J. Austral. Math. Soc. **19** (Series B), (1976), 259–288.

A TRANSFORMATION OF COOKE'S TREATMENT OF SOME TRIPLE INTEGRAL EQUATIONS

E. R. LOVE and D. L. CLEMENTS

(Received 19 June 1975)

Abstract

The reduction of an important class of triple integral equations to a pair of simultaneous Fredholm equations has been carried out by Cooke [1]. In this paper, Cooke's equations are transformed to new uncoupled Fredholm equations which, for certain important cases, are shown to be simpler than Cooke's and also superior for the purposes of solution by iteration.

1. Introduction

Cooke [1], using the method of Noble [2] for dual integral equations, reduced the triple integral equations (1.1), (1.2) and (1.3) below to two simultaneous Fredholm equations of the second kind. These integral equations occur in potential problems with different boundary conditions on an annulus $a < r < b$, its inside $0 < r < a$ and its outside $b < r < \infty$. The triple equations are

$$\int_0^\infty A(\lambda) J_n(r\lambda)\, d\lambda = q_1(r) \quad \text{for} \quad r < a, \tag{1.1}$$

$$\int_0^\infty \lambda^{2\alpha} A(\lambda) J_n(r\lambda)\, d\lambda = p_0(r) \quad \text{for} \quad a < r < b, \tag{1.2}$$

$$\int_0^\infty A(\lambda) J_n(r\lambda)\, d\lambda = q_2(r) \quad \text{for} \quad r > b, \tag{1.3}$$

where everything is known except the function $A(\lambda)$. They constitute essentially a Fredholm equation of the first kind but on a semi-infinite interval. The constants n and α are restricted only to satisfy $n \geqq 0$ and $0 < \alpha < 1$ or $-1 < \alpha < 0$. The cases n positive integral and $\alpha = \pm\frac{1}{2}$ are important; and probably the axisymmetric case $n = 0$ is the most important.

In this paper we transform Cooke's integral equations of the second kind, for all $n \geqq 0$ and either $0<\alpha<1$ or $-1<\alpha<0$, into a pair of uncoupled integral equations of the second kind. For $0<\alpha<1$, the new equations are shown to be satisfactory for solution by iteration for all a/b sufficiently small. For values of α not too near to 1, a/b may in fact be as large as $\frac{3}{4}$; and in most cases (see Tables A and B) it may be much closer to 1. For comparison we examine also the effectiveness of iteration of Cooke's equations.

The known terms in our equations, that is, the terms involving p_0, q_1 and q_2, are simpler than those in Cooke's, involving single integrals rather than double. The kernels are less simple than Cooke's, except in the case $n=0$, when they are even simpler. In the general case they involve a hypergeometric function. In the important case $n=0$ the equations are

$$f_1(r)+\frac{2\sin\pi\alpha}{\pi}\int_b^\infty \frac{s}{s^2-r^2}f_2(s)\,ds = g_1(r) \quad \text{for} \quad r<a, \tag{1.4}$$

$$f_2(r)+\frac{2\sin\pi\alpha}{\pi}\int_0^a \frac{s}{r^2-s^2}\frac{r^{2\alpha}}{s^{2\alpha}}f_1(s)\,ds = g_2(r) \quad \text{for} \quad r>b, \tag{1.5}$$

where f_1 and f_2 are unknown while g_1 and g_2 are known functions which assume different forms in the intervals $-1<\alpha<0$ and $0<\alpha<1$.

We further reduce our equations to two uncoupled equations. These are proper Fredholm equations of the second kind on a finite interval, with bounded kernels. In the important case $n=0$ these equations are

$$f_+(x)+\frac{2\sin\pi\alpha}{\pi}\int_0^k \frac{(xt)^{\frac{1}{2}-\alpha}}{1-x^2t^2}f_+(t)\,dt = g_+(x) \quad \text{for} \quad 0<x<k, \tag{1.6}$$

$$f_-(x)-\frac{2\sin\pi\alpha}{\pi}\int_0^k \frac{(xt)^{\frac{1}{2}-\alpha}}{1-x^2t^2}f_-(t)\,dt = g_-(x) \quad \text{for} \quad 0<x<k, \tag{1.7}$$

where $f_\pm$ are unknown and $g_\pm$ are known functions, and $k=\sqrt{(a/b)}$, so that $0<k<1$.

Our analysis is formal, except that assumptions are made explicit in two cases where they are clearly necessary. An example of the risks in formal analysis is seen in §5, where the expression for the value of an integral needs an unexpected modification when $n=0$. It is interesting to see how this exception disappears in the final transformed equations; if there had been no exception in §5, there would have been one in the final equations.

Finally, before proceeding with the analysis, we note that a number of authors (for example, Gubenko and Mossakovskii [3], Williams [4] and Clements and Love [5]) have studied boundary-value problems within the class considered in this paper but with less generality and from other starting points. The work of these authors does, in special cases (notably when $n = 0$ and $\alpha = \pm\frac{1}{2}$), have points of contact with the present work.

2. Cooke's equations

A Fredholm equations for $\int_0^\infty \lambda^{2\alpha} A(\lambda) J_n(\lambda r)\, d\lambda.$

In [1, §4] Cooke obtains equations for the two unknown functions

$$p_1(r) = \int_0^\infty \lambda^{2\alpha} A(\lambda) J_n(\lambda r)\, d\lambda \quad \text{for} \quad r < a, \tag{2.1}$$

$$p_2(r) = \int_0^\infty \lambda^{2\alpha} A(\lambda) J_n(\lambda r)\, d\lambda \quad \text{for} \quad r > b, \tag{2.2}$$

when $A(\lambda)$ is required to satisfy the triple equations (1.1) to (1.3).

We always suppose $r \geqq 0$. We have made changes in Cooke's notation for various minor reasons, including the desirability of using the same letter p in (1.2), (2.1) and (2.2). It will in fact be convenient to write

$$\begin{aligned} p(r) &= p_1(r) \quad \text{for} \quad r < a, \\ p(r) &= p_0(r) \quad \text{for} \quad a < r < b, \\ p(r) &= p_2(r) \quad \text{for} \quad r > b. \end{aligned} \tag{2.3}$$

Thus we write p_1, p_0, p_2, q_1 and q_2 for Cooke's f_1, g, f_2, f and h respectively.

For the case $0 < \alpha < 1$, Cooke's equations for $p_1(r)$ and $p_2(r)$ are

$$p_1(r) + \frac{2\sin\pi\alpha}{\pi}\int_b^\infty \frac{r^n s^{-n+1}}{s^2 - r^2}\frac{(s^2 - a^2)^\alpha}{(a^2 - r^2)^\alpha}\, p_2(s)\, ds = k_1(r) + h_1(r) \quad \text{for} \quad r < a, \tag{2.4}$$

$$p_2(r) + \frac{2\sin\pi\alpha}{\pi}\int_0^a \frac{r^{-n} s^{n+1}}{r^2 - s^2}\frac{(b^2 - s^2)^\alpha}{(r^2 - b^2)^\alpha}\, p_1(s)\, ds = k_2(r) + h_2(r) \quad \text{for} \quad r > b, \tag{2.5}$$

where k_1, h_1, k_2, h_2 are the known functions

$$k_1(r) = -\frac{2\sin\pi\alpha}{\pi}\int_a^b \frac{r^n s^{-n+1}}{s^2 - r^2}\frac{(s^2 - a^2)^\alpha}{(a^2 - r^2)^\alpha}\, p_0(s)\, ds, \tag{2.6}$$

$$k_2(r) = -\frac{2\sin\pi\alpha}{\pi}\int_a^b \frac{r^{-n} s^{n+1}}{r^2 - s^2}\frac{(b^2 - s^2)^\alpha}{(r^2 - b^2)^\alpha}\, p_0(s)\, ds, \tag{2.7}$$

$$h_1(r) = -\frac{2^{2\alpha} r^{n-1}}{\Gamma(1-\alpha)^2} \frac{d}{dr} \int_r^a \frac{s^{-2n+2\alpha}}{(s^2-r^2)^\alpha} \times \left(\frac{d}{ds} \int_0^s \frac{t^{n+1}}{(s^2-t^2)^\alpha} q_1(t)\, dt \right) ds, \tag{2.8}$$

$$h_2(r) = -\frac{2^{2\alpha} r^{-n-1}}{\Gamma(1-\alpha)^2} \frac{d}{dr} \int_b^r \frac{s^{2n+2\alpha}}{(r^2-s^2)^\alpha} \times \left(\frac{d}{ds} \int_s^\infty \frac{t^{-n+1}}{(t^2-s^2)^\alpha} q_2(t)\, dt \right) ds. \tag{2.9}$$

For the case $-1 < \alpha < 0$, Cooke's equations are exactly (2.4) and (2.5) except that h_1 and h_2 are given by

$$h_1(r) = \frac{2^{2-2\beta} r^n}{\Gamma(\beta)^2} \int_r^a \frac{s^{-2n-2\beta+1}}{(s^2-r^2)^{1-\beta}}\, ds \int_0^s \frac{t^{n+1}}{(s^2-t^2)^{1-\beta}} q_1(t)\, dt, \tag{2.10}$$

$$h_2(r) = \frac{2^{2-2\beta} r^{-n}}{\Gamma(\beta)^2} \int_b^r \frac{s^{2n-2\beta+1}}{(r^2-s^2)^{1-\beta}}\, ds \int_s^\infty \frac{t^{-n+1}}{(t^2-s^2)^{1-\beta}} q_2(t)\, dt. \tag{2.11}$$

We have put $\alpha = -\beta$, so that $0 < \beta < 1$; this is intended to help in checking convergence of the integrals.

B Fredholm equations for $\int_0^\infty A(\lambda) J_n(r\lambda)\, d\lambda$.

In [1, §5] Cooke obtains equations from which the unknown function

$$q(r) = \int_0^\infty A(\lambda) J_n(r\lambda)\, d\lambda \quad \text{for} \quad a < r < b \tag{2.12}$$

can be found when $A(\lambda)$ is required to satisfy (1.1) to (1.3). To do this he replaces the unknown function $A(\lambda)$ by two more, $A_1(\lambda)$ and $A_2(\lambda)$, whose sum is $A(\lambda)$. Another relation is imposed on $A_1(\lambda)$ and $A_2(\lambda)$ later; this relation is implied in the equations [1, (30) and (31), or (32) and (33)]. We do not need the details of it here; we need only the transforms

$$q_3(r) = \int_0^\infty A_1(\lambda) J_n(\lambda r)\, d\lambda \quad \text{for} \quad r < b, \tag{2.13}$$

$$q_4(r) = \int_0^\infty A_2(\lambda) J_n(\lambda r)\, d\lambda \quad \text{for} \quad r > a. \tag{2.14}$$

For the case $0 < \alpha < 1$, Cooke shows that q_3 and q_4 satisfy the simultaneous integral equations

$$q_3(r) + \frac{2\sin\pi\alpha}{\pi}\int_b^\infty \frac{r^n s^{-n+1}}{s^2 - r^2}\frac{(b^2 - r^2)^\alpha}{(s^2 - b^2)^\alpha}[q_4(s) - q_2(s)]\,ds$$
$$= e_1(r) \quad \text{for} \quad r < b, \tag{2.15}$$

$$q_4(r) + \frac{2\sin\pi\alpha}{\pi}\int_0^a \frac{r^{-n} s^{n+1}}{r^2 - s^2}\frac{(r^2 - a^2)^\alpha}{(a^2 - s^2)^\alpha}[q_3(s) - q_1(s)]\,ds$$
$$= e_2(r) \quad \text{for} \quad r > a, \tag{2.16}$$

where e_1 and e_2 are known functions which take the forms

$$e_1(r) = \frac{2^{2-2\alpha}}{\Gamma(\alpha)^2}\int_r^b \frac{r^n s^{-2n-2\alpha+1}}{(s^2 - r^2)^{1-\alpha}}\,ds\int_0^s \frac{t^{n+1}}{(s^2 - t^2)^{1-\alpha}}p_3(t)\,dt, \tag{2.17}$$

$$e_2(r) = \frac{2^{2-2\alpha}}{\Gamma(\alpha)^2}\int_a^r \frac{r^{-n} s^{2n-2\alpha+1}}{(r^2 - s^2)^{1-\alpha}}\,ds\int_s^\infty \frac{t^{-n+1}}{(t^2 - s^2)^{1-\alpha}}p_4(t)\,dt. \tag{2.18}$$

Here p_3 and p_4 are known functions such that $p_3(t) + p_4(t) = p_0(t)$ for $a < t < b$. They are defined by some dissection such as [1, §2, where g_1 and g_2 are the counterparts of p_3 and p_4] or [5, §5, where V_3 and V_4 are the counterparts of p_3 and p_4].

