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A NOTE ON COMPACT SETS IN SPACES OF SUBSETS

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A simple characterisation is given of compact sets of the space $\mathcal{K}(X)$, of nonempty compact subsets of a complete metric space X, with the Hausdorff metric d_H. It is used to give a new proof of the Blaschke selection theorem for compact starshaped sets.

Let $(\mathcal{E}^n, D)$ be the metric space of fuzzy sets on $\mathbf{R}^n$, with D the supremum over Hausdorff distances between corresponding level sets. Compact sets of this space have recently been characterised [**2**]: roughly speaking, a closed set $\mathcal{U}$ of $(\mathcal{E}^n, D)$ is compact if and only if

(1) fuzzy sets in $\mathcal{U}$ have uniformly bounded support, and
(2) the level set maps $\alpha \mapsto [u]^\alpha = \{x \in \mathbf{R}^n : \alpha \leqslant u(x) \leqslant 1\}$, $0 < \alpha \leqslant 1$, are uniformly equi-leftcontinuous for all $u \in \mathcal{U}$.

The Blaschke selection theorem and its converse follow as a simple corollary: that compact sets of the metric space $(\mathcal{K}^n_{co}, d_H)$ of nonempty compact convex subsets of $\mathbf{R}^n$, where d_H is the induced Hausdorff metric, are the closed, uniformly bounded subsets of $\mathcal{K}^n_{co}$. Such subsets are represented as fuzzy sets by their characteristic functions. These are independent of α, so condition (2) is automatically satisfied.

On the other hand, it is well-known that if (X, d) is a compact space, then so too is $(\mathcal{K}(X), d_H)$, the space of nonempty compact subsets of X with metric d_H [**3**]. Thus B compact in X gives $\mathcal{K}(B)$ compact in $\mathcal{K}(X)$. However, converse theorems (which then characterise compact subsets of $\mathcal{K}(X)$) have apparently not been stated. Such results are not without application, because frequently a subset of $\mathcal{K}(X)$ is studied, rather than the whole space, as in formulating concepts of random sets [**4**, **5**]. We present below a direct proof of such a characterisation, which is more convenient than the usual one involving total boundedness. As a direct application, Corollary 2 is a new proof of the Blaschke theorem and its converse for starshaped sets, without gauge functions [**1**].

DEFINITION: A nonempty subset $\mathcal{U}$ of $(\mathcal{K}(X), d_H)$ is said to be compactly bounded (in X) if there exists a compact subset W of X such that $U \subseteq W$ for every $U \in \mathcal{U}$.

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PROPOSITION. *Let* (X, d) *be a complete metric space. Then a nonempty closed subset* $\mathcal{C}$ *of* $(\mathcal{K}(X), d_H)$*, the metric space of nonempty compact subsets of* X*, is compact if and only if* $\mathcal{C}$ *is compactly bounded.*

COROLLARY 1. *A nonempty closed subset* $\mathcal{C}$ *of* $(\mathcal{K}(\mathbf{R}^n), d_H)$ *is compact if and only if* $\mathcal{C}$ *is uniformly bounded in* $\mathbf{R}^n$*.*

COROLLARY 2. *Let* $\mathcal{V}$ *be a nonempty closed subset of the class of nonempty compact starshaped sets in* $\mathbf{R}^n$*. Then* $\mathcal{V}$ *is compact if and only if* $\mathcal{V}$ *is uniformly bounded in* $\mathbf{R}^n$*.*

PROOF OF PROPOSITION: Write $W(\mathcal{C}) = \bigcup\{C : C \in \mathcal{C}\}$. If $W(\mathcal{C}) \subseteq V$, where V is compact in X, then $\mathcal{C} \subseteq \mathcal{K}(V) \subseteq \mathcal{K}(X)$, and $\mathcal{C}$ is closed in the subspace topology on $\mathcal{K}(V)$. Thus $\mathcal{C}$ is compact in $\mathcal{K}(V)$, and in $\mathcal{K}(X)$. Conversely, if $\mathcal{C}$ is compact in $(\mathcal{K}(X), d_H)$ then it is also sequentially compact. Let $\{x_i\}$ be a sequence in $W(\mathcal{C})$. Then there is a corresponding sequence of sets $\{C_i\}$ in $\mathcal{C}$, such that $x_i \in C_i$ for each i. Compactness of $\mathcal{C}$ gives a subsequence $\{C_{i(j)}\}$ converging to $C_0 \in \mathcal{C}$. So $d(x_{i(j)}, C_0) \to 0$ as $i(j) \to \infty$. Since C_0 is itself compact, there is a further subsequence $\{x_{k'}\} \subseteq \{x_{i(j)}\}$ converging to some $x_0 \in C_0$. Thus $W(\mathcal{C})$ is a sequentially compact subset of X, and $\mathcal{C}$ is compactly bounded. ∎

PROOF OF COROLLARY 2: Let $\{C_k\}$ be a bounded sequence of compact starshaped sets in $\mathcal{V}$. A subsequence $\{C_{k'}\} \subseteq \{C_k\}$ converges if and only if, for any bounded sequence of points $p_k \in \mathbf{R}^n$, a subsequence of $\{C_k - p_k\}$ converges. Thus it suffices to consider only sequences where each C_k is starshaped with respect to the origin O. From Corollary 1 it only remains to show that the (relabelled) subsequence $\{C_k\}$ has limit $C \in \mathcal{K}(\mathbf{R}^n)$ which is starshaped. If C were not starshaped, there would exist $y \in C$, $z \notin C$ such that $z \in$ the line segment Oy. Recall that the Hausdorff semidistance $\rho(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|$ has the properties

$$\rho(A,B) = 0 \quad \text{if and only if} \quad A \subseteq B$$
$$\rho(A,C) \leqslant \rho(A,C) + \rho(B,C)$$

for sets in $\mathcal{K}(\mathbf{R}^n)$. By compactness, there exists $\eta > 0$ such that $\rho(z, C) = \eta$. For k sufficiently large, $\rho(C_k, C) \leqslant d_H(C_k, C) < \eta/3$, so select $y_k \in C_k$ such that $\|y - y_k\| < \eta/3$. Then there exists $z_k \in Oy_k \subset C_k$ such that $\|z - z_k\| < \eta/3$. Thus

$$\eta = \rho(z, C) \leqslant \rho(z, z_k) + \rho(z_k, C_k) + \rho(C_k, C) < \eta/3 + 0 + \eta/3,$$

a contradiction, and C is starshaped. ∎

Note. This characterisation of compact sets is not true for $(2^X, d_H)$, where 2^X is the set of nonempty closed subsets of X, when X is not compact. For example, $\mathcal{C} = \{C_\gamma = [\gamma, \infty) : 0 \leqslant \gamma \leqslant 1\}$ is a compact subset of $(2^{\mathbf{R}}, d_H)$, as $d_H(C_\gamma, C_\beta) = |\gamma - \beta|$ and thus $\mathcal{C}$ is totally bounded in $2^{\mathbf{R}}$. However, $\bigcup_{0 \leqslant \gamma \leqslant 1} C_\gamma = \mathbf{R}^+$ is not a compact subset in $\mathbf{R}$ and $\mathcal{C}$ is not compactly bounded.

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