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ON RING-THEORETIC (IN)FINITENESS OF BANACH ALGEBRAS OF OPERATORS ON BANACH SPACES

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Abstract. Let $\mathscr{B}(\mathfrak{X})$ denote the Banach algebra of all bounded linear operators on a Banach space $\mathfrak{X}$. We show that $\mathscr{B}(\mathfrak{X})$ is finite if and only if no proper, complemented subspace of $\mathfrak{X}$ is isomorphic to $\mathfrak{X}$, and we show that $\mathscr{B}(\mathfrak{X})$ is properly infinite if and only if $\mathfrak{X}$ contains a complemented subspace isomorphic to $\mathfrak{X} \oplus \mathfrak{X}$. We apply these characterizations to find Banach spaces $\mathfrak{X}_1, \mathfrak{X}_2$, and $\mathfrak{X}_3$ such that $\mathscr{B}(\mathfrak{X}_1)$ is finite, $\mathscr{B}(\mathfrak{X}_2)$ is infinite, but not properly infinite, and $\mathscr{B}(\mathfrak{X}_3)$ is properly infinite. Moreover, we prove that every unital, properly infinite ring has a continued bisection of the identity, and we give examples of Banach spaces $\mathfrak{Y}_1$ and $\mathfrak{Y}_2$ such that $\mathscr{B}(\mathfrak{Y}_1)$ and $\mathscr{B}(\mathfrak{Y}_2)$ are infinite without being properly infinite, $\mathscr{B}(\mathfrak{Y}_1)$ has a continued bisection of the identity, and $\mathscr{B}(\mathfrak{Y}_2)$ has no continued bisection of the identity. Finally, we exhibit a unital C^*-algebra which is finite and has a continued bisection of the identity.

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1. Finite, infinite, and properly infinite Banach algebras of operators on Banach spaces. Throughout this note, we use the term *operator* to denote a bounded, linear map between Banach spaces. Unless otherwise specified, all Banach spaces are assumed to be over the same scalar field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. For a Banach space $\mathfrak{X}$, we write $\mathscr{B}(\mathfrak{X})$ for the collection of all operators on $\mathfrak{X}$; this is a unital Banach algebra with identity $I_{\mathfrak{X}}$ (the identity operator on $\mathfrak{X}$). We denote by im T the image of an operator T.

DEFINITION 1.1. Let $\mathscr{R}$ be a ring.

(i) Two elements S and T in $\mathscr{R}$ are *orthogonal* if $ST = 0$ and $TS = 0$.

(ii) An element P in $\mathscr{R}$ is *idempotent* if $P^2 = P$.

(iii) Two idempotent elements P and Q in $\mathscr{R}$ are *algebraically equivalent*, written $P \approx Q$, if $P = ST$ and $Q = TS$ for some elements S and T in $\mathscr{R}$.

REMARK 1.2. Let P and Q be idempotent elements in a ring $\mathscr{R}$.

(i) The elements P and Q are orthogonal if and only if $P + Q$ is idempotent.

(ii) Suppose that P and Q are algebraically equivalent. Take S and T as in Definition 1.1(iii). Then $P = SQT$ and $Q = TPS$, so that P and Q generate the same ideal in $\mathscr{R}$.

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Definition 1.3. Let $\mathfrak{R}$ be a unital ring. We write $1_{\mathfrak{R}}$ for the identity of $\mathfrak{R}$.

(i) $\mathfrak{R}$ is *finite* if $P \approx 1_{\mathfrak{R}}$ implies that $P = 1_{\mathfrak{R}}$ for each idempotent element P in $\mathfrak{R}$.

(ii) $\mathfrak{R}$ is *infinite* if $\mathfrak{R}$ is not finite.

(iii) $\mathfrak{R}$ is *properly infinite* if there are idempotent elements P_1 and P_2 in $\mathfrak{R}$ that are orthogonal and satisfy $P_n \approx 1_{\mathfrak{R}}$ for $n = 1, 2$.

It is clear that a unital, properly infinite ring is infinite. In a commutative ring, the relation $\approx$ is trivial (that is, it coincides with $=$), and so each unital, commutative ring is finite.

Historically, the relation $\approx$ and the concepts of finite, infinite, and properly infinite rings have roots back to Murray and von Neumann's seminal studies of projections in rings of operators acting on a Hilbert space (see **[10]**). Despite the great success of Murray and von Neumann's programme, not much attention has been paid to these ideas in the case of operators acting on general Banach spaces; that is, Banach spaces that are *not* Hilbert spaces. However, following the recent advances in Banach space theory, especially the deep work of Gowers and Maurey (e.g., see **[4]**, **[3]**, and **[5]**), the time is now ripe for a more detailed study of the invariants of Banach spaces and their algebras of operators, including concepts such as finiteness, infiniteness, and proper infiniteness. This note addresses some of the most fundamental questions of this kind.