For the case $-1 < \alpha < 0$, Cooke's equations are exactly (2.15) and (2.16) with e_1 and e_2 given instead by

$$e_1(r) = \frac{-2^{-2\alpha} r^{n-1}}{\Gamma(1+\alpha)^2}\frac{d}{dr}\int_r^b \frac{s^{-2n-2\alpha}}{(s^2 - r^2)^{-\alpha}}\,ds$$
$$\times\frac{d}{ds}\int_0^\infty \frac{t^{n+1}}{(s^2 - t^2)^{-\alpha}}p_3(t)\,dt, \tag{2.19}$$

$$e_2(r) = \frac{2^{-2\alpha} r^{-n-1}}{\Gamma(1+\alpha)^2}\frac{d}{dr}\int_a^r \frac{s^{2n-2\alpha}}{(r^2 - s^2)^{-\alpha}}\,ds$$
$$\times\frac{d}{ds}\int_s^\infty \frac{t^{-n+1}}{(t^2 - s^2)^{-\alpha}}p_4(t)\,dt. \tag{2.20}$$

3. Reduction of Cooke's second pair of equations

We now show that we need only consider the pair (2.4) and (2.5), because the pair (2.15) and (2.16) can be transformed into equations like (2.4) and (2.5).

First, (2.15) and (2.16) need only be solved in $r < a$ and $r > b$ respectively, because back substitution of $q_3(s)$ in $s < a$ and $q_4(s)$ in $s > b$ into the integrals in (2.15) and (2.16) would then give $q_3(r)$ and $q_4(r)$ in $a < r < b$. Putting

$$\tau_1(r) = q_3(r) - q_1(r) \quad \text{in} \quad r < a \tag{3.1}$$

and

$$\tau_2(r) = q_4(r) - q_2(r) \quad \text{in} \quad r > b, \tag{3.2}$$

(2.15) and (2.16) become

$$\tau_1(r) + \frac{2\sin\pi\alpha}{\pi}\int_b^\infty \frac{r^n s^{-n+1}}{s^2 - r^2}\frac{(b^2 - r^2)^\alpha}{(s^2 - b^2)^\alpha}\tau_2(s)\,ds$$

$$= e_1(r) - q_1(r) \quad \text{for} \quad r < a, \tag{3.3}$$

$$\tau_2(r) + \frac{2\sin\pi\alpha}{\pi}\int_0^a \frac{r^{-n} s^{n+1}}{r^2 - s^2}\frac{(r^2 - a^2)^\alpha}{(a^2 - s^2)^\alpha}\tau_1(s)\,ds$$

$$= e_2(r) - q_2(r) \quad \text{for} \quad r > b. \tag{3.4}$$

We define new unknown functions $\phi_1(r)$ and $\phi_2(r)$ by

$$\phi_1(r) = [(a^2 - r^2)(b^2 - r^2)]^{-\alpha}\tau_1(r) \quad \text{for} \quad r < a, \tag{3.5}$$

$$\phi_2(r) = [(r^2 - a^2)(r^2 - b^2)]^{-\alpha}\tau_2(r) \quad \text{for} \quad r > b. \tag{3.6}$$

Use of these expressions in (3.3) and (3.4) yields

$$\phi_1(r) + \frac{2\sin\pi\alpha}{\pi}\int_b^\infty \frac{r^n s^{-n+1}}{s^2 - r^2}\frac{(s^2 - a^2)^\alpha}{(a^2 - r^2)^\alpha}\phi_2(s)\,ds$$

$$= \frac{e_1(r) - q_1(r)}{[(a^2 - r^2)(b^2 - r^2)]^\alpha} \quad \text{for} \quad r < a, \tag{3.7}$$

$$\phi_2(r) + \frac{2\sin\pi\alpha}{\pi}\int_0^a \frac{r^{-n} s^{n+1}}{r^2 - s^2}\frac{(b^2 - s^2)^\alpha}{(r^2 - b^2)^\alpha}\phi_1(s)\,ds$$

$$= \frac{e_2(r) - q_2(r)}{[(r^2 - a^2)(r^2 - b^2)]^\alpha} \quad \text{for} \quad r > b. \tag{3.8}$$

Except for the known functions on the right, equations (3.7) and (3.8) are identical with (2.4) and (2.5). Hence, in the subsequent analysis, we shall concentrate on the pair (2.4) and (2.5).

The integrals occurring in (2.4) and (2.5), and in (2.15) and (2.16), are either infinite or have unbounded integrands, so that they may involve computational hazards. The transformation described in this paper produces equations with continuous kernels on a finite interval, except if $n + \frac{1}{2} < \alpha$, as is seen in (9.7) to (9.9) and in (1.6) to (1.7).

https://doi.org/10.1017/S0334270000001156 Published online by Cambridge University Press

4. An integral

If $0 < \alpha < 1$, $n \geqq 0$, $s > r > 0$ *and* F *is the hypergeometric function,*

$$\int_s^\infty \frac{1}{(t^2-s^2)^\alpha}\frac{\partial}{\partial t}\left(\frac{t^{-2n}}{(t^2-r^2)^{1-\alpha}}\right)dt$$
$$= -\frac{\Gamma(n+1)\Gamma(1-\alpha)}{\Gamma(n+1-\alpha)}\frac{1}{s^{2n+2}}F\left(1-\alpha, n+1;\, n+1-\alpha;\, \frac{r^2}{s^2}\right). \quad (4.1)$$

To prove this, we first perform the differentiation and then make the substitutions $r = 1/u$, $s = 1/v$, $t = 1/w$. The integral becomes

$$-\int_s^\infty \frac{2}{(t^2-s^2)^\alpha}\frac{n(t^2-r^2)+(1-\alpha)t^2}{t^{2n+1}(t^2-r^2)^{2-\alpha}}dt$$
$$= -2u^{2-2\alpha}v^{2\alpha}\int_0^v \frac{w^{2n+1}}{(v^2-w^2)^\alpha}\frac{n(u^2-w^2)+(1-\alpha)u^2}{(u^2-w^2)^{2-\alpha}}dw$$
$$= u^{2n+3-2\alpha}v^{2\alpha}\int_0^v \frac{w^{2n+1}}{(v^2-w^2)^\alpha}\frac{\partial}{\partial u}\left(\frac{(u^2-w^2)^{\alpha-1}}{u^{2n}}\right)dw$$
$$= u^{2n+3-2\alpha}v^{2\alpha}\frac{\partial}{\partial u}\int_0^v \frac{w^{2n+1}}{(v^2-w^2)^\alpha}\frac{(u^2-w^2)^{\alpha-1}}{u^{2n}}dw$$
$$= u^{2n+3-2\alpha}v^{2n+2}\frac{\partial}{\partial u}\left\{\frac{u^{-2n-2+2\alpha}}{2}\int_0^1 \frac{x^n}{(1-x)^\alpha}\left(1-\frac{v^2}{u^2}x\right)^{\alpha-1}dx\right\} \quad (4.2)$$
$$= u^{2n+3-2\alpha}v^{2n+2}$$
$$\times\frac{\partial}{\partial u}\left\{\frac{\Gamma(n+1)\Gamma(1-\alpha)}{2\Gamma(n+2-\alpha)}u^{-2n-2+2\alpha}F\left(1-\alpha, n+1;\, n+2-\alpha;\, \frac{v^2}{u^2}\right)\right\}$$
$$= -\frac{\Gamma(n+1)\Gamma(1-\alpha)}{2\Gamma(n+2-\alpha)}\frac{r^{2\alpha-2n-1}}{s^{2n+2}}$$
$$\times\frac{\partial}{\partial r}\left\{r^{2n+2-2\alpha}F\left(1-\alpha, n+1;\, n+2-\alpha;\, \frac{r^2}{s^2}\right)\right\}$$
$$= -\frac{\Gamma(n+1)\Gamma(1-\alpha)}{\Gamma(n+2-\alpha)}\frac{r^{2\alpha-2n}}{s^{2\alpha+2}}$$
$$\times\frac{\partial}{\partial(r^2/s^2)}\left\{\left(\frac{r^2}{s^2}\right)^{n+1-\alpha}F\left(1-\alpha, n+1;\, n+2-\alpha;\, \frac{r^2}{s^2}\right)\right\}.$$

This gives (4.1) by use of [6, 2.8(22)]. At (4.2) we have put $w = v\sqrt{x}$ and then used Euler's integral [6, 2.1(10)].

https://doi.org/10.1017/S0334270000001156 Published online by Cambridge University Press

5. Another integral

If $0 < \alpha < 1$, $n > 0$, $r > s > 0$ *and* F *is the hypergeometric function,*

$$\int_0^s \frac{1}{(s^2-t^2)^\alpha}\frac{\partial}{\partial t}\left(\frac{t^{2n}}{(r^2-t^2)^{1-\alpha}}\right)dt$$
$$= \frac{\Gamma(n+1)\Gamma(1-\alpha)}{\Gamma(n+1-\alpha)}\frac{s^{2n-2\alpha}}{r^{2-2\alpha}}F\left(1-\alpha, n+1; n+1-\alpha; \frac{s^2}{r^2}\right); \qquad (5.1)$$

while if $n = 0$ *this equation holds with an extra factor* s^2/r^2 *on the right.*

For the case $n = 0$ we may change to a variable u by the substitution

$$t = \left(\frac{s^2 - r^2u^2}{1-u^2}\right)^{1/2}, \quad r^2 - t^2 = \frac{r^2-s^2}{1-u^2}, \quad s^2-t^2 = \frac{(r^2-s^2)u^2}{1-u^2}.$$

The integral becomes

$$\int_0^s \frac{2(1-\alpha)t}{(s^2-t^2)^\alpha(r^2-t^2)^{2-\alpha}}dt = \frac{2(1-\alpha)}{r^2-s^2}\int_0^{s/r} u^{1-2\alpha}du = \frac{(s/r)^{2-2\alpha}}{r^2-s^2}; \qquad (5.2)$$

this is the result stated, because the hypergeometric function in (5.1) degenerates when $n = 0$ to $r^2/(r^2-s^2)$.

Now suppose that $n > 0$. Using Euler's formula for homogeneous functions and then the substitution $t = s\sqrt{u}$, the integral is equal to

$$\int_0^s \frac{1}{(s^2-t^2)^\alpha t}\left\{\frac{2(n-1+\alpha)t^{2n}}{(r^2-t^2)^{1-\alpha}} - r\frac{\partial}{\partial r}\left(\frac{t^{2n}}{(r^2-t^2)^{1-\alpha}}\right)\right\}dt$$
$$= \frac{n-1+\alpha}{s^{2\alpha-2n}r^{2-2\alpha}}\int_0^1 \frac{u^{n-1}(1-u)^{-\alpha}}{\{1-(s^2/r^2)u\}^{1-\alpha}}du$$
$$+ \frac{1-\alpha}{s^{2\alpha-2n}r^{2-2\alpha}}\int_0^1 \frac{u^{n-1}(1-u)^{-\alpha}}{\{1-(s^2/r^2)u\}^{2-\alpha}}du$$
$$= \frac{n-1+\alpha}{s^{2\alpha-2n}r^{2-2\alpha}}\frac{\Gamma(n)\Gamma(1-\alpha)}{\Gamma(n+1-\alpha)}F\left(1-\alpha, n; n+1-\alpha; \frac{s^2}{r^2}\right)$$
$$+ \frac{1-\alpha}{s^{2\alpha-2n}r^{2-2\alpha}}\frac{\Gamma(n)\Gamma(1-\alpha)}{\Gamma(n+1-\alpha)}F\left(2-\alpha, n; n+1-\alpha; \frac{s^2}{r^2}\right) \qquad (5.3)$$

$$= \frac{s^{2n-2\alpha}}{r^{2-2\alpha}}\frac{n\Gamma(n)\Gamma(1-\alpha)}{\Gamma(n+1-\alpha)}F\left(1-\alpha, n+1; n+1-\alpha; \frac{s^2}{r^2}\right); \qquad (5.4)$$

and (5.1) follows from this. For (5.3) we have used Euler's integral [6, 2.1(10)]; and for (5.4), [6, 2.8(32)].

