BULL. AUSTRAL. MATH. SOC. VOL. 68 (2003) [325–328]
22D15, 43A10, 43A20, 43A60, 46E15, 46H25, 46M20, 47B47

CONNES-AMENABILITY AND NORMAL, VIRTUAL DIAGONALS FOR MEASURE ALGEBRAS, II

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We prove that the following are equivalent for a locally compact group G:

(i) G is amenable;

(ii) $M(G)$ is Connes-amenable;

(iii) $M(G)$ has a normal, virtual diagonal.

THE RESULT

A Banach algebra $\mathfrak{A}$ is called *dual* if it is a dual space such that multiplication in $\mathfrak{A}$ is separately w^*-continuous. A Banach bimodule over a dual Banach algebra is called *normal* if it is a dual space such that the module operations are separately w^*-continuous (see [**6, 7, 8**]).

DEFINITION 1: A dual Banach algebra $\mathfrak{A}$ is called *Connes-amenable* if, for every normal Banach $\mathfrak{A}$-bimodule E, every w^*-continuous derivation $D\colon \mathfrak{A} \to E$ is inner.

The notion of Connes-amenability was introduced for von Neumann algebras in [**5**] (the name "Connes-amenability" seems to originate from [**3**]); it is equivalent to a number of important von Neumann algebraic properties such as injectivity, semidiscreteness, and being approximately finite-dimensional (see [**7**, Chapter 6] for a self-contained exposition and references to the original literature).

For arbitrary dual Banach algebras, Connes-amenability was first considered in [**6**], and in [**8**] it was shown that $M(G)$, the measure algebra of a locally compact group G, is Connes-amenable if and only if G is amenable.

For any dual Banach algebra $\mathfrak{A}$, let $\mathcal{L}^2_{w^*}(\mathfrak{A}, \mathbb{C})$ denote the separately w^*-continuous bilinear functionals on $\mathfrak{A}$. It is easy to see that the multiplication map $\Delta: \mathfrak{A} \otimes \mathfrak{A} \to \mathfrak{A}$ extends to $\mathcal{L}^2_{w^*}(\mathfrak{A}, \mathbb{C})^*$ as a continuous $\mathfrak{A}$-bimodule homomorphism.

DEFINITION 2: Let $\mathfrak{A}$ be a dual Banach algebra. A *normal, virtual diagonal* for $\mathfrak{A}$ is an element $\mathrm{M} \in \mathcal{L}^2_{w^*}(\mathfrak{A}, \mathbb{C})^*$ such that

$$a \cdot \mathrm{M} = \mathrm{M} \cdot a \quad \text{and} \quad a\Delta\mathrm{M} = a \qquad (a \in \mathfrak{A}).$$

Received 18th March, 2003
Research supported by NSERC under grant no. 227043-00.

If $\mathfrak{A}$ has a normal, virtual diagonal, then it is Connes-amenable ([**1**, **2**]). The converse holds if $\mathfrak{A}$ is a von Neumann algebra ([**2**]). It is an open question — likely with a negative answer — if Connes-amenability and the existence of normal, virtual diagonals are equivalent for arbitrary dual Banach algebras.

In [**6**] and [**8**], we gave partial positive answers for $\mathfrak{A} = M(G)$:

1. If G is compact, then $M(G)$ has a normal, virtual diagonal ([**6**, Proposition 5.2] and [**8**, Proposition 3.3]).
2. If G is discrete, then $M(G) = \ell^1(G)$ is Connes-amenable if and only if it has a normal, virtual diagonal ([**8**, Corollary 5.4]).

In this note, we shall prove the following theorem, and thus extend [**8**, Theorem 5.3].

Theorem 1. *The following are equivalent for a locally compact group G:*

(i) *G is amenable.*

(ii) *$M(G)$ is Connes-amenable.*

(iii) *$M(G)$ has a normal, virtual diagonal.*

The proof

For convenience, we quote the following well-known characterisation of amenable locally compact groups (see [**7**, Lemma 7.1.1], for example).

Lemma 1. *A locally compact group G is amenable if and only if there is a net $(f_\alpha)_\alpha$ of non-negative functions in the unit sphere of $L^1(G)$ such that*

$$(*) \qquad \sup_{x \in K} \|\delta_x * f_\alpha - f_\alpha\| \to 0$$

for each compact subset K of G.

Proof of the Theorem: In view of [**8**, Theorem 5.3], it is sufficient to show that (i) implies (iii).