Indeed, in this section we shall characterize the Banach spaces $\mathfrak{X}$ for which $\mathscr{B}(\mathfrak{X})$ is finite and properly infinite, respectively. As an application, we shall show that all three cases of Definition 1.3 can occur for $\mathscr{B}(\mathfrak{X})$; that is, we shall give examples of Banach spaces $\mathfrak{X}_1, \mathfrak{X}_2$, and $\mathfrak{X}_3$ such that $\mathscr{B}(\mathfrak{X}_1)$ is finite, $\mathscr{B}(\mathfrak{X}_2)$ is infinite, but not properly infinite, and $\mathscr{B}(\mathfrak{X}_3)$ is properly infinite.

We begin with a standard lemma. For completeness, we include a proof.

Lemma 1.4. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces, and let $P\colon \mathfrak{X} \to \mathfrak{X}$ and $Q\colon \mathfrak{Y} \to \mathfrak{Y}$ be idempotent operators. There are operators $S\colon \mathfrak{Y} \to \mathfrak{X}$ and $T\colon \mathfrak{X} \to \mathfrak{Y}$ such that $P = ST$ and $Q = TS$ if and only if the images of P and Q are isomorphic as Banach spaces.*

Proof. Suppose that $S\colon \mathfrak{Y} \to \mathfrak{X}$ and $T\colon \mathfrak{X} \to \mathfrak{Y}$ are operators with $P = ST$ and $Q = TS$. Then $S(\operatorname{im} Q) \subseteq \operatorname{im} P$, $T(\operatorname{im} P) \subseteq \operatorname{im} Q$, and the restriction

$$\tilde{S}\colon y \mapsto Sy, \quad \operatorname{im} Q \to \operatorname{im} P,$$

is an isomorphism with inverse

$$\tilde{T}\colon x \mapsto Tx, \quad \operatorname{im} P \to \operatorname{im} Q.$$

Conversely, suppose that $U\colon \operatorname{im} P \to \operatorname{im} Q$ is an isomorphism. Then the operators

$$S\colon x \mapsto UPx, \quad \mathfrak{X} \to \mathfrak{Y}, \quad \text{and} \quad T\colon y \mapsto U^{-1}Qy, \quad \mathfrak{Y} \to \mathfrak{X},$$

satisfy $ST = P$ and $TS = Q$. □

Corollary 1.5. *Let $\mathfrak{X}$ be a Banach space. Two idempotent operators on $\mathfrak{X}$ are algebraically equivalent if and only if their images are isomorphic as Banach spaces.*

In particular, the Banach algebra $\mathscr{B}(\mathfrak{X})$ is finite if and only if no proper, complemented subspace of $\mathfrak{X}$ is isomorphic to $\mathfrak{X}$.

It is well known (and immediate from Corollary 1.5) that the Banach algebra $\mathscr{B}(\mathfrak{X})$ is finite whenever $\mathfrak{X}$ is a finite-dimensional Banach space. Surprisingly, there are infinite-dimensional Banach spaces $\mathfrak{X}$ for which $\mathscr{B}(\mathfrak{X})$ is finite. This is a consequence of the following important theorem of Gowers and Maurey. Part (i) of the theorem and part (ii) in the case of complex scalars are proved in [**4**]; a full proof independent of the choice of scalar field is given in [**5**]. Before stating the theorem, we recall the fundamental definition of Gowers and Maurey: a Banach space $\mathfrak{X}$ is *hereditarily indecomposable* if $\mathfrak{X}$ is infinite-dimensional and no closed subspace $\mathfrak{Y}$ of $\mathfrak{X}$ admits an idempotent operator $P: \mathfrak{Y} \to \mathfrak{Y}$ such that neither the image of P nor the kernel of P is finite-dimensional.

THEOREM 1.6. (W. T. Gowers and B. Maurey)

(i) *Hereditarily indecomposable Banach spaces exist.*

(ii) *A hereditarily indecomposable Banach space is not isomorphic to any proper subspace of itself.*

COROLLARY 1.7. *Let $\mathfrak{X}$ be a hereditarily indecomposable Banach space. Then the Banach algebra $\mathcal{B}(\mathfrak{X})$ is finite.*

The following characterization of unital, properly infinite rings is elementary, but often useful. It is probably well known; as before, we outline a (short) proof for completeness.