6. Transformation of (2.4) when α is positive

We change to new unknown functions f_1 and f_2 related to the old unknown functions p_1 and p_2 in (2.4) and (2.5) by

$$f_1(r) = \int_r^a \frac{\rho^{-n+1} p_1(\rho)}{(\rho^2 - r^2)^{1-\alpha}} \, d\rho \quad \text{for} \quad r < a, \tag{6.1}$$

$$f_2(r) = \int_b^r \frac{\rho^{n+1} p_2(\rho)}{(r^2 - \rho^2)^{1-\alpha}} \, d\rho \quad \text{for} \quad r > b. \tag{6.2}$$

We rewrite (2.4), using (2.3) and (2.6), as

$$h_1(\rho) - p_1(\rho) = \frac{2 \sin \pi\alpha}{\pi} \int_a^\infty \frac{\rho^n s^{-n+1}}{s^2 - \rho^2} \frac{(s^2 - a^2)^\alpha}{(a^2 - \rho^2)^\alpha} p(s) \, ds \quad \text{for} \quad \rho < a.$$

Then by (6.1)

$$H_1(r) - f_1(r) = \frac{2 \sin \pi\alpha}{\pi} \int_r^a \frac{\rho^{-n+1}}{(\rho^2 - r^2)^{1-\alpha}} \, d\rho \times \int_a^\infty \frac{\rho^n s^{-n+1}}{s^2 - \rho^2} \frac{(s^2 - a^2)^\alpha}{(a^2 - \rho^2)^\alpha} p(s) \, ds, \tag{6.3}$$

where

$$H_1(r) = \int_r^a \frac{\rho^{-n+1} h_1(\rho)}{(\rho^2 - r^2)^{1-\alpha}} \, d\rho \quad \text{for} \quad r < a. \tag{6.4}$$

Formally changing the order of integration, and using (A2), (6.3) gives

$$\begin{aligned} H_1(r) - f_1(r) &= \frac{2 \sin \pi\alpha}{\pi} \int_a^\infty \frac{s^{-n+1} p(s)}{(s^2 - a^2)^{-\alpha}} \, ds \times \int_r^a \frac{\rho \, d\rho}{(s^2 - \rho^2)(a^2 - \rho^2)^\alpha (\rho^2 - r^2)^{1-\alpha}} \\ &= \int_a^\infty \frac{s^{-n+1} p(s)}{(s^2 - r^2)^{1-\alpha}} \, ds \\ &= \int_a^b \frac{s^{-n+1} p_0(s)}{(s^2 - r^2)^{1-\alpha}} \, ds + \int_b^\infty \frac{s^{-n+1} p_2(s)}{(s^2 - r^2)^{1-\alpha}} \, ds \quad \text{for} \quad r < a. \end{aligned} \tag{6.5}$$

https://doi.org/10.1017/S0334270000001156 Published online by Cambridge University Press

Inverting (6.2) by (A1), and substituting the result in the last term of (6.5), appropriately since $s > b$, we obtain

$$\int_b^\infty \frac{s^{-n+1}p_2(s)}{(s^2-r^2)^{1-\alpha}}\,ds$$

$$= \frac{2\sin\pi\alpha}{\pi}\int_b^\infty \frac{s^{-2n}}{(s^2-r^2)^{1-\alpha}}\left(\frac{d}{ds}\int_b^s \frac{tf_2(t)}{(s^2-t^2)^\alpha}\,dt\right)ds$$

$$= -\frac{2\sin\pi\alpha}{\pi}\int_b^\infty \left(\frac{\partial}{\partial s}\frac{s^{-2n}}{(s^2-r^2)^{1-\alpha}}\right)ds\int_b^s \frac{tf_2(t)}{(s^2-t^2)^\alpha}\,dt \tag{6.6}$$

integrating by parts and *assuming that the integrated terms vanish*. This assumption is easily seen to be correct if f_2 is bounded and n is positive; but it is false if $f_2(t) = (t^2-b^2)^{\alpha-1}$, for instance.

Formally changing the order of integration, and using (4.1) with s and t interchanged, the right side of (6.6) is equal to

$$-\frac{2\sin\pi\alpha}{\pi}\int_b^\infty tf_2(t)\,dt\int_t^\infty \frac{1}{(s^2-t^2)^\alpha}\frac{\partial}{\partial s}\left(\frac{s^{-2n}}{(s^2-r^2)^{1-\alpha}}\right)ds$$

$$= \frac{2\Gamma(n+1)}{\Gamma(n+1-\alpha)\Gamma(\alpha)}\int_b^\infty \frac{f_2(t)}{t^{2n+1}}F\left(1-\alpha, n+1; n+1-\alpha; \frac{r^2}{t^2}\right)dt. \tag{6.7}$$

Before rewriting (6.5) we simplify H_1. We continue to keep $r < a$ as we have already done throughout this section. Writing

$$Q_1(s) = \frac{d}{ds}\int_0^s \frac{t^{n+1}}{(s^2-t^2)^\alpha}q_1(t)\,dt \quad \text{for} \quad s < a, \tag{6.8}$$

we have, by (6.4) and (2.8),

$$H_1(r) = -\frac{2^{2\alpha}}{\Gamma(1-\alpha)^2}\int_r^a \frac{1}{(\rho^2-r^2)^{1-\alpha}}\left(\frac{d}{d\rho}\int_\rho^a \frac{s^{2\alpha-2n}}{(s^2-\rho^2)^\alpha}Q_1(s)\,ds\right)d\rho$$

$$= \frac{2^{2\alpha-1}\Gamma(\alpha)}{\Gamma(1-\alpha)}r^{2\alpha-2n-1}Q_1(r); \tag{6.9}$$

this simplification following from (A1) with b, s, t replaced by a, ρ, s. Finally, by (6.6), (6.7), (6.8) and (6.9), (6.5) becomes

$$f_1(r) + \frac{2\Gamma(n+1)}{\Gamma(n+1-\alpha)\Gamma(\alpha)}\int_b^\infty F\left(1-\alpha, n+1;\, n+1-\alpha;\, \frac{r^2}{s^2}\right)s^{-2n-1}f_2(s)\,ds$$

$$= -\int_a^b \frac{t^{-n+1}p_0(t)}{(t^2-r^2)^{1-\alpha}}\,dt + \frac{2^{2\alpha-1}\Gamma(\alpha)}{\Gamma(1-\alpha)}r^{2\alpha-2n-1}\frac{d}{dr}\int_0^r \frac{t^{n+1}q_1(t)}{(r^2-t^2)^\alpha}\,dt, \tag{6.10}$$

to hold for $r < a$.

https://doi.org/10.1017/S0334270000001156 Published online by Cambridge University Press

In the axisymmetric case $n = 0$, equation (6.10) reduces to

$$f_1(r) + \frac{2\sin\pi\alpha}{\pi}\int_b^\infty \frac{s}{s^2 - r^2} f_2(s)\, ds$$
$$= -\int_a^b \frac{tp_0(t)}{(t^2 - r^2)^{1-\alpha}}\, dt + \frac{\Gamma(\alpha)}{\Gamma(1-\alpha)}(2r)^{2\alpha-1}\frac{d}{dr}\int_0^r \frac{tq_1(t)}{(r^2 - t^2)^\alpha}\, dt. \quad (6.11)$$

7. Transformation of (2.5) when α is positive

As in §6 we transform to the new unknown functions f_1 and f_2 defined by (6.1) and (6.2). But we keep $r > b$ throughout this section; and we also suppose that $n > 0$. We rewrite (2.5), using (2.3) and (2.7), as

$$h_2(\rho) - p_2(\rho) = \frac{2\sin\pi\alpha}{\pi}\int_0^b \frac{\rho^{-n}s^{n+1}}{\rho^2 - s^2}\frac{(b^2 - s^2)^\alpha}{(\rho^2 - b^2)^\alpha} p(s)\, ds \quad \text{for} \quad \rho > b. \quad (7.1)$$

Then by (6.2)

$$H_2(r) - f_2(r) = \frac{2\sin\pi\alpha}{\pi}\int_b^r \frac{\rho^{n+1}}{(r^2 - \rho^2)^{1-\alpha}}\, d\rho$$
$$\times \int_0^b \frac{\rho^{-n}s^{n+1}}{\rho^2 - s^2}\frac{(b^2 - s^2)^\alpha}{(\rho^2 - b^2)^\alpha} p(s)\, ds \quad (7.2)$$

where

$$H_2(r) = \int_b^r \frac{\rho^{n+1}h_2(\rho)}{(r^2 - \rho^2)^{1-\alpha}}\, d\rho \quad \text{for} \quad r > b. \quad (7.3)$$

Formally changing the order of integration, and using (A2), (7.2) gives

$$H_2(r) - f_2(r)$$
$$= \frac{2\sin\pi\alpha}{\pi}\int_0^b \frac{s^{n+1}p(s)}{(b^2 - s^2)^{-\alpha}}\, ds \int_b^r \frac{\rho\, d\rho}{(\rho^2 - s^2)(r^2 - \rho^2)^{1-\alpha}(\rho^2 - b^2)^\alpha}$$
$$= \int_0^b \frac{s^{n+1}p(s)}{(r^2 - s^2)^{1-\alpha}}\, ds$$
$$= \int_0^a \frac{s^{n+1}p_1(s)}{(r^2 - s^2)^{1-\alpha}}\, ds + \int_a^b \frac{s^{n+1}p_0(s)}{(r^2 - s^2)^{1-\alpha}}\, ds \quad \text{for} \quad r > b. \quad (7.4)$$

Inverting (6.1) by (A1), and substituting the result in the first term of (7.4),

$$\int_0^a \frac{s^{n+1}p_1(s)}{(r^2 - s^2)^{1-\alpha}}\, ds = -\frac{2\sin\pi\alpha}{\pi}\int_0^a \frac{s^{2n}}{(r^2 - s^2)^{1-\alpha}}\left(\frac{d}{ds}\int_s^a \frac{tf_1(t)}{(t^2 - s^2)^\alpha}\, dt\right) ds$$
$$= \frac{2\sin\pi\alpha}{\pi}\int_0^a \left(\frac{\partial}{\partial s}\frac{s^{2n}}{(r^2 - s^2)^{1-\alpha}}\right) ds \int_s^a \frac{tf_1(t)}{(t^2 - s^2)^\alpha}\, dt, \quad (7.5)$$

https://doi.org/10.1017/S0334270000001156 Published online by Cambridge University Press

integrating by parts and *assuming that the integrated terms vanish.* As in §6 this is correct if f_1 is bounded, but may not be otherwise. But also it would be quite improbable that the integrated term would vanish at the lower terminal if n were permitted to be 0.

Formally changing the order of integration, and using (5.1) as we may since $n > 0$, the right side of (7.5) is equal to

$$\frac{2\sin\pi\alpha}{\pi}\int_0^a tf_1(t)\,dt\int_0^t \frac{1}{(t^2-s^2)^\alpha}\frac{\partial}{\partial s}\left(\frac{s^{2n}}{(r^2-s^2)^{1-\alpha}}\right)ds$$
$$=\frac{2\Gamma(n+1)}{\Gamma(n+1-\alpha)\Gamma(\alpha)}$$
$$\times\int_0^a \frac{t^{2n+1-2\alpha}}{r^{2-2\alpha}} f_1(t)F\left(1-\alpha, n+1; n+1-\alpha; \frac{t^2}{r^2}\right)dt. \tag{7.6}$$

Before rewriting (7.4) we simplify H_2. Writing

$$Q_2(s)=\frac{d}{ds}\int_s^\infty \frac{t^{-n+1}}{(t^2-s^2)^\alpha}q_2(t)\,dt \quad \text{for} \quad s>b, \tag{7.7}$$

we have, by (7.3) and (2.9),

$$H_2(r)=-\frac{2^{2\alpha}}{\Gamma(1-\alpha)^2}\int_b^r \frac{1}{(r^2-\rho^2)^{1-\alpha}}\left(\frac{d}{d\rho}\int_b^\rho \frac{s^{2n+2\alpha}}{(\rho^2-s^2)^\alpha}Q_2(s)ds\right)d\rho$$
$$=-\frac{2^{2\alpha-1}\Gamma(\alpha)}{\Gamma(1-\alpha)}r^{2n-1+2\alpha}Q_2(r), \tag{7.8}$$

this simplification following from (A1) with a, s, t replaced by b, ρ, s.

