DEFINITION 6. Suppose a Markov transition matrix M is given. A *path* from i_1 to i_k is a finite M-admissible sequence $i_1, i_2, \dots, i_k$. A *cycle* is a path with $M_{i_k i_1} = 1$. By *elementary path* or *elementary cycle* we mean a path or a cycle for which $i_s \neq i_t$, when $s \neq t$.

DEFINITION 7. Definition 7 Let $(X; T_1, \dots, T_n)$ be an iterated function system, where T_i's are Lipschitz maps with $Lip(T_i) = r_i$, and M be a Markov transition matrix. $(X; T_1, \dots, T_n)$ is *called cyclically* contracting if for any elementary cycle $i_1, \dots, i_k$ we have $r_{i_1} r_{i_2} \dots r_{i_m} < 1$.

In [3] we find the following theorem.

THEOREM. (Feiste) *Let $(X; T_1, \dots, T_n)$ be an iterated function system, where T_i's are Lipschitz maps. If $(X; T_1, \dots, T_n)$ is cyclically contracting with respect to an irreducible Markov transition matrix M then there is a unique m-tuple $B = (B_1, \dots, B_n)$ of compact subsets $B_i \subset X$ with*

$$B_i = \bigcup_{M_{ij}=1} T_i(B_j)$$

for all $i \in \{1, \dots, n\}$.

Bandt proved in [1] that

$$B_i = \{a | a = \lim_{m \to \infty} T_i \circ T_{i_2} \circ \dots \circ T_{i_m} x; (i, i_2, \dots, i_m, \dots) \in \Sigma_M^+, x \in X\} .$$

Let $A_M = \bigcup_{i=1}^n B_i$. We also call it the Markov attractor of $(X; T_1, \dots, T_n)$, though the attractor A may not exist in this situation. If $B_i \bigcap B_j = \emptyset$ we also call $(X; T_1, \dots, T_n)$ disjoint iterated function system.

Now we generalize Theorem 1 to the case $(X; T_1, \dots, T_n)$ is cyclically contracting.

Suppose that M is an irreducible Markov transition matrix, $\{s_1, s_2, \dots, s_n\}$ is a group of positive constants such that for any elementary cycle $i_1, i_2, \dots, i_k$ we

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have $s_{i_1}s_{i_2}\dots s_{i_k} < 1$, and l is a constant satisfying $\|MS^l\| = 1$. As in Section 2, we define a metric d'' on Σ_M^+ by

$$d''(\mathbf{i},\mathbf{j}) = \begin{cases} (s_{i_1}\dots s_{i_k})^l v_{i_k}, & i_1 = j_1,\ \dots,\ i_k = j_k;\ \ i_{k+1} \neq j_{k+1} \\ \max\{s_i\}, & i_1 \neq j_1, \end{cases}$$

where $\mathbf{i} = (i_1, i_2, \dots)$, $\mathbf{j} = (j_1, j_2, \dots)$ are M-admissible sequences, and $\mathbf{v} = (v_1, v_2, \dots, v_n)'$ has the same meaning as in the proof of Proposition 1.

Suppose $\mathbf{i} = (i_1, \dots, i_k, i_{k+1}, \dots)$, $\mathbf{j} = (i_1, \dots, i_k, j_{k+1}, \dots)$ and $\mathbf{t} = (i_1, \dots, i_r, t_{r+1}, \dots)$ are M-admissible squences with $i_{r+1} \neq t_{r+1}$, $i_{k+1} \neq j_{k+1}$ and $r < k$. We have

$$\begin{aligned} d''(\mathbf{i},\mathbf{t}) &= (s_{i_1}\dots s_{i_r})^l v_{i_r} = (s_{i_1}\dots s_{i_r})^l \sum_{\nu=1}^{n} (MS^l)_{i_r\nu}^{k-r} v_\nu \\ &\geq (s_{i_1}\dots s_{i_k})^l v_{i_k} = d''(\mathbf{i},\mathbf{j}). \end{aligned}$$

Hence $d''(\mathbf{i},\mathbf{j}) \leq d''(\mathbf{i},\mathbf{t}) + d''(\mathbf{t},\mathbf{j})$. Thus d'' is really a metric on Σ_M^+. But this time d'' is not a metric on Σ_n^+. In [1] it is shown that $(s_{i_1}\dots s_{i_k})^l v_{i_k} \to 0$, as $k \to \infty$. In the same way as in Section 2 we can also prove

PROPOSITION 2. $\dim(\Sigma_M^+,\ d'') = 1$.

Using Lemma 2, with $\delta = l^{-1}$ for the first part, and $\delta = u^{-1}$ for the second, we obtain

THEOREM 3. *Let M be an irreducible Markov transition matrix, and let $\{X; T_1, \dots, T_n\}$ be a disjoint iterated function system cyclically contracting with respect to M satisfying*

$$s_i d(x,y) \leq d(T_i x, T_i y) \leq \bar{s}_i d(x,y).$$

Then

$$l \leq \dim(A_M) \leq u$$

where $\|MS^l\| = 1$ and $\|M\bar{S}^u\| = 1$.

As in Section 3, we can generalise Theorem 3 to the case $\|M\| > 0$, where M need not be irreducible. Finally we obtain

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THEOREM 4. *Suppose that M is a Markov transition matrix with at least one non-zero eigenvalue, and that $(X; T_1, \ldots, T_n)$ is a disjoint iterated function system cyclically contracting with respect to M satisfying*

$$s_i d(x, y) \le d(T_i x, T_i y) \le \bar{s}_i d(x, y).$$

Then we have

$$l \le \dim(A_M) \le u$$

where $\|MS^l\| = 1$ and $\|M\bar{S}^u\| = 1$.

NOTE ADDED IN PROOF. The technique used in the proofs of Theorem 1 and 3 can also be applied to fractals constructed with sofic systems (see [1]). Let $\{X; T_1, \ldots, T_n\}$ be an iterated function system. Let $Q_i (i = 1, \ldots, m)$ be a non-empty subset of $\{1, \ldots, n\} \times \{1, \ldots, m\}$ and $F = \{(k_1 k_2 \ldots) |$ there are $i_0, i_1, \ldots \in \{1, \ldots, m\}$, with $(k_s, i_s) \in Q_{i_{s-1}}\}$. If $\{X; T_1, \ldots, T_n\}$ is cyclically contracting with F, there exist non-empty compact subsets $C_1, \ldots, C_n$ such that $C_i = \cup\{T_k(C_j) | (k, j) \in Q_i\}$. Construct an $m \times m$ matrix such that $M(\mathbf{r}, \alpha)$ by letting $M_{ij} = \sum\{r_k^\alpha | (k, j) \in Q_i\}$, for $\mathbf{r} = (r_1, \ldots, r_N)$ with $r_i > 0$. Assume $M(\mathbf{r}, \alpha)$ is irreducible. Then there is a unique α, such that $\|M(\mathbf{r}, \alpha)\| = 1$, if $r_i = Lip(T_i)$.

THEOREM. *Suppose $\{X; T_1, \ldots, T_N\}$ is cyclically contracting with F and satisfies $s_i d(x, y) \le d(T_i(x), T_i(y)) \le \bar{s}_i d(x, y)$. Suppose for each i, $C_i = \cup_{(k,j) \in Q_i} T_k(C_j)$ is a disjoint union. Then $l \le \dim(C_i) \le u$, where l and u are determined by $\|M(s, l)\| = 1$ and $\|M(\bar{s}, u)\| = 1$, and where $s = (s_1, s_2, \ldots, s_N)$, $\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_N)$.*

Details of the proof of the theorem will appear in my Ph.D thesis.

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