LEMMA 1.8. *Let $\mathcal{R}$ be a unital ring and let $\delta_{m,n}$ denote Kronecker's delta symbol. The following assertions are equivalent:*

(a) *$\mathcal{R}$ is properly infinite;*

(b) *there are elements S_1, S_2, T_1 and T_2 in $\mathcal{R}$ such that $T_nS_m = \delta_{m,n}1_{\mathcal{R}}$, for $m, n = 1, 2$;*

(c) *there is a sequence $(P_n)_{n=1}^{\infty}$ of idempotent elements in $\mathcal{R}$ that are pairwise orthogonal and satisfy $P_n \approx 1_{\mathcal{R}}$, for each $n \in \mathbb{N}$;*

(d) *there are sequences $(S_n)_{n=1}^{\infty}$ and $(T_n)_{n=1}^{\infty}$ in $\mathcal{R}$ such that $T_nS_m = \delta_{m,n}1_{\mathcal{R}}$, for each $m, n \in \mathbb{N}$.*

Proof. The implications '(d) $\Rightarrow$ (b) $\Rightarrow$ (a)' and '(d) $\Rightarrow$ (c) $\Rightarrow$ (a)' are evident. Therefore we need only prove that (a) implies (d). Suppose that $P_1, P_2 \in \mathcal{R}$ are orthogonal and idempotent and satisfy $P_1 \approx 1_{\mathcal{R}} \approx P_2$. Take $U_1, V_1, U_2, V_2 \in \mathcal{R}$ such that $U_mV_m = P_m$ and $V_mU_m = 1_{\mathcal{R}}$, for $m = 1, 2$, and set $S_n := U_2^{n-1}U_1$ and $T_n := V_1V_2^{n-1}$, for each $n \in \mathbb{N}$. Then a straightforward induction argument shows that (d) is satisfied. □

PROPOSITION 1.9. *Let $\mathfrak{X}$ be a Banach space. The Banach algebra $\mathcal{B}(\mathfrak{X})$ is properly infinite if and only if $\mathfrak{X}$ contains a complemented subspace isomorphic to $\mathfrak{X} \oplus \mathfrak{X}$.*

Proof. For $n = 1, 2$, let $J_n: \mathfrak{X} \to \mathfrak{X} \oplus \mathfrak{X}$ and $K_n: \mathfrak{X} \oplus \mathfrak{X} \to \mathfrak{X}$ be the canonical nth coordinate embedding and projection, respectively, so that $J_1(x) = (x, 0)$, $J_2(x) = (0, x)$, $K_1(x_1, x_2) = x_1$, and $K_2(x_1, x_2) = x_2$, for $x, x_1, x_2 \in \mathfrak{X}$.

Suppose that $\mathcal{B}(\mathfrak{X})$ is properly infinite. Take operators S_1, S_2, T_1, and T_2 on $\mathfrak{X}$ such that $T_nS_m = \delta_{m,n}I_{\mathfrak{X}}$, for $m, n = 1, 2$. Then the operators

$$U := S_1K_1 + S_2K_2: \mathfrak{X} \oplus \mathfrak{X} \to \mathfrak{X} \quad \text{and} \quad V := J_1T_1 + J_2T_2: \mathfrak{X} \to \mathfrak{X} \oplus \mathfrak{X}$$

satisfy $VU = I_{\mathfrak{X}\oplus\mathfrak{X}}$. It follows that the operator UV on $\mathfrak{X}$ is idempotent, and Lemma 1.4 shows that $\operatorname{im}(UV)$ is isomorphic to $\operatorname{im}(VU) = \mathfrak{X} \oplus \mathfrak{X}$.

Conversely, suppose that $\mathfrak{X}$ contains a complemented subspace isomorphic to $\mathfrak{X} \oplus \mathfrak{X}$. By Lemma 1.4, there are operators $S: \mathfrak{X} \oplus \mathfrak{X} \to \mathfrak{X}$ and $T: \mathfrak{X} \to \mathfrak{X} \oplus \mathfrak{X}$ with

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$TS = I_{\mathfrak{X}\oplus\mathfrak{X}}$, and so the operators $S_m := SJ_m$ and $T_n := K_nT$ on $\mathfrak{X}$ satisfy $T_nS_m = \delta_{m,n}I_{\mathfrak{X}}$, for $m, n = 1,2$. This proves that $\mathscr{B}(\mathfrak{X})$ is properly infinite. $\square$

EXAMPLE 1.10. All the classical Banach spaces, including c_0, $C([0, 1])$, ℓ_p, and $L_p([0, 1])$, where $1 \leqslant p \leqslant \infty$, are isomorphic to the direct sum of two copies of themselves. Hence, they are examples of Banach spaces $\mathfrak{X}$ such that $\mathscr{B}(\mathfrak{X})$ is properly infinite.

The following observation is immediate from the definition.

LEMMA 1.11. *Let $\mathscr{R}$ and $\mathscr{S}$ be unital rings. Suppose that $\mathscr{R}$ is properly infinite and that there is a unital ring homomorphism from $\mathscr{R}$ into $\mathscr{S}$. Then $\mathscr{S}$ is properly infinite.* $\square$

We shall use this to prove that the Banach algebra $\mathscr{B}(\mathfrak{J}_p)$, where $1 < p < \infty$ and $\mathfrak{J}_p$ is the pth James space, is infinite, but not properly infinite. We begin with the formal definition of the Banach space $\mathfrak{J}_p$.