Finally, by (7.5), (7.6), (7.7) and (7.8), (7.4) becomes

$$f_2(r)+\frac{2\Gamma(n+1)}{\Gamma(n+1-\alpha)\Gamma(\alpha)}$$
$$\times\int_0^a F\left(1-\alpha, n+1;\, n+1-\alpha\,;\, \frac{s^2}{r^2}\right)\frac{s^{2n+1-2\alpha}}{r^{2-2\alpha}}f_1(s)\,ds$$
$$=-\int_a^b \frac{t^{n+1}p_0(t)}{(r^2-t^2)^{1-\alpha}}\,dt$$
$$-\frac{2^{2\alpha-1}\Gamma(\alpha)}{\Gamma(1-\alpha)}r^{2\alpha+2n-1}\frac{d}{dr}\int_r^\infty \frac{t^{-n+1}q_2(t)}{(t^2-r^2)^\alpha}\,dt, \tag{7.9}$$

to hold for $r > b$.

https://doi.org/10.1017/S0334270000001156 Published online by Cambridge University Press

8. The axisymmetric case when α is positive

The transformed equation (6.10) has been demonstrated in §6 to hold in the axisymmetric case $n = 0$, but this case was excluded in §7 to avoid the exception to the integral (5.1) in §5. However, nearly all of §7 still holds if $n = 0$, and we now consider what remaining adjustments are needed in that case.

The hypergeometric function in (7.6) simplifies, when $n = 0$, to $r^2/(r^2 - t^2)$. But because of the rider to (5.1) it must be replaced by $t^2/(r^2 - t^2)$. So it appears that (7.9) needs correcting when $n = 0$. However another correction is also needed. In the integration by parts which gives (7.5), the integrated terms which we have assumed to vanish are

$$-\frac{2\sin\pi\alpha}{\pi}\left[\frac{s^{2n}}{(r^2 - s^2)^{1-\alpha}}\int_s^a \frac{tf_1(t)}{(t^2 - s^2)^\alpha}\,dt\right]_{s=0}^{s=a-0}.$$

Still assuming that this vanishes at the upper terminal, its value at the lower when $n = 0$ is

$$\frac{2\sin\pi\alpha}{\pi}\frac{1}{r^{2-2\alpha}}\int_0^a t^{1-2\alpha}f_1(t)\,dt\,; \tag{8.1}$$

whereas for $n > 0$ this value is annulled by the factor s^{2n}. Thus, when $n = 0$, (7.6) must be corrected both by changing the hypergeometric function to $t^2/(r^2 - t^2)$ and by adding (8.1). It thus becomes

$$\begin{aligned}\frac{2}{\Gamma(1-\alpha)\Gamma(\alpha)}&\int_0^a \frac{t^2}{r^2 - t^2}\frac{t^{1-2\alpha}}{r^{2-2\alpha}}f_1(t)\,dt\\ &+\frac{2\sin\pi\alpha}{\pi}\int_0^a \frac{t^{1-2\alpha}}{r^{2-2\alpha}}f_1(t)\,dt\\ &=\frac{2\sin\pi\alpha}{\pi}\int_0^a \frac{r^2}{r^2 - t^2}\frac{t^{1-2\alpha}}{r^{2-2\alpha}}f_1(t)\,dt,\end{aligned} \tag{8.2}$$

and this must replace the integral term on the left of (7.9) when $n = 0$. But (8.2) is exactly what the integral term on the left of (7.9) reduces to when $n = 0$! So that, after all, (7.9) still holds in the axisymmetric case.

9. Further reduction of the equations

We rewrite (6.10) and (7.9) thus:

$$f_1(r) + 2\Gamma(n,\alpha)\int_b^\infty F\left(\frac{r^2}{s^2}\right)\frac{1}{s^{2n+1}}f_2(s)\,ds = g_1(r) \quad \text{for} \quad r < a, \tag{9.1}$$

$$f_2(r) + 2\Gamma(n,\alpha)\int_0^a F\left(\frac{s^2}{r^2}\right)\frac{s^{2n+1-2\alpha}}{r^{2-2\alpha}}f_1(s)\,ds = g_2(r) \quad \text{for} \quad r > b. \tag{9.2}$$

https://doi.org/10.1017/S0334270000001156 Published online by Cambridge University Press

Here g_1 and g_2 are the known right sides of (6.10) and (7.9), $\Gamma(n, \alpha)$ is the reciprocal of $B(n+1-\alpha, \alpha)$, and we have temporarily suppressed mention of the three parameters $1-\alpha$, $n+1$, $n+1-\alpha$ in the hypergeometric function F.

Let $c = \sqrt{(ab)}$ and $k = \sqrt{(a/b)}$; so that $a < c < b$ and $0 < k < 1$.

In (9.1) put $r = cx$ and $s = c/t$; obtaining for $0 < x < k$,

$$f_1(cx) + 2\Gamma(n, \alpha) \int_0^k F(x^2t^2) c^{-2n} t^{2n-1} f_2(c/t)\, dt = g_1(cx).$$

In (9.2) put $r = c/x$ and $s = ct$; obtaining, also for $0 < x < k$,

$$f_2(c/x) + 2\Gamma(n, \alpha) \int_0^k F(x^2t^2) c^{2n} x^{2-2\alpha} t^{2n+1-2\alpha} f_1(ct)\, dt = g_2(c/x).$$

With suitable multipliers these equations become

$$c^n x^{n+\frac{1}{2}-\alpha} f_1(cx) + 2\Gamma(n, \alpha) \int_0^k F(x^2t^2)(xt)^{n+\frac{1}{2}-\alpha} c^{-n} t^{n-\frac{3}{2}+\alpha} f_2(c/t)\, dt = c^n x^{n+\frac{1}{2}-\alpha} g_1(cx), \tag{9.3}$$

$$c^{-n} x^{n-\frac{3}{2}+\alpha} f_2(c/x) + 2\Gamma(n, \alpha) \int_0^k F(x^2t^2)(xt)^{n+\frac{1}{2}-\alpha} c^n t^{n+\frac{1}{2}-\alpha} f_1(ct)\, dt = c^{-n} x^{n-\frac{3}{2}+\alpha} g_2(c/x). \tag{9.4}$$

Adding and subtracting (9.3) and (9.4), and writing

$$f_\pm(x) = c^n x^{n+\frac{1}{2}-\alpha} f_1(cx) \pm c^{-n} x^{n-\frac{3}{2}+\alpha} f_2(c/x), \tag{9.5}$$

$$g_\pm(x) = c^n x^{n+\frac{1}{2}-\alpha} g_1(cx) \pm c^{-n} x^{n-\frac{3}{2}+\alpha} g_2(c/x), \tag{9.6}$$

we obtain uncoupled equations for f_+ and f_-:

$$f_+(x) + \int_0^k K(x, t) f_+(t)\, dt = g_+(x) \quad \text{for} \quad 0 < x < k, \tag{9.7}$$

$$f_-(x) - \int_0^k K(x, t) f_-(t)\, dt = g_-(x) \quad \text{for} \quad 0 < x < k, \tag{9.8}$$

where

$$K(x, t) = 2\Gamma(n, \alpha) F(x^2t^2)(xt)^{n+\frac{1}{2}-\alpha} \tag{9.9}$$

$$= \frac{2\Gamma(n+1)}{\Gamma(n+1-\alpha)\Gamma(\alpha)} (xt)^{n+\frac{1}{2}-\alpha} F(1-\alpha, n+1;\, n+1-\alpha;\, x^2t^2) \tag{9.10}$$

$$= \frac{2\Gamma(n+1)}{\Gamma(n+1-\alpha)\Gamma(\alpha)} \frac{(xt)^{n+\frac{1}{2}-\alpha}}{1-x^2t^2} F(-\alpha, n;\, n+1-\alpha;\, x^2t^2). \tag{9.11}$$

Here (9.11) is got from (9.10) by use of [6, 2.9(2)].

This kernel K is positive and symmetric, on the square $\{0 \leqq x \leqq k$ and $0 \leqq t \leqq k\}$, for $n \geqq 0$ and $0 < \alpha < 1$. The positiveness is seen from the hypergeometric series arising from (9.10), in which all the coefficients are positive. The kernel is also continuous on the square if $n+\frac{1}{2}-\alpha \geqq 0$; if not, it is in $\mathbf{L}^2$ as (10.2) shows.

10. Effectiveness of iteration for (9.7) and (9.8)

When $g_\pm \in L^2$, a sufficient condition for iteration of (9.7) and (9.8) to converge is that $\|K\|<1$, where $\|K\|$ is the $\mathbf{L}^2$ norm

$$\|K\| = \left(\int_0^k dx \int_0^k |K(x,t)|^2\, dt\right)^{1/2}. \tag{10.1}$$

This condition is also sufficient for existence and uniqueness of solutions in L^2.

The hypergeometric function in (9.11) is positive because that in (9.10) is positive; and it is a decreasing function of x^2t^2 since all coefficients in its power series, except the constant term 1, are negative because of the factor $-\alpha$. So

$$|K(x,t)| \leqq 2\Gamma(n,\alpha)\frac{(xt)^{n+\frac{1}{2}-\alpha}}{1-x^2t^2},$$

$$\|K\| \leqq \frac{2\Gamma(n,\alpha)}{1-k^4}\left(\int_0^k x^{2n+1-2\alpha}\,dx \int_0^k t^{2n+1-2\alpha}\,dt\right)^{1/2}$$

$$= \frac{\Gamma(n,\alpha)}{1-k^4}\,\frac{k^{2(n+1-\alpha)}}{n+1-\alpha}. \tag{10.2}$$

For iteration to be effective it is thus sufficient if the right side of (10.2) is less than 1. Given n and α such that $n \geqq 0$ and $0<\alpha<1$, then, it is evident that iteration is effective for sufficiently small k; for the majorant in (10.2) is an increasing function of k which tends to nought as $k \to 0$ and to infinity as $k \to 1$.