Let $(f_\alpha)_\alpha$ be a net as specified in the lemma. Define a net $(m_\alpha)_\alpha$ in $M(G \times G)$ by letting

$$\langle f, m_\alpha \rangle := \int_G f(x, x^{-1}) f_\alpha(x)\, dx \qquad (f \in \mathcal{C}_0(G \times G)),$$

where dx denotes integration with respect to left Haar measure on G. Let $y \in G$, and note that, for $f \in \mathcal{C}_0(G \times G)$,

$$\begin{aligned} \langle f, (\delta_y \otimes \delta_e) * m_\alpha \rangle &= \int_G f(yx, x^{-1}) f_\alpha(x)\, dx \\ &= \int_G f(x, x^{-1}y) f_\alpha(y^{-1}x)\, dx \qquad (\text{substitute } y^{-1}x \text{ for } x) \\ &= \int_G f(x, x^{-1}y)(\delta_y * f_\alpha)(x)\, dx \end{aligned}$$

https://doi.org/10.1017/S0004972700037709 Published online by Cambridge University Press

and

$$\langle f, m_\alpha * (\delta_e \otimes \delta_y)\rangle = \int_G f(x, x^{-1}y) f_\alpha(x)\, dx.$$

It follows from $(*)$ that

$$(**) \qquad \sup_{y \in K} \left\| (\delta_y \otimes \delta_e) * m_\alpha - m_\alpha * (\delta_e \otimes \delta_y) \right\| \to 0$$

for each compact subset K of G.

By [8, Proposition 3.1], we may identify the Banach $M(G)$-bimodules $\mathcal{L}^2_{w^*}(M(G), \mathbb{C})$ and

$$\mathcal{SC}_0(G \times G) := \left\{ f \in \ell^\infty(G \times G) : f(\cdot, x), f(x, \cdot) \in \mathcal{C}_0(G) \text{ for each } x \in G \right\}.$$

Let $\mathcal{U}$ be an ultrafilter on the index set of $(m_\alpha)_\alpha$ that dominates the order filter. Define $\mathrm{M} \in \mathcal{SC}_0(G \times G)^*$ by letting

$$\langle f, \mathrm{M}\rangle := \lim_{\mathcal{U}} \int_{G \times G} f(x, y)\, dm_\alpha(x, y) \qquad \left(f \in \mathcal{SC}_0(G \times G)\right)$$

(since all functions in $\mathcal{SC}_0(G \times G)$ are measurable with respect to any Borel measure by [4], the integrals do exist). It is routinely seen that $\Delta\mathrm{M} = \delta_e$.

Let $\mu \in M(G)$ and let $f \in \mathcal{SC}_0(G \times G)$. Then we have:

$$\begin{aligned}
&\left|\langle f, \mu \cdot \mathrm{M} - \mathrm{M} \cdot \mu\rangle\right| \\
&\quad = \left|\langle f \cdot \mu - \mu \cdot f, \mathrm{M}\rangle\right| \\
&\quad = \left|\lim_{\mathcal{U}} \int_{G \times G} \left(\int_G \left(f(zx, y) - f(x, yz)\right) d\mu(z)\right) dm_\alpha(x, y)\right| \\
&\quad = \left|\lim_{\mathcal{U}} \int_G \left(\int_{G \times G} \left(f(zx, y) - f(x, yz)\right) dm_\alpha(x, y)\right) d\mu(z)\right| \qquad \text{(by Fubini's theorem)} \\
&\quad \leqslant \lim_{\mathcal{U}} \int_G \left|\int_{G \times G} \left(f(zx, y) - f(x, yz)\right) dm_\alpha(x, y)\right| d|\mu|(z) \\
&\quad = \lim_{\mathcal{U}} \int_G \left|\int_{G \times G} f(x, y)\, d\left((\delta_z \otimes \delta_e) * m_\alpha - m_\alpha * (\delta_e \otimes \delta_z)\right)(x, y)\right| d|\mu|(z) \\
&\quad \leqslant \lim_{\mathcal{U}} \int_G \|f\| \left\|(\delta_z \otimes \delta_e) * m_\alpha - m_\alpha * (\delta_e \otimes \delta_z)\right\| d|\mu|(z) \\
&\quad \to 0 \qquad \text{(by } (**) \text{ and the inner regularity of } |\mu|).
\end{aligned}$$

It follows that M is a normal, virtual diagonal for $M(G)$. ∎

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