DEFINITION 1.12. Let $1 < p < \infty$. For each sequence $\alpha = (\alpha_n)_{n=1}^{\infty}$ of scalars, set

$$\|\alpha\|_{\mathcal{J}_p} := \sup\left\{\left(\sum_{j=1}^{k-1}\left|\alpha_{n_j} - \alpha_{n_{j+1}}\right|^p\right)^{1/p} \,\middle|\, k, n_1, \ldots, n_k \in \mathbb{N}, k \geqslant 2, n_1 < n_2 < \cdots < n_k\right\}.$$

The pth *James space* is

$$\mathfrak{J}_p := \{\alpha \in c_0 \mid \|\alpha\|_{\mathfrak{J}_p} < \infty\}.$$

This is a Banach space with respect to the coordinatewise-defined vector-space operations and the norm $\|\cdot\|_{\mathfrak{J}_p}$. A fundamental property of $\mathfrak{J}_p$ is that it is *quasi-reflexive*; that is, the canonical image of $\mathfrak{J}_p$ in its bidual space $\mathfrak{J}_p^{**}$ has codimension 1; for $p = 2$, this is shown by James in [**6**]. An immediate consequence of the quasi-reflexivity of $\mathfrak{J}_p$ is that the ideal $\mathscr{W}(\mathfrak{J}_p)$ of weakly compact operators has codimension 1 in $\mathscr{B}(\mathfrak{J}_p)$, and hence the quotient homomorphism of $\mathscr{B}(\mathfrak{J}_p)$ onto $\mathscr{B}(\mathfrak{J}_p)/\mathscr{W}(\mathfrak{J}_p)$ induces a continuous algebra epimorphism $\varphi\colon \mathscr{B}(\mathfrak{J}_p) \to \mathbb{K}$, where we recall that $\mathbb{K}$ denotes the scalar field. Specifically, φ is given by

$$\varphi\big(\zeta I_{\mathfrak{J}_p} + W\big) = \zeta \qquad (\zeta \in \mathbb{K},\ W \in \mathscr{W}(\mathfrak{J}_p)).$$

PROPOSITION 1.13. *For each real number $p > 1$, the Banach algebra $\mathscr{B}(\mathfrak{J}_p)$ is infinite, but not properly infinite.*

Proof. The Banach algebra $\mathscr{B}(\mathfrak{J}_p)$ is infinite because $\mathfrak{J}_p$ clearly contains a proper, complemented subspace isomorphic to $\mathfrak{J}_p$. To be precise, consider the left- and right-shift operators

$$L\colon (\alpha_n)_{n=1}^{\infty} \mapsto (\alpha_n)_{n=2}^{\infty},\quad \mathfrak{J}_p \to \mathfrak{J}_p, \quad \text{and} \quad R\colon (\alpha_n)_{n=1}^{\infty} \mapsto (0, \alpha_1, \alpha_2, \ldots),\quad \mathfrak{J}_p \to \mathfrak{J}_p.$$

They satisfy $LR = I_{\mathfrak{J}_p}$ and $RL \neq I_{\mathfrak{J}_p}$, so that RL is an idempotent operator which is algebraically equivalent to the identity operator without being equal to it.

On the other hand, Lemma 1.11 implies that $\mathscr{B}(\mathfrak{J}_p)$ is not properly infinite because it admits a unital ring homomorphism φ onto the finite ring $\mathbb{K}$. $\square$

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2. Continued bisections of the identity. In this section we examine the relationship between the existence of a continued bisection of the identity in a unital ring and the properties studied in §1. Indeed, we prove that every unital, properly infinite ring has a continued bisection of the identity, and we give examples to show that no other implications hold in general: a unital ring that is either finite or infinite without being properly infinite may or may not have a continued bisection of the identity.

DEFINITION 2.1. Let $\mathfrak{R}$ be a unital ring. A *continued bisection of the identity* in $\mathfrak{R}$ is a pair $((P_n)_{n=1}^{\infty}, (Q_n)_{n=1}^{\infty})$ of sequences of idempotent elements in $\mathfrak{R}$ satisfying

(i) $1_{\mathfrak{R}} = P_1 + Q_1$;
(ii) $P_n = P_{n+1} + Q_{n+1}$ for each $n \in \mathbb{N}$;
(iii) for each $n \in \mathbb{N}$, the elements P_n and Q_n generate the same ideal in $\mathfrak{R}$.

REMARK 2.2. Let $((P_n)_{n=1}^{\infty}, (Q_n)_{n=1}^{\infty})$ be a continued bisection of the identity in a unital ring $\mathfrak{R}$.