For quantitative estimates of the effectiveness of iteration some less crude inequality than (10.2) is desirable, particularly when n and $1-\alpha$ are small. We establish one inequality as follows. By (9.9) and (10.1),

$$\|K\|^2 = 4\Gamma(n,\alpha)^2 \int_0^k dx \int_0^k (xt)^{2n+1-2\alpha} F(x^2t^2)^2\, dt$$

$$\leqq 4\Gamma(n,\alpha)^2 \int_0^k x^{2n+1-2\alpha} F(x^2k^2)\, dx \int_0^k t^{2n+1-2\alpha} F(k^2t^2)\, dt \tag{10.3}$$

$$= \left(2\Gamma(n,\alpha)\int_0^k x^{2n+1-2\alpha} F(k^2x^2)\, dx\right)^2. \tag{10.4}$$

In (10.3) we have used the fact that the hypergeometric function in (9.10) is an increasing function of its fourth variable because the corresponding

https://doi.org/10.1017/S0334270000001156 Published online by Cambridge University Press

hypergeometric series has positive coefficients. Equally the hypergeometric function in (10.4) is positive, and so

$$\|K\| \leqq 2\Gamma(n,\alpha)\int_0^k x^{2n+1-2\alpha}F(1-\alpha, n+1; n+1-\alpha; k^2x^2)\,dx$$

$$= 2\Gamma(n,\alpha)\int_0^k \frac{x^{2n+1-2\alpha}}{1-k^2x^2}F(-\alpha, n; n+1-\alpha; k^2x^2)\,dx \tag{10.5}$$

using [6, 2.9(2)]. By the hypergeometric series we have, since $k^2x^2 < 1$,

$$F(-\alpha, n; n+1-\alpha; k^2x^2) \leqq 1 - \frac{\alpha n}{n+1-\alpha}k^2x^2$$

$$= 1 - k^2x^2 + \frac{(n+1)(1-\alpha)}{n+1-\alpha}k^2x^2,$$

whence (10.5) gives

$$\|K\| \leqq 2\Gamma(n,\alpha)\int_0^k \left(x^{2n+1-2\alpha} + x^{2n+2-2\alpha}\frac{(n+1)(1-\alpha)}{n+1-\alpha}\frac{k^2x}{1-k^2x^2}\right)dx$$

$$\leqq 2\Gamma(n,\alpha)\int_0^k \left(x^{2n+1-2\alpha} + k^{2n+2-2\alpha}\frac{(n+1)(1-\alpha)}{n+1-\alpha}\frac{k^2x}{1-k^2x^2}\right)dx \tag{10.6}$$

$$= \Gamma(n,\alpha)\left(\frac{k^{2n+2-2\alpha}}{n+1-\alpha} + k^{2n+2-2\alpha}\frac{(n+1)(1-\alpha)}{n+1-\alpha}\log\frac{1}{1-k^4}\right).$$

The fact that $2n+2-2\alpha > 0$ is essential in (10.6), both for the inequality that gives the second term and for the convergence of the integral of the first term. We thus obtain the inequality sought, namely

$$\|K\| \leqq \frac{\Gamma(n+2)}{\Gamma(n+2-\alpha)\Gamma(\alpha)}\left(\frac{1}{n+1} + (1-\alpha)\log\frac{1}{1-k^4}\right)k^{2(n+1-\alpha)}. \tag{10.7}$$

Table A shows values of $a/b = k^2$ such that iteration converges for these and all lesser values. Only two-decimal-place values are considered.

TABLE A
(for (9.7) and (9.8) using (10.7))

n \ α	0.1	0.3	0.5	0.7	0.9
0	0.99	0.96	0.86	0.76	0.66
0.5	0.99	0.96	0.89	0.84	0.84
1	0.99	0.96	0.90	0.86	0.87
1.5	0.99	0.96	0.91	0.87	0.88
2	0.99	0.96	0.91	0.87	0.89

There is no certainty that the entries would continue to increase with n if this table were prolonged. For larger values of n we use (10.9), a different inequality from (10.7), found by replacing (10.6) by (10.8) as follows.

$$\|K\| \leqq 2\Gamma(n,\alpha)\int_0^k \left(x^{2n+1-2\alpha} + x^{2n+3-2\alpha}\frac{(n+1)(1-\alpha)}{n+1-\alpha}\frac{k^2}{1-k^4}\right)dx \tag{10.8}$$

$$= \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)\Gamma(\alpha)}\left(\frac{k^{2n+2-2\alpha}}{n+1-\alpha} + \frac{k^{2n+4-2\alpha}}{n+2-\alpha}\frac{(n+1)(1-\alpha)}{n+1-\alpha}\frac{k^2}{1-k^4}\right)$$

$$= \left(\frac{\Gamma(n,\alpha)}{n+1-\alpha} + \frac{\Gamma(n+1,\alpha)}{n+2-\alpha}(1-\alpha)\frac{k^4}{1-k^4}\right)k^{2(n+1-\alpha)}. \tag{10.9}$$

The expression $\Gamma(n,\alpha)/(n+1-\alpha)$ is a decreasing function of n in $n \geqq 0$ because the derivative of its logarithm is

$$\frac{\partial}{\partial n}\log\frac{\Gamma(n+1)}{\Gamma(n+2-\alpha)\Gamma(\alpha)} = \psi(n+1) - \psi(n+2-\alpha) < 0,$$

the ψ-function being increasing. Thus the main bracket in (10.9) is decreasing in $n \geqq 0$. So also is the other factor $k^{2(n+1-\alpha)}$. Consequently (10.9) itself is a decreasing function of n in $n \geqq 0$.

Table B, like Table A, shows two-decimal-place values of a/b such that iteration converges for these and all lesser values.

TABLE B
(for (9.7) and (9.8) using (10.9))

α	0.1	0.3	0.5	0.7	0.9
$n \geqq 2$	0.98	0.95	0.92	0.90	0.90

No doubt improved values of a/b could be calculated from (10.9) for larger values of n in this table, because of the monotony established above. But it may be noted that (10.9) is no substitute for (10.7) for the smaller values of n in Table A; for instance, for $n = 0$ and $\alpha = 0.5$ calculation from (10.9) gives 0.68 instead of the entry 0.86 in Table A.

11. Iteration of Cooke's equations

In order to discuss the prospects of iteration of Cooke's equations (2.4) and (2.5), we first reduce them to uncoupled equations by the method of §9; compare also [1, §7]. Following the procedure of §9, we put $r = cx$ and $s = c/t$

https://doi.org/10.1017/S0334270000001156 Published online by Cambridge University Press

in (2.4), and then $r = c/x$ and $s = ct$ in (2.5); this gives, corresponding to (9.3) and (9.4),

$$x^{\frac{1}{2}}p_1(cx) + \frac{2\sin\pi\alpha}{\pi}\int_0^k \frac{(xt)^{n+\frac{1}{2}}}{1-x^2t^2}\left(\frac{1-k^2t^2}{k^2-x^2}\right)^\alpha \frac{p_2(c/t)}{t^{2\alpha+\frac{3}{2}}}\,dt = x^{\frac{1}{2}}i_1(cx), \quad (11.1)$$

$$\frac{p_2(c/x)}{x^{2\alpha+\frac{3}{2}}} + \frac{2\sin\pi\alpha}{\pi}\int_0^k \frac{(xt)^{n+\frac{1}{2}}}{1-x^2t^2}\left(\frac{1-k^2t^2}{k^2-x^2}\right)^\alpha t^{\frac{1}{2}}p_1(ct)\,dt = \frac{i_2(c/x)}{x^{2\alpha+\frac{3}{2}}}, \quad (11.2)$$

where $i_1 = k_1 + h_1$ and $i_2 = k_2 + h_2$. Defining p_+ and p_- by

$$p_\pm(x) = x^{\frac{1}{2}}p_1(cx) \pm x^{-2\alpha-\frac{3}{2}}p_2(c/x), \quad (11.3)$$

and i_+ and i_- similarly, we obtain the two uncoupled equations

$$p_\pm(x) \pm \int_0^k K_1(x,t)p_\pm(t)\,dt = i_\pm(x) \quad \text{for} \quad 0 < x < k; \quad (11.4)$$

where

$$K_1(x,t) = \frac{2\sin\pi\alpha}{\pi}\frac{(xt)^{n+\frac{1}{2}}}{1-x^2t^2}\left(\frac{1-k^2t^2}{k^2-x^2}\right)^\alpha \quad (11.5)$$

for $0 < x < k$ and $0 < t < k$.

12. Effectiveness of iteration with equations (11.4)

Assuming that $i_\pm \in L^2$, convergence of iteration of both equations (11.4) is assured if $\|K_1\| < 1$; the meaning of $\|K_1\|$ being given by (10.1) and (11.5). We now discuss whether this condition is satisfied.

When $\alpha \geqq \frac{1}{2}$ the condition $\|K_1\| < 1$ is never satisfied, whatever the values of k and n such that $0 < k < 1$ and $n \geqq 0$. We prove this simply by showing that $\|K_1\| = \infty$ in all these cases. By (11.5), $\|K_1\|^2$ is a positive multiple of

$$\int_0^k \frac{x^{2n+1}}{(k^2-x^2)^{2\alpha}}\,dx \int_0^k \frac{(1-k^2t^2)^{2\alpha}}{(1-x^2t^2)^2}\,t^{2n+1}\,dt$$

$$\geqq \int_0^k \frac{x^{2n+1}}{(k^2-x^2)^{2\alpha}}\,dx \int_0^k (1-k^4)^{2\alpha}t^{2n+1}\,dt$$

$$= \frac{(1-k^4)^{2\alpha}k^{2n+2}}{2n+2}\int_0^k \frac{x^{2n+1}}{(k^2-x^2)^{2\alpha}}\,dx = \infty \quad \text{since} \quad 2\alpha \geqq 1.$$

In particular, the proposed condition is not satisfied in the important case $\alpha = \frac{1}{2}$.

When $0 < \alpha < \frac{1}{2}$, on the other hand, iteration of (11.4) converges for sufficiently small k. To show this, we obtain an inequality for $\|K_1\|$ as in §10.

From (11.5) we have, for $0 \leqq x \leqq k$,

$$\left(\frac{\pi}{\sin\pi\alpha}\right)^2\int_0^k K_1(x,t)^2\,dt = 4\int_0^k \frac{(xt)^{2n+1}}{(1-x^2t^2)^2}\frac{(1-k^2t^2)^{2\alpha}}{(k^2-x^2)^{2\alpha}}\,dt$$

$$\leqq 4\int_0^k \frac{(xt)^{2n+1}}{(1-x^2t^2)^2}\frac{(1-x^2t^2)^{2\alpha}}{(k^2-x^2)^{2\alpha}}\,dt$$

$$=\frac{4x^{2n+1}}{(k^2-x^2)^{2\alpha}}\int_0^k \frac{t^{2n+1}}{(1-x^2t^2)^{2-2\alpha}}\,dt$$

$$\leqq\frac{4x^{2n+1}}{(k^2-x^2)^{2\alpha}}\int_0^k \frac{t^{2n+1}}{(1-x^2k^2)^{2-2\alpha}}\,dt$$

$$=\frac{2k}{n+1}\frac{(kx)^{2n+1}}{(k^2-x^2)^{2\alpha}(1-k^2x^2)^{2-2\alpha}}.$$

Putting $x = k\sqrt{u}$ and remembering that $2\alpha < 1$, we now have

$$\left(\frac{\pi}{\sin\pi\alpha}\right)^2\|K_1\|^2 \leqq \frac{2k}{n+1}\int_0^k \frac{(kx)^{2n+1}}{(k^2-x^2)^{2\alpha}(1-k^2x^2)^{2-2\alpha}}\,dx$$

$$=\frac{k^{4(n+1-\alpha)}}{n+1}\int_0^1 u^n(1-u)^{-2\alpha}(1-k^4u)^{2\alpha-2}\,du$$

$$=\frac{k^{4(n+1-\alpha)}}{n+1}\frac{\Gamma(n+1)\Gamma(1-2\alpha)}{\Gamma(n+2-2\alpha)}F(2-2\alpha, n+1; n+2-2\alpha; k^4) \tag{12.1}$$

$$=\frac{k^{4(n+1-\alpha)}}{n+1}\frac{\Gamma(n+1)\Gamma(1-2\alpha)}{\Gamma(n+2-2\alpha)}(1-k^4)^{-1}F(1-2\alpha, n; n+2-2\alpha; k^4) \tag{12.2}$$

$$\leqq\frac{k^{4(n+1-\alpha)}}{n+1}\frac{\Gamma(n+1)\Gamma(1-2\alpha)}{\Gamma(n+2-2\alpha)}(1-k^4)^{-1}\frac{\Gamma(n+2-2\alpha)\Gamma(1)}{\Gamma(n+1)\Gamma(2-2\alpha)}$$

$$=\frac{1}{(n+1)(1-2\alpha)}\frac{k^{4(n+1-\alpha)}}{1-k^4}.$$

At (12.1) we have used Euler's integral for the hypergeometric function [6, 2.1(10)] and at (12.2) the relation [6, 2.9(2)]. The hypergeometric function in (12.2) is an increasing function of its fourth variable k^4, since the corresponding hypergeometric series has positive coefficients; so in its next appearance we have replaced k^4 by 1 and written the value of the hypergeometric function given by [6, 2.1(14)]. We now have the desired inequality

$$\|K_1\| \leqq \frac{\sin\pi\alpha}{\pi}\frac{1}{\{(n+1)(1-2\alpha)\}^{\frac12}}\frac{k^{2(n+1-\alpha)}}{(1-k^4)^{\frac12}}. \tag{12.3}$$

Table C, like Tables A and B, shows two-decimal-place values of a/b such that iteration converges for these and all lesser values. But of course this

https://doi.org/10.1017/S0334270000001156 Published online by Cambridge University Press

table refers to iteration of equations (11.4), and therefore of (2.4) and (2.5); and it is necessarily restricted to $0 < \alpha < \frac{1}{2}$.

TABLE C
(for (11.4) using (12.3))

n \ α	0.1	0.2	0.3	0.4	$\geqq 0.5$
0	0.99	0.97	0.92	0.80	-
0.5	0.99	0.98	0.94	0.87	-
1	0.99	0.98	0.96	0.91	-
1.5	0.99	0.98	0.97	0.92	-
$\geqq 2$	0.99	0.99	0.97	0.94	-

Notice that Table C differs from Tables A and B in layout, as it contains columns for $\alpha = 0.2$ and 0.4. Notice also that the entries for $n = 2$ serve also for $n > 2$; this is because the right side at (12.3) is a decreasing function of n.

13. Iteration of (11.4) with another norm

A norm which is sometimes more accommodating than (10.1) is

$$\|K\|_* = \sup_{0<x<k} \int_0^k |K(x,t)|\,dt; \tag{13.1}$$

and if $i_\pm$ are bounded, another sufficient condition for convergence of iteration of equations (11.4) is that $\|K_1\|_* < 1$. However, this condition is not satisfied for any of the values of k, α and n contemplated, namely $0 < k < 1$, $0 < \alpha < 1$ and $n \geqq 0$. For

$$\frac{\pi}{2\sin\pi\alpha}\int_0^k |K_1(x,t)|\,dt \doteq \frac{x^{n+\frac{1}{2}}}{(k^2-x^2)^\alpha}\int_0^k \frac{(1-k^2t^2)^\alpha}{1-x^2t^2}\,t^{n+\frac{1}{2}}\,dt$$

$$\geqq \frac{x^{n+\frac{1}{2}}(1-k^4)^\alpha}{(k^2-x^2)^\alpha}\int_0^k t^{n+\frac{1}{2}}\,dt = \frac{(1-k^4)^\alpha k^{n+\frac{3}{2}}}{n+\frac{3}{2}}\,\frac{x^{n+\frac{1}{2}}}{(k^2-x^2)^\alpha};$$

the last expression is unbounded on $0 < x < k$, so that $\|K_1\|_*$ is infinite. Thus convergence of iteration when $\alpha \geqq \frac{1}{2}$ is not assured with this norm either.

Of course the condition $\|K_1\|_* < 1$ may not be necessary for convergence of iteration; it is only known to be sufficient. But the failure, when $\alpha \geqq \frac{1}{2}$, of both the tests we have applied suggests no great likelihood that iteration with K_1 is effective in any practical sense.

Plainly the denominator factor $(k^2 - x^2)^\alpha$ in (11.5) is the main cause of this trouble. There are of course other sufficient conditions for convergence of iteration, and it is conceivable that iteration with an $L^{p,q}$ norm such as

https://doi.org/10.1017/S0334270000001156 Published online by Cambridge University Press

$$\|K_1\|_{p,q} = \left\{ \int_0^k \left(\int_0^k |K_1(x,t)|^q \, dt \right)^{p/q} dx \right\}^{1/p}, \tag{13.2}$$

where $p \geqq 1$ and $q \geqq 1$, might converge if $0 < \alpha < 1/p$; at least (13.2) is finite under this condition.

14. Transformation of (2.4) and (2.5) when α is negative

In this section we show how (2.4) and (2.5) can be transformed to equations of the type of (6.10) and (7.9) when $-1 < \alpha < 0$. In this case the transformed equations may, in general, be no better than (2.4) and (2.5) for numerical computation by methods other than iteration; for, when $-1 < \alpha < 0$, the latter equations have positive continuous kernels and hence should be quite satisfactory for the calculation of $p_1(r)$ and $p_2(r)$. However, in the important case $n = 0$ the transformed equations have kernels which are simpler than those in (2.4) and (2.5). Also, the right hand sides of the transformed equations are simpler than the corresponding terms in (2.4) and (2.5), whatever the value of n.

Putting $\alpha = -\beta$ in (2.4) and (2.5) we obtain

$$p_1(r) - \frac{2\sin\pi\beta}{\pi} \int_b^\infty \frac{r^n s^{-n+1}}{s^2 - r^2} \frac{(a^2 - r^2)^\beta}{(s^2 - a^2)^\beta} p_2(s) \, ds$$
$$= k_1(r) + h_1(r) \quad \text{for} \quad r < a, \tag{14.1}$$

$$p_2(r) - \frac{2\sin\pi\beta}{\pi} \int_0^a \frac{r^{-n} s^{n+1}}{r^2 - s^2} \frac{(r^2 - b^2)^\beta}{(b^2 - s^2)^\beta} p_1(s) \, ds$$
$$= k_2(r) + h_2(r) \quad \text{for} \quad r > b, \tag{14.2}$$

and $0 < \beta < 1$. Transforming these equations by the Abel equations

$$g_1(r) = \frac{1}{r} \frac{d}{dr} \int_r^a \frac{\rho^{-n+1} p_1(\rho)}{(\rho^2 - r^2)^\beta} \, d\rho \qquad \text{for} \quad r < a, \tag{14.3}$$

$$p_1(r) = \frac{-2\sin\pi\beta}{\pi} \int_r^a \frac{r^n t g_1(t)}{(t^2 - r^2)^{1-\beta}} \, dt \quad \text{for} \quad r < a, \tag{14.4}$$

$$g_2(r) = \frac{1}{r} \frac{d}{dr} \int_b^r \frac{\rho^{1+n} p_2(\rho)}{(\rho^2 - r^2)^\beta} \, d\rho, \qquad \text{for} \quad r > b \tag{14.5}$$

$$p_2(r) = \frac{2\sin\pi\beta}{\pi} \int_b^r \frac{r^{-n} t g_2(t)}{(r^2 - t^2)^{1-\beta}} \, dt \qquad \text{for} \quad r > b, \tag{14.6}$$

it follows that

https://doi.org/10.1017/S0334270000001156 Published online by Cambridge University Press

$$g_1(r)-\frac{2\sin\pi\beta}{\pi}\frac{1}{r}\frac{d}{dr}\int_r^a\frac{(a^2-\rho^2)^\beta\rho\,d\rho}{(\rho^2-r^2)^\beta}$$

$$\times\int_b^\infty\frac{s^{1-n}(s^2-a^2)^{-\beta}}{s^2-\rho^2}p_2(s)\,ds=d_1(r)\qquad\text{for}\quad r<a,\qquad(14.7)$$

$$g_2(r)-\frac{2\sin\pi\beta}{\pi}\frac{1}{r}\frac{d}{dr}\int_b^r\frac{(\rho^2-b^2)^\beta\rho\,d\rho}{(r^2-\rho^2)^\beta}$$

$$\times\int_0^a\frac{s^{1+n}(b^2-s^2)^{-\beta}}{\rho^2-s^2}p_1(s)\,ds=d_2(r)\qquad\text{for}\quad r>b,\qquad(14.8)$$

where

$$d_1(r)=\frac{1}{r}\frac{d}{dr}\int_r^a\frac{\rho^{1-n}[k_1(\rho)+h_1(\rho)]}{(\rho^2-r^2)^\beta}d\rho\qquad\text{for}\quad r<a,\qquad(14.9)$$

$$d_2(r)=\frac{1}{r}\frac{d}{dr}\int_b^r\frac{\rho^{1+n}[k_2(\rho)+h_2(\rho)]}{(r^2-\rho^2)^\beta}d\rho\qquad\text{for}\quad r>b.\qquad(14.10)$$

Changing the order of integration in (14.7) and (14.8) and using (A2) we obtain

$$g_1(r)-d_1(r)=-2\beta\int_b^\infty\frac{s^{1-n}}{(s^2-r^2)^{1+\beta}}p_2(s)\,ds\quad\text{for}\quad r<a,\qquad(14.11)$$

$$g_2(r)-d_2(r)=2\beta\int_0^a\frac{s^{1+n}}{(r^2-s^2)^{1+\beta}}p_1(s)\,ds\qquad\text{for}\quad r>b.\qquad(14.12)$$

Substituting for $p_1(r)$ and $p_2(r)$ from (14.4) and (14.6), changing the order of integration and using (A2) we may write (14.11) and (14.12) in the form

$$g_1(r)-d_1(r)$$

$$=-\frac{4\beta\sin\pi\beta}{\pi}\int_b^\infty g_2(s)s\,ds\int_s^\infty\frac{t^{1-2n}}{(t^2-r^2)^{1+\beta}(t^2-s^2)^{1-\beta}}dt$$

$$=-\frac{2\beta\Gamma(1+n)}{\Gamma(1-\beta)\Gamma(1+n+\beta)}\int_b^\infty F\left(1+\beta,1+n;1+n+\beta;\frac{r^2}{s^2}\right)s^{-1-2n}g_2(s)\,ds$$

$$\text{for}\quad r<a,\qquad(14.13)$$

$$g_2(r)-d_2(r)$$

$$=-\frac{4\beta\sin\pi\beta}{\pi}\int_0^\infty g_1(s)s\,ds\int_0^s\frac{t^{1+2n}}{(r^2-t^2)^{1+\beta}(s^2-t^2)^{1-\beta}}dt$$

$$=-\frac{2\beta\Gamma(1+n)}{\Gamma(1-\beta)\Gamma(1+n+\beta)}\int_0^a F\left(1+\beta,1+n;1+n+\beta;\frac{s^2}{r^2}\right)\frac{s^{2n+1+2\beta}}{r^{2+2\beta}}g_1(s)\,ds$$

$$\text{for}\quad r>b.\qquad(14.14)$$

https://doi.org/10.1017/S0334270000001156 Published online by Cambridge University Press

We now consider the functions $d_1(r)$ and $d_2(r)$. From (2.6) it follows that

$$\frac{1}{r}\frac{d}{dr}\int_r^a \frac{\rho^{-n+1}k_1(\rho)}{(\rho^2-r^2)^\beta}\,d\rho = \frac{2\sin\pi\beta}{\pi}\frac{1}{r}\frac{d}{dr}\int_r^a \frac{(a^2-\rho^2)^\beta}{(\rho^2-r^2)^\beta}\,\rho d\rho \times\int_a^b \frac{s^{-n+1}(s^2-a^2)^{-\beta}}{s^2-\rho^2}\,p_0(s)\,ds \quad \text{for} \quad r<a.$$

Changing the order of integration and using (A2)

$$\frac{1}{r}\frac{d}{dr}\int_r^a \frac{\rho^{-n+1}k_1(\rho)}{(\rho^2-r^2)^\beta}\,d\rho = -2\beta\int_a^b \frac{s^{-n+1}}{(s^2-r^2)^{1+\beta}}\,p_0(s)\,ds. \tag{14.15}$$

Also, using (2.10)

$$\frac{1}{r}\frac{d}{dr}\int_r^a \frac{\rho^{-n+1}h_1(\rho)}{(\rho^2-r^2)^\beta}\,d\rho$$

$$= \frac{2^{2-2\beta}}{\Gamma(\beta)^2}\frac{1}{r}\frac{d}{dr}\int_r^a \frac{\rho}{(\rho^2-r^2)^\beta}\,d\rho\int_\rho^a \frac{s^{-2n-2\beta+1}}{(s^2-\rho^2)^{1-\beta}}\,ds \times\int_0^s \frac{t^{n+1}}{(s^2-t^2)^{1-\beta}}\,q_1(t)\,dt \tag{14.16}$$

$$= -2^{1-2\beta}\frac{\Gamma(1-\beta)}{\Gamma(\beta)}\,r^{-2n-2\beta}\int_0^r \frac{t^{n+1}}{(r^2-t^2)^{1-\beta}}\,q_1(t)\,dt, \tag{14.17}$$

where (A1) has been used to obtain (14.17) from (14.16). Use of (14.9), (14.15) and (14.17) now yields, for $r<a$

$$d_1(r) = 2\alpha\int_a^b \frac{t^{-n+1}}{(t^2-r^2)^{1-\alpha}}\,p_0(t)\,dt + 2^{1+2\alpha}\frac{\alpha\,\Gamma(1+\alpha)}{\Gamma(1-\alpha)}\,r^{-2n+2\alpha}\int_0^r \frac{t^{n+1}}{(r^2-t^2)^{1+\alpha}}\,q_1(t)\,dt, \tag{14.18}$$

where we have replaced β by $-\alpha$.