(i) It follows from Remark 1.2(i) that P_n and Q_n are orthogonal for each $n \in \mathbb{N}$.
(ii) An easy inductive argument shows that the ideal generated by P_n (and hence by Q_n) is all of $\mathfrak{R}$ for each $n \in \mathbb{N}$.

Continued bisections of the identity originate from Johnson's study of automatic continuity of homomorphisms from $\mathfrak{B}(\mathfrak{X})$ for a Banach space $\mathfrak{X}$. (See [**7**]; an up-to-date account of applications of continued bisections of the identity in automatic continuity theory can be found in [**1**].) The following result is a ring-theoretic counterpart of Johnson's fundamental observation that $\mathfrak{B}(\mathfrak{X})$ has a continued bisection of the identity whenever $\mathfrak{X}$ is isomorphic to $\mathfrak{X} \oplus \mathfrak{X}$.

PROPOSITION 2.3. *A unital, properly infinite ring has a continued bisection of the identity.*

Proof. Let $\mathfrak{R}$ be a unital, properly infinite ring. Take sequences $(S_n)_{n=1}^{\infty}$ and $(T_n)_{n=1}^{\infty}$ in $\mathfrak{R}$ such that $T_n S_m = \delta_{m,n} 1_{\mathfrak{R}}$ for $m, n \in \mathbb{N}$. Then the elements

$$P_n := 1_{\mathfrak{R}} - \sum_{m=1}^{n} S_m T_m \in \mathfrak{R} \quad \text{and} \quad Q_n := S_n T_n \in \mathfrak{R} \qquad (n \in \mathbb{N})$$

are idempotent and satisfy

$$T_{n+1} P_n S_{n+1} = 1_{\mathfrak{R}} \quad \text{and} \quad T_n Q_n S_n = 1_{\mathfrak{R}} \qquad (n \in \mathbb{N}).$$

This implies that condition (iii) in Definition 2.1 is fulfilled. Conditions (i)–(ii) are easy to check, and so $((P_n)_{n=1}^{\infty}, (Q_n)_{n=1}^{\infty})$ is a continued bisection of the identity in $\mathfrak{R}$. □

LEMMA 2.4. *Let $\mathfrak{R}$ and $\mathfrak{S}$ be unital rings, and let $\varphi: \mathfrak{R} \to \mathfrak{S}$ be a unital ring homomorphism. Suppose that $((P_n)_{n=1}^{\infty}, (Q_n)_{n=1}^{\infty})$ is a continued bisection of the identity in $\mathfrak{R}$. Then $((\varphi(P_n))_{n=1}^{\infty}, (\varphi(Q_n))_{n=1}^{\infty}$ is a continued bisection of the identity in $\mathfrak{S}$.*

Proof. Conditions (i)–(ii) in Definition 2.1 are clear, whereas condition (iii) follows from Remark 2.2(ii). □

Since the scalar field $\mathbb{K}$ has no continued bisection of the identity, we obtain the following improvement of Proposition 1.13.

COROLLARY 2.5. *For each real number $p > 1$, the Banach algebra $\mathfrak{B}(\mathfrak{J}_p)$ is infinite, but it has no continued bisection of the identity.*

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The converse of Proposition 2.3 does not hold. We shall use the following striking theorem of Figiel, proved in **[2]**, to give a counterexample. For $1 \leqslant p \leqslant \infty$ and a sequence $(\mathfrak{X}_n)_{n=1}^\infty$ of Banach spaces, we write $(\oplus_{n=1}^\infty \mathfrak{X}_n)_{\ell_p}$ for the direct sum of $\mathfrak{X}_1, \mathfrak{X}_2, \ldots$ in the sense of ℓ_p, and for each $N \in \mathbb{N}$, we denote by ℓ_p^N the N-dimensional vector space over $\mathbb{K}$ equipped with the ℓ_p-norm.

Theorem 2.6. (T. Figiel) *For each strictly decreasing sequence $(q_n)_{n=1}^\infty$ of real numbers greater than 2 and each real number p with $1 < p \leqslant \lim_{n\to\infty} q_n$, there exists a sequence $(N_n)_{n=1}^\infty$ of natural numbers such that the Banach space*

$$\mathfrak{F} := \left(\bigoplus_{n=1}^{\infty} \ell_{q_n}^{N_n}\right)_{\ell_p} \tag{2.1}$$

has the following property: for each natural number m, the direct sum of m copies of $\mathfrak{F}$ does not contain any subspace isomorphic to the direct sum of $m+1$ copies of $\mathfrak{F}$. □

The following lemma is based on an argument from **[11]**.