Similarly, from (2.7) and using (A2)

$$\frac{1}{r}\frac{d}{dr}\int_b^r \frac{\rho^{1+n}k_2(\rho)}{(r^2-\rho^2)^\beta}\,d\rho$$

$$= \frac{2\sin\pi\beta}{\pi}\frac{1}{r}\frac{d}{dr}\int_b^r \frac{\rho^{1+n}}{(r^2-\rho^2)^\beta}\,d\rho\int_a^b \frac{\rho^{-n}s^{n+1}}{\rho^2-s^2}\left(\frac{b^2-s^2}{\rho^2-b^2}\right)^{-\beta} p_0(s)\,ds$$

$$= 2\beta\int_a^b \frac{s^{n+1}}{(r^2-s^2)^{1+\beta}}\,p_0(s)\,ds. \tag{14.19}$$

Also, using (A1) and (2.11)

https://doi.org/10.1017/S0334270000001156 Published online by Cambridge University Press

$$\frac{1}{r}\frac{d}{dr}\int_b^r \frac{\rho^{1+n}h_2(\rho)}{(r^2-\rho^2)^{\beta}}\,d\rho$$

$$=\frac{2^{2-2\beta}}{\Gamma(\beta)^2}\frac{1}{r}\frac{d}{dr}\int_b^r \frac{\rho^{1+n}}{(r^2-\rho^2)^{\beta}}\,d\rho\int_b^{\rho}\frac{\rho^{-n}s^{2n-2\beta+1}}{(\rho^2-s^2)^{1-\beta}}\,ds\int_s^{\infty}\frac{t^{-n+1}}{(t^2-s^2)^{1-\beta}}\,q_2(t)\,dt$$

$$=\frac{2^{1-2\beta}}{\Gamma(\beta)^2}\frac{\pi}{\sin\pi\beta}\,r^{2n-2\beta}\int_r^{\infty}\frac{t^{-n+1}}{(t^2-r^2)^{1-\beta}}\,q_2(t)\,dt. \tag{14.20}$$

Hence, using (14.10), (14.19), (14.20) and replacing β by $-\alpha$, for $r>b$

$$d_2(r)=-2\alpha\int_a^b\frac{t^{n+1}}{(r^2-t^2)^{1-\alpha}}\,p_0(t)\,dt$$
$$-2^{1+2\alpha}\frac{\alpha\Gamma(1+\alpha)}{\Gamma(1-\alpha)}\,r^{2n+2\alpha}\int_r^{\infty}\frac{t^{-n+1}}{(t^2-r^2)^{1+\alpha}}\,q_2(t)\,dt, \tag{14.21}$$

Finally, in equations (14.13) and (14.14) we put $\beta=-\alpha$, $g_1(r)=-2\alpha f_1(r)$ and $g_2(r)=2\alpha f_2(r)$ to obtain

$$f_1(r)+\frac{2\Gamma(1+n)}{\Gamma(\alpha)\Gamma(1+n-\alpha)}\int_b^{\infty}F\left(1-\alpha,1+n;1+n-\alpha;\frac{r^2}{s^2}\right)s^{-2n-1}f_2(s)\,ds$$
$$=-\int_a^b\frac{t^{-n+1}}{(t^2-r^2)^{1-\alpha}}\,p_0(t)\,dt$$
$$-\frac{2^{2\alpha}\Gamma(1+\alpha)}{\Gamma(1-\alpha)}\,r^{-2n+2\alpha}\int_0^r\frac{t^{n+1}}{(r^2-t^2)^{1+\alpha}}\,q_1(t)\,dt \quad\text{for}\quad r<a, \tag{14.22}$$

$$f_2(r)+\frac{2\Gamma(1+n)}{\Gamma(\alpha)\Gamma(1+n-\alpha)}\int_0^a F\left(1-\alpha,1+n;1+n-\alpha;\frac{s^2}{r^2}\right)\frac{s^{2n+1-2\alpha}}{r^{2-2\alpha}}\,f_1(s)\,ds$$
$$=-\int_a^b\frac{t^{n+1}}{(r^2-t^2)^{1-\alpha}}\,p_0(t)\,dt$$
$$-\frac{2^{2\alpha}\Gamma(1+\alpha)}{\Gamma(1-\alpha)}\,r^{2n+2\alpha}\int_r^{\infty}\frac{t^{-n+1}}{(t^2-r^2)^{1+\alpha}}\,q_2(t)\,dt \quad\text{for}\quad r>b. \tag{14.23}$$

It is apparent that, except for the final terms on the right sides, these coupled Fredholm equations are identical with the pair (6.10) and (7.9). When $n=0$ they reduce to

$$f_1(r)+\frac{2\sin\pi\alpha}{\pi}\int_b^{\infty}\frac{s}{s^2-r^2}\,f_2(s)\,ds$$
$$=-\int_a^b\frac{tp_0(t)}{(t^2-r^2)^{1-\alpha}}\,dt$$
$$-\frac{2^{2\alpha}\Gamma(1+\alpha)}{\Gamma(1-\alpha)}\,r^{2\alpha}\int_0^r\frac{t}{(r^2-t^2)^{1+\alpha}}\,q_1(t)\,dt \quad\text{for}\quad r<a, \tag{14.24}$$

https://doi.org/10.1017/S0334270000001156 Published online by Cambridge University Press

$$f_2(r)+\frac{2\sin\pi\alpha}{\pi}\int_0^a \frac{r^{2\alpha}s^{1-2\alpha}}{r^2-s^2}f_1(s)\,ds$$
$$=-\int_a^b \frac{tp_0(t)}{(r^2-t^2)^{1-\alpha}}\,dt$$
$$-\frac{2^{2\alpha}\Gamma(1+\alpha)}{\Gamma(1-\alpha)}r^{2\alpha}\int_r^\infty \frac{t}{(t^2-r^2)^{1+\alpha}}q_2(t)\,dt \quad \text{for} \quad r>b. \tag{14.25}$$

Thus, in the important case $n=0$, equations (14.22) and (14.23) have kernels which are simpler than the kernels in (2.4) and (2.5). Also their right hand sides are simpler than those of (2.4) and (2.5), whether $n=0$ or not.

15. An application — the annulus crack

In order to demonstrate an application of equations (6.10) and (7.9) we consider the problem of determining the stress field in an infinite homogeneous isotropic material with a circular annulus crack. Referred to axisymmetric cylindrical coordinates r, z we suppose the crack occupies the region $a<r<b$ on $z=0$. Because of the symmetry involved, we need only consider the upper half-space $z>0$. The components of stress and displacement may be expressed in terms of one harmonic function $\Phi(r,z)$ (see Green and Zerna [7], page 171). In particular, the components of displacement and stress u_z and σ_{zz} are related to Φ by the equations

$$2\mu u_z = z\frac{\partial^2\Phi}{\partial z^2}-2(1-\eta)\frac{\partial\Phi}{\partial z}, \tag{15.1}$$

$$\sigma_{zz} = z\frac{\partial^3\Phi}{\partial z^3}-\frac{\partial^2\Phi}{\partial z^2}, \tag{15.2}$$

where μ is the shear modulus and η is Poisson's ratio. The boundary conditions on $z=0$ are

$$u_z=0 \qquad \text{for} \quad r<a \quad \text{and} \quad r>b, \tag{15.3}$$

$$\sigma_{zz} = -p_0(r) \quad \text{for} \quad a<r<b, \tag{15.4}$$

where $p_0(r)$ is given. Also all components of stress and displacement must vanish as $(r^2+z^2)^{\frac{1}{2}}\to\infty$. Hence we need to find a harmonic function, vanishing at infinity and such that, on $z=0$,

$$\frac{\partial\Phi}{\partial z}=0 \qquad \text{for} \quad r<a \quad \text{and} \quad r>b, \tag{15.5}$$

$$\frac{\partial^2\Phi}{\partial z^2}=p_0(r) \quad \text{for} \quad a<r<b. \tag{15.6}$$

A suitable choice of $\Phi(r, z)$ is

$$\Phi(r, z) = \int_0^\infty \lambda^{-1} e^{-\lambda z} A(\lambda) J_0(\lambda r)\, d\lambda, \tag{15.7}$$

where $A(\lambda)$ is a function to be determined, of suitable behaviour near 0 and ∞ to ensure convergence. With this choice of Φ the condition at infinity is automatically satisfied while the conditions (15.5) and (15.6) yield

$$\int_0^\infty A(\lambda) J_0(\lambda r)\, d\lambda = 0 \qquad \text{for} \quad r < a, \tag{15.8}$$

$$\int_0^\infty \lambda A(\lambda) J_0(\lambda r)\, d\lambda = p_0(r) \qquad \text{for} \quad a < r < b, \tag{15.9}$$

$$\int_0^\infty A(\lambda) J_0(\lambda r)\, d\lambda = 0 \qquad \text{for} \quad r > b. \tag{15.10}$$

These equations are the case of (1.1) to (1.3) with $n = 0$, $\alpha = \frac{1}{2}$, $q_1(r) = 0$ and $q_2(r) = 0$, so that the relevant results of the previous sections are applicable.

The physical quantity of greatest interest in this problem is the normal stress σ_{zz} on the plane $z = 0$ near the crack edges at $r = a$ and $r = b$. Now σ_{zz} on the plane $z = 0$ is given by the expression

$$\sigma_{zz} = -\int_0^\infty \lambda A(\lambda) J_0(\lambda r)\, d\lambda, \tag{15.11}$$

and hence we require information about the value of this integral as $r \to a-$ and $r \to b+$. Referring to (2.1) and (2.2), this depends on the behaviour of $p_1(r)$ as $r \to a-$ and of $p_2(r)$ as $r \to b+$.

This information is most readily obtained from the transformed coupled Fredholm equations (6.10) and (7.9) since, using (6.1), (6.2) and (A1), we obtain, when $n = 0$ and $\alpha = \frac{1}{2}$,

$$\begin{aligned} p_1(r) &= -\frac{2}{\pi}\frac{1}{r}\frac{d}{dr}\int_r^a \frac{t f_1(t)}{(t^2 - r^2)^{\frac{1}{2}}}\, dt \\ &= \frac{2}{\pi}\frac{f_1(a)}{(a^2 - r^2)^{\frac{1}{2}}} - \frac{2}{\pi}\frac{1}{r}\frac{d}{dr}\int_r^a \frac{t[f_1(t) - f_1(a)]}{(t^2 - r^2)^{\frac{1}{2}}}\, dt \qquad \text{for} \quad r < a \end{aligned} \tag{15.12}$$

and

$$\begin{aligned} p_2(r) &= \frac{2}{\pi}\frac{1}{r}\frac{d}{dr}\int_b^r \frac{t f_2(t)}{(r^2 - t^2)^{\frac{1}{2}}}\, dt \\ &= \frac{2}{\pi}\frac{f_2(b)}{(r^2 - b^2)^{\frac{1}{2}}} + \frac{2}{\pi}\frac{1}{r}\frac{d}{dr}\int_b^r \frac{t[f_2(t) - f_2(b)]}{(r^2 - t^2)^{\frac{1}{2}}}\, dt \qquad \text{for} \quad r > b, \end{aligned} \tag{15.13}$$

assuming the existence of $f_1(a)$ and $f_2(b)$, which will be established presently. In view of (2.1), (2.2) and (15.11) to (15.13) it is likely, and will also be justified below, that, on $z = 0$,

$$\sigma_{zz} \sim -\frac{2}{\pi}\frac{f_1(a)}{(a^2-r^2)^{\frac{1}{2}}} \quad \text{as} \quad r \to a-, \tag{15.14}$$

$$\sigma_{zz} \sim -\frac{2}{\pi}\frac{f_2(b)}{(r^2-b^2)^{\frac{1}{2}}} \quad \text{as} \quad r \to b+. \tag{15.15}$$

Hence it is apparent that the Fredholm equations (6.10) and (7.9) immediately give the asymptotic behaviour of the stress field near the edges of the crack.

In the important case of constant applied normal stress $p_0(r) = p_0$ (this case is important because, by superimposing a uniform stress field $\sigma_{zz} = p_0$ we may obtain the stress field round a stress free annular crack which is being opened by a uniform normal tension p_0 at infinity) equations (6.10) and (7.9) simplify to

$$f_1(r) + \frac{2}{\pi}\int_b^\infty \frac{s}{s^2-r^2} f_2(s)\,ds = [-(b^2-r^2)^{\frac{1}{2}} + (a^2-r^2)^{\frac{1}{2}}]p_0 \quad \text{for} \quad r < a, \tag{15.16}$$

$$f_2(r) + \frac{2}{\pi}\int_0^a \frac{r}{r^2-s^2} f_1(s)\,ds = [(r^2-b^2)^{\frac{1}{2}} - (r^2-a^2)^{\frac{1}{2}}]p_0 \quad \text{for} \quad r > b \tag{15.17}$$

and the corresponding uncoupled equations are

$$f_+(x) + \frac{2}{\pi}\int_0^k \frac{1}{1-x^2t^2} f_+(t)\,dt = \left\{\left(a+\frac{c}{x^2}\right)\left(1-\frac{x^2}{k^2}\right)^{1/2} - \left(b+\frac{c}{x^2}\right)(1-k^2x^2)^{\frac{1}{2}}\right\}p_0 \quad \text{for} \quad 0 < x < k, \tag{15.18}$$

$$f_-(x) - \frac{2}{\pi}\int_0^k \frac{1}{1-x^2t^2} f_-(t)\,dt = \left\{\left(a-\frac{c}{x^2}\right)\left(1-\frac{x^2}{k^2}\right)^{1/2} - \left(b-\frac{c}{x^2}\right)(1-k^2x^2)^{\frac{1}{2}}\right\}p_0 \quad \text{for} \quad 0 < x < k, \tag{15.19}$$

where $r = cx$ and $s = c/t$ have been substituted in (15.16), and $r = c/x$ and $s = ct$ in (15.17) and

$$f_\pm(x) = f_1(cx) \pm \frac{f_2(c/x)}{x}, \tag{15.20}$$

https://doi.org/10.1017/S0334270000001156 Published online by Cambridge University Press

so that

$$f_1(a) = \tfrac{1}{2}\{f_+(k) + f_-(k)\}, \quad f_2(b) = \tfrac{1}{2}k\{f_+(k) - f_-(k)\}. \qquad (15.21),\ (15.22)$$

Since $0 < k < 1$, the kernels of (15.18) and (15.19) are continuous; and it is easily seen by power series expansions that their right hand sides are also continuous in $0 \leqq x \leqq k$ if suitably defined at $x = 0$. These equations consequently have unique continuous solutions for $f_+(x)$ and $f_-(x)$ in $0 \leqq x \leqq k$. This with (15.21) and (15.22), justifies the existence of $f_1(a)$ and $f_2(b)$ assumed in (15.12) and (15.13), and also proves the continuity of f_+ and f_- and of f_1 and f_2.

To justify (15.14) and (15.15) as valid deductions from (15.12) and (15.13), we need to know that $f_1(r)$ and $f_2(r)$ have continuous derivatives in half-neighbourhoods of a and b respectively. This property follows from the continuous differentiability of $f_+(x)$ and $f_-(x)$ in a half-neighbourhood of k, which is apparent from (15.18) and (15.19).

We can now justify (15.14) by showing that the last term in (15.12) tends to 0 as $r \to a-$. Integrating by parts,

$$\int_r^a \frac{t\{f_1(t) - f_1(a)\}}{(t^2 - r^2)^{\frac{1}{2}}}\,dt = -\int_r^a (t^2 - r^2)^{\frac{1}{2}} f_1'(t)\,dt;$$

$$\frac{1}{r}\frac{d}{dr}\int_r^a \frac{t\{f_1(t) - f_1(a)\}}{(t^2 - r^2)^{\frac{1}{2}}}\,dt = \frac{1}{r}\int_r^a \frac{r}{(t^2 - r^2)^{\frac{1}{2}}} f_1'(t)\,dt$$

$$= O\left\{\int_r^a \frac{1}{(t^2 - r^2)^{\frac{1}{2}}}\,dt\right\} = O\left\{\operatorname{arcosh}\frac{a}{r}\right\}$$

using the boundedness of $f_1'(t)$. The last expression tends to 0 as $r \to a-$ justifying (15.14). A similar justification for (15.15) could be given.

Appendix: evaluations of integrals

A1. *Abel's Integral Equation.* Standard pairs of inversion formulae, supposing that $0 < \alpha < 1$ and $0 \leqq a < r < b \leqq \infty$, are:

$$\int_a^r \frac{g(t)}{(r^2 - t^2)^{1-\alpha}}\,dt = f(r) \text{ and } g(r) = \frac{2\sin \pi\alpha}{\pi}\frac{d}{dr}\int_a^r \frac{tf(t)}{(r^2 - t^2)^{\alpha}}\,dt,$$

$$\int_r^b \frac{g(t)}{(t^2 - r^2)^{1-\alpha}}\,dt = f(r) \text{ and } g(r) = -\frac{2\sin \pi\alpha}{\pi}\frac{d}{dr}\int_r^b \frac{tf(t)}{(t^2 - r^2)^{\alpha}}\,dt.$$

From these follow

$$f(r)=\frac{2\sin\pi\alpha}{\pi}\int_a^r\frac{1}{(r^2-s^2)^{1-\alpha}}\left\{\frac{d}{ds}\int_a^s\frac{tf(t)}{(s^2-t^2)^\alpha}\,dt\right\}ds,$$

$$f(r)=-\frac{2\sin\pi\alpha}{\pi}\int_r^b\frac{1}{(s^2-r^2)^{1-\alpha}}\left\{\frac{d}{ds}\int_s^b\frac{tf(t)}{(t^2-s^2)^\alpha}\,dt\right\}ds,$$

$$g(r)=\frac{2\sin\pi\alpha}{\pi}\frac{d}{dr}\left\{\int_a^r\frac{s}{(r^2-s^2)^\alpha}\,ds\int_a^s\frac{1}{(s^2-t^2)^{1-\alpha}}\,g(t)\,dt\right\},$$

$$g(r)=-\frac{2\sin\pi\alpha}{\pi}\frac{d}{dr}\left\{\int_r^b\frac{s}{(s^2-r^2)^\alpha}\,ds\int_s^b\frac{1}{(t^2-s^2)^{1-\alpha}}\,g(t)\,dt\right\}.$$

A2. If $0<\alpha<1$, $s\geqq 0$ and $0<a<b<\infty$ then

$$\int_a^b\frac{(b^2-\rho^2)^{-\alpha}(\rho^2-a^2)^{\alpha-1}}{s^2-\rho^2}\rho\,d\rho=\begin{cases}\dfrac{\pi}{2\sin\pi\alpha}\dfrac{(s^2-a^2)^{\alpha-1}}{(s^2-b^2)^\alpha} & \text{if}\quad s>b,\\[2ex] \dfrac{-\pi}{2\sin\pi\alpha}\dfrac{(a^2-s^2)^{\alpha-1}}{(b^2-s^2)^\alpha} & \text{if}\quad s<a.\end{cases}$$

$$\int_a^b\left(\frac{b^2-\rho^2}{\rho^2-a^2}\right)^\alpha\frac{1}{s^2-\rho^2}\rho\,d\rho=\begin{cases}\dfrac{\pi}{2\sin\pi\alpha}\left[1-\left(\dfrac{s^2-b^2}{s^2-a^2}\right)^\alpha\right] & \text{if}\quad s>b,\\[2ex] \dfrac{\pi}{2\sin\pi\alpha}\left[1-\left(\dfrac{b^2-s^2}{a^2-s^2}\right)^\alpha\right] & \text{if}\quad s<a.\end{cases}$$

$$\int_a^b\left(\frac{\rho^2-a^2}{b^2-\rho^2}\right)^\alpha\frac{1}{\rho^2-s^2}\rho\,d\rho=\begin{cases}\dfrac{\pi}{2\sin\pi\alpha}\left[1-\left(\dfrac{s^2-a^2}{s^2-b^2}\right)^\alpha\right] & \text{if}\quad s>b,\\[2ex] \dfrac{\pi}{2\sin\pi\alpha}\left[1-\left(\dfrac{a^2-s^2}{b^2-s^2}\right)^\alpha\right] & \text{if}\quad s<a.\end{cases}$$

The first pair of these formulae can be obtained by the substitutions

$$\rho^2=a^2+(b^2-a^2)u,\quad u=\frac{1}{1+v},\quad v=\frac{s^2-b^2}{s^2-a^2}w.$$

In the second pair we write $\gamma=1-\alpha$, and use

$$(b^2-\rho^2)^\alpha=(b^2-\rho^2)^{1-\gamma}=\{(b^2-s^2)+(s^2-\rho^2)\}(b^2-\rho^2)^{-\gamma}.$$

This gives the sum of two integrals, one of which is evaluated immediately from the first pair of formulae and the other from the beta-gamma relation.

The third pair can be treated similarly.

A3. If $\alpha>0$, $s>r>0$ and $n>-1$,

https://doi.org/10.1017/S0334270000001156 Published online by Cambridge University Press

$$\int_0^r u^{2n+1}(r^2-u^2)^{\alpha-1}(s^2-u^2)^{-(1+\alpha)}\,du$$

$$=\frac{\Gamma(\alpha)\Gamma(1+n)}{2\Gamma(1+n+\alpha)}r^{2n+2\alpha}s^{-2-2\alpha}F\left(1+\alpha,1+n;1+n+\alpha;\frac{r^2}{s^2}\right)\quad \text{if}\quad s>r,$$

$$\int_s^\infty t^{1-2n}(t^2-s^2)^{\alpha-1}(t^2-r^2)^{-(1+\alpha)}dt$$

$$=\frac{\Gamma(\alpha)\Gamma(1+n)}{2\Gamma(1+n+\alpha)}s^{-2-2n}F\left(1+\alpha,1+n;1+n+\alpha;\frac{r^2}{s^2}\right)\quad \text{if}\quad s>r.$$

The first of these integrals follows immediately from Euler's integral for the hypergeometric function. The second can be obtained from the first by making the substitution $u = rs/t$.

References

[1] J. C. Cooke, 'Some further triple integral equation solutions,' *Proc. Edinburgh Math. Soc.* (II) 13 (1963), 303–316.

[2] B. Noble, 'The solution of Bessel function dual integral equations by a multiplying factor method,' *Proc. Camb Phil. Soc.* 59 (1963), 351–362.

[3] V. S. Gubenko and V. I. Mossakovskii, 'Pressure of an axially symmetric circular die on an elastic half-space,' *Prikl. Mat. Meh.* 24 (1960), 330–334 (Russian); translated as *J. Appl. Math. Mech.* 24 (1960), 477–486.

[4] W. E. Williams, 'Integral equation formulation of some three part boundary value problems,' *Proc. Edinburgh Math. Soc.* (II) 13 (1963), 317–323.

[5] D. L. Clements and E. R. Love, 'Potential problems involving an annulus,' *Proc. Camb. Phil. Soc.* 76 (1974), 313–325.

[6] A. Erdélyi and others, '*Higher Transcendental Functions*,' vol. 1, Bateman Manuscript Project (McGraw-Hill, 1953).

[7] A. E. Green and W. Zerna, '*Theoretical Elasticity*' (Oxford, 1954).

Department of Mathematics,
University of Melbourne,
Melbourne,
Australia.

Department of Applied Mathematics,
University of Adelaide,
Adelaide,
Australia.

https://doi.org/10.1017/S0334270000001156 Published online by Cambridge University Press