Lemma 2.7. *Let $\mathfrak{X}$ be a Banach space with an unconditional basis $(e_n)_{n=1}^\infty$. Suppose that $(e_n)_{n=1}^\infty$ is equivalent to the basic sequence $(e_n)_{n=2}^\infty$. Then the Banach algebra $\mathfrak{B}(\mathfrak{X})$ is infinite and has a continued bisection of the identity.*

Proof. Let $(f_n)_{n=1}^\infty$ denote the biorthogonal functionals associated with the basis $(e_n)_{n=1}^\infty$. The fact that $(e_n)_{n=1}^\infty$ is equivalent to $(e_n)_{n=2}^\infty$ means that $\mathfrak{X}$ admits left- and right-shift operators

$$L\colon x \mapsto \sum_{m=1}^\infty \langle x, f_{m+1}\rangle e_m, \quad \mathfrak{X} \to \mathfrak{X}, \quad \text{and} \quad R\colon x \mapsto \sum_{m=1}^\infty \langle x, f_m\rangle e_{m+1}, \quad \mathfrak{X} \to \mathfrak{X}.$$

It follows that $\mathfrak{B}(\mathfrak{X})$ is infinite because $LR = I_{\mathfrak{X}}$ and $RL \neq I_{\mathfrak{X}}$.

The unconditionality of $(e_n)_{n=1}^\infty$ implies that there are idempotent operators

$$P_n\colon x \mapsto \sum_{m=1}^\infty \langle x, f_{2^n m}\rangle, e_{2^n m}, \quad \mathfrak{X} \to \mathfrak{X}, \quad Q_n\colon x \mapsto \sum_{m=1}^\infty \langle x, f_{2^n m - 2^{n-1}}\rangle e_{2^n m - 2^{n-1}}, \quad \mathfrak{X} \to \mathfrak{X},$$

for each $n \in \mathbb{N}$. Direct calculations show that these operators satisfy conditions (i)–(ii) in Definition 2.1, and that $P_n = R^{2^{n-1}} Q_n L^{2^{n-1}}$. This implies that $Q_n = L^{2^{n-1}} P_n R^{2^{n-1}}$ because $LR = I_{\mathfrak{X}}$, and so condition (iii) in Definition 2.1 is fulfilled, too. □

Example 2.8. For each $n \in \mathbb{N}$, let $\mathfrak{F}_n$ be a non-zero, finite-dimensional Banach space with a normalized basis $(e_1^{(n)}, \ldots, e_{N_n}^{(n)})$, and let $(f_1^{(n)}, \ldots, f_{N_n}^{(n)})$ denote the associated biorthogonal functionals. Define left- and right-shift operators L_n and R_n on $\mathfrak{F}_n$ by

$$L_n(x) := \begin{cases} 0 & \text{if } N_n = 1, \\ \displaystyle\sum_{m=1}^{N_n - 1} \left\langle x, f_{m+1}^{(n)}\right\rangle e_m^{(n)} & \text{otherwise,} \end{cases} \qquad R_n(x) := \begin{cases} 0 & \text{if } N_n = 1, \\ \displaystyle\sum_{m=1}^{N_n - 1} \left\langle x, f_m^{(n)}\right\rangle e_{m+1}^{(n)} & \text{otherwise,} \end{cases}$$

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for each $x \in \mathfrak{F}_n$. Set

$$C_n := \sup\left\{ \left\| \sum_{m=1}^{N_n} \xi_m \langle x, f_m^{(n)} \rangle e_m^{(n)} \right\| \;\middle|\; x \in \operatorname{ball} \mathfrak{F}_n,\ (\xi_1, \dots, \xi_{N_n}) \in \operatorname{ball} \ell_\infty^{N_n} \sum_{m=1}^{N_n} \right\},$$

where ball $\mathfrak{F}_n$ and ball $\ell_\infty^{N_n}$ denote the closed unit balls of $\mathfrak{F}_n$ and $\ell_\infty^{N_n}$, respectively. Suppose that

$$\sup\{C_n \mid n \in \mathbb{N}\} < \infty. \tag{2.2}$$

Then, for each real number p $\geqslant 1$,

$$\begin{aligned}&\big(J_1(e_1^{(1)}), J_1(e_2^{(1)}), \dots, J_1(e_{N_1}^{(1)}), J_2(e_1^{(2)}), J_2(e_2^{(2)}), \dots, J_2(e_{N_2}^{(2)}), \dots, J_n(e_1^{(n)}),\\ &\quad J_n(e_2^{(n)}), \dots, J_n(e_{N_n}^{(n)}), \dots\big)\end{aligned} \tag{2.3}$$

is an unconditional basis of the Banach space $\mathfrak{F} := (\bigoplus_{n=1}^\infty \mathfrak{F}_n)_{\ell_p}$, where $J_n\colon \mathfrak{F}_n \to \mathfrak{F}$ denotes the canonical nth coordinate embedding for each $n \in \mathbb{N}$. Let $K_n\colon \mathfrak{F} \to \mathfrak{F}_n$ be the canonical nth coordinate projection for each $n \in \mathbb{N}$, and suppose that

$$\sup\{\|L_n\|, \|R_n\| \mid n \in \mathbb{N}\} < \infty. \tag{2.4}$$

Then there are operators

$$L'\colon x \mapsto (L_n K_n x)_{n=1}^\infty, \quad \mathfrak{F} \to \mathfrak{F}, \quad \text{and} \quad R'\colon x \mapsto (R_n K_n x_n)_{n=1}^\infty, \quad \mathfrak{F} \to \mathfrak{F}.$$

Moreover, we can define operators

$$L''\colon x \mapsto \big(\langle f_1^{(n+1)}, K_{n+1}x\rangle e_{N_n}^{(n)}\big)_{n=1}^\infty, \quad \mathfrak{F} \to \mathfrak{F},$$

and

$$R''\colon x \mapsto \big(0, \langle f_{N_1}^{(1)}, K_1 x\rangle e_1^{(2)}, \langle f_{N_2}^{(2)}, K_2 x\rangle e_1^{(3)}, \dots, \langle f_{N_{n-1}}^{(n-1)}, K_{n-1}x\rangle e_1^{(n)}, \dots\big), \quad \mathfrak{F} \to \mathfrak{F},$$

because the norms of the biorthogonal functionals $f_m^{(n)}, n \in \mathbb{N}, 1 \leqslant m \leqslant N_n$, are uniformly bounded by (2.2). Direct calculations show that $L' + L''$ and $R' + R''$ act as left- and right-shift operators on $\mathfrak{F}$ with respect to the basis (2.3). It follows that the conditions in Lemma 2.7 are satisfied, and consequently $\mathcal{B}(\mathfrak{F})$ is infinite and has a continued bisection of the identity.

The discussion above applies in particular to $\mathfrak{F}_n = \ell_{q_n}^{N_n}$, where $1 \leqslant q_n \leqslant \infty$, $N_n \in \mathbb{N}$, and $(e_1^{(n)}, \dots, e_{N_n}^{(n)})$ is the canonical basis of $\ell_{q_n}^{N_n}$ for each $n \in \mathbb{N}$. Indeed, in this case we have that $C_n = 1 = \|L_n\| = \|R_n\|$ (unless $N_n = 1$, in which case $\|L_n\| = \|R_n\| = 0$), and so conditions (2.2) and (2.4) are satisfied. $\square$

Corollary 2.9. *Let $(q_n)_{n=1}^\infty$, p, and $(N_n)_{n=1}^\infty$ be chosen in accordance with Theorem* 2.6, *and define the Banach space $\mathfrak{F}$ by* (2.1). *Then the Banach algebra $\mathcal{B}(\mathfrak{F})$ is infinite and has a continued bisection of the identity, but it is not properly infinite.*

Proof. Proposition 1.9 and Theorem 2.6 show that $\mathcal{B}(\mathfrak{F})$ is not properly infinite. It follows from Example 2.8 that $\mathcal{B}(\mathfrak{F})$ is infinite and has a continued bisection of the identity, as observed by Loy and Willis in [**9**, p. 327]. $\square$

Having an unconditional basis guarantees a wealth of idempotent operators, and so one would expect that, for a Banach space $\mathfrak{X}$ with an unconditional basis, the Banach algebra $\mathscr{B}(\mathfrak{X})$ is infinite and has a continued bisection of the identity. This is, however, not necessarily the case; an impressive construction by Gowers provides a counterexample, as we shall show in Proposition 2.11, below. It follows in particular that the assumption in Lemma 2.7 that the unconditional basis $(e_n)_{n=1}^{\infty}$ be equivalent to $(e_n)_{n=2}^{\infty}$ is not superfluous (although it can certainly be relaxed somewhat).

LEMMA 2.10. *A unital, commutative ring has no continued bisection of the identity.*

Proof. Assume towards a contradiction that $((P_n)_{n=1}^{\infty}, (Q_n)_{n=1}^{\infty})$ is a continued bisection of the identity in a unital, commutative ring $\mathscr{R}$. Since $\mathscr{R}$ is commutative, an element S in $\mathscr{R}$ is invertible if and only if the ideal generated by S is all of $\mathscr{R}$. It follows from Remark 2.2(ii) that P_n and Q_n are invertible for each $n \in \mathbb{N}$. However, the identity is the only element that is both idempotent and invertible, and so $P_n = Q_n = 1_{\mathscr{R}}$ for each $n \in \mathbb{N}$, contradicting Definition 2.1(i). □

PROPOSITION 2.11. *There is a Banach space $\mathfrak{G}$ with the following properties:*

(i) *$\mathfrak{G}$ has an unconditional basis;*
(ii) *the Banach algebra $\mathscr{B}(\mathfrak{G})$ is finite and has no continued bisection of the identity.*

Proof. In [**3**], Gowers constructs a Banach space $\mathfrak{G}$ that has an unconditional basis and which is not isomorphic to any proper subspace of itself. This implies that $\mathscr{B}(\mathfrak{G})$ is finite by Corollary 1.5. The properties of $\mathfrak{G}$ are investigated further by Gowers and Maurey in [**5**, Section (5.1)]. It is a consequence of their work that the commutative Banach algebra ℓ_∞/c_0 is a quotient of $\mathscr{B}(\mathfrak{G})$ (see [**8**, Corollary 8.3] for a proof of this), and so Lemmas 2.4 and 2.10 imply that $\mathscr{B}(\mathfrak{G})$ has no continued bisection of the identity. □

We have no example of a Banach space $\mathfrak{X}$ such that the Banach algebra $\mathscr{B}(\mathfrak{X})$ is finite and has a continued bisection of the identity. In fact, it might seem a natural conjecture that a unital ring with a continued bisection of the identity is necessarily infinite. This conjecture is, however, *false* as we shall now show. The counterexample that we shall exhibit is even a C^*-algebra. For that reason we shall work with complex scalars only for the remainder of this section.

To be specific, a unital, finite C^*-algebra with a continued bisection of the identity is the UHF-algebra $M_{2^\infty}(\mathbb{C})$, also known as the CAR-algebra. By definition, $M_{2^\infty}(\mathbb{C})$ is the inductive limit (in the category of C^*-algebras) of the sequence

$$M_2(\mathbb{C}) \xrightarrow{\varphi_1} M_4(\mathbb{C}) \xrightarrow{\varphi_2} M_8(\mathbb{C}) \xrightarrow{\varphi_3} \cdots \xrightarrow{\varphi_{n-1}} M_{2^n}(\mathbb{C}) \xrightarrow{\varphi_n} M_{2^{n+1}}(\mathbb{C}) \xrightarrow{\varphi_{n+1}} \cdots,$$

where $M_{2^n}(\mathbb{C})$ is the C^*-algebra of complex $(2^n \times 2^n)$-matrices, and where the connecting *-homomorphisms are given by

$$\varphi_n\colon A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad M_{2^n}(\mathbb{C}) \to M_{2^{n+1}}(\mathbb{C}),$$

for each $n \in \mathbb{N}$. It follows from the definition of the inductive limit that there are unital *-monomorphisms $\mu_n\colon M_{2^n}(\mathbb{C}) \to M_{2^\infty}(\mathbb{C})$ such that $\mu_n = \mu_{n+1} \circ \varphi_n$ for each $n \in \mathbb{N}$. For a proof of this and further results about inductive limits and UHF-algebras, we refer to [**12**, Chapters 6–7].

In the realm of C^*-algebras, the concepts of being finite, infinite, and properly infinite are usually defined using projections (that is, *self-adjoint* idempotent elements) rather than general idempotent elements; see for instance [**12**, Exercise 4.6 and Definition 5.1.1]. However, it follows from [**12**, Exercise 3.11(i)–(ii)] that the C^*-algebraic definitions are equivalent to ours.

PROPOSITION 2.12. *The unital C^*-algebra $M_{2^\infty}(\mathbb{C})$ is finite and has a continued bisection of the identity.*

Proof. It is well known that $M_{2^\infty}(\mathbb{C})$ is finite; for example, see [**12**, Exercise 7.5].

For each $n \in \mathbb{N}$, let E_n be the complex $(2^n \times 2^n)$-matrix with a 1 in position (1, 1) and zeros everywhere else, and let F_n be the complex $(2^n \times 2^n)$-matrix with a 1 in position $(2^{n-1} + 1, 2^{n-1} + 1)$ and zeros everywhere else. Then E_n and F_n are projections in $M_{2^n}(\mathbb{C})$ with $E_n \approx F_n$. It follows that $P_n := \mu_n(E_n)$ and $Q_n := \mu_n(F_n)$ are projections in $M_{2^\infty}(\mathbb{C})$ with $P_n \approx Q_n$. In particular, P_n and Q_n generate the same ideal in $M_{2^\infty}(\mathbb{C})$ by Remark 1.2(ii). Direct calculations show that

$$P_{n+1} + Q_{n+1} = \mu_{n+1}(E_{n+1} + F_{n+1}) = (\mu_{n+1} \circ \varphi_n)(E_n) = \mu_n(E_n) = P_n \quad (n \in \mathbb{N})$$

and

$$P_1 + Q_1 = \mu_1(E_1 + F_1) = \mu_1\big(1_{M_2(\mathbb{C})}\big) = 1_{M_{2^\infty}(\mathbb{C})},$$

and so $((P_n)_{n=1}^\infty, (Q_n)_{n=1}^\infty)$ is a continued bisection of the identity in $M_{2^\infty}(\mathbb{C})$. □

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