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On convex lattice polygons

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Let Π be a convex lattice polygon with b boundary points and c (≥ 1) interior points. We show that for any given c, the number b satisfies $b \leq 2c + 7$, and identify the polygons for which equality holds.

A *lattice polygon* Π is a simple polygon whose vertices are points of the integral lattice. We let $A = A(\Pi)$ denote the area of Π, $b(\Pi)$ the number of lattice points on the boundary of Π, and $c(\Pi)$ the number of lattice points interior to Π.

In 1899, Pick [2] proved that

$$A(\Pi) = \tfrac{1}{2}b(\Pi) + c(\Pi) - 1 .$$

Nosarzewska [1] and more recently Wills [4], have established inequalities relating the area, perimeter, and number of interior points of a convex lattice polygon. It is our purpose here to establish a simple necessary condition for Π to be convex.

We set $f(\Pi) = b(\Pi) - 2c(\Pi)$. Using Pick's formula we can obtain alternative expressions for $f(\Pi)$:

$$\tfrac{1}{2}f(\Pi) = b(\Pi) - A(\Pi) - 1$$

and

$$\tfrac{1}{2}f(\Pi) = A(\Pi) - 2c(\Pi) + 1 .$$

Lattice polygons which can be obtained from one another using integral unimodular transformations or translations are said to be *equivalent*. The property of convexity, and the quantities A, b, c, and f are easily seen to be invariant under equivalence.

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The illustrated triangle, Δ (Figure 1) is a lattice polygon of special interest. We observe that

$$A(\Delta) = \frac{9}{2}\ , \quad b(\Delta) = 9\ , \quad c(\Delta) = 1\ ,$$

and

$$f(\Delta) = 7\ .$$

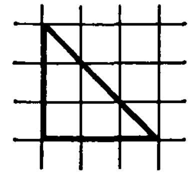

FIGURE 1

THEOREM. *Let* Π *be a convex lattice polygon with at least one interior point. If* Π *is equivalent to* Δ *, then* $f(\Pi) = 7$ *. Otherwise* $f(\Pi) \le 6$.

In the proof of this theorem, we shall make use of the following lemma.

LEMMA. *Let* AB, CD *be segments lying along the* x*-axis, having integral endpoints, and lengths* h, k *respectively. Let* p *be a positive integer such that* $p > h + k$ *. Then there exist points* P, R *on* AB, CD *respectively having integral coordinates, and such that distance* PR *satisfies*

$$PR = mp + u \quad (m \text{ a non-negative integer}) \text{ where } |u| \le \tfrac{1}{2}(p-h-k)\ .$$

Proof. Let AB be the segment $[0, h]$, and let $A'B'$ be the setment $[p, p+h]$ obtained by translating AB through p . We may translate CD through integral multiples of p to the position $[t, t+k]$, where $0 \le t < p$. In fact, we may assume that $h < t < p - k$, else CD overlaps one of the segments AB, $A'B'$, and we have our result with $u = 0$.

Hence we may assume that points A, B, C, D, A', B' lie in this order along the x-axis. Let $BC = x$, $DA' = y$. Then

$$(BA' =)\ p - h = x + k + y\ ;$$

that is,

$$x + y = p - h - k\ .$$

Clearly it is impossible for both x and y to be greater than $\tfrac{1}{2}(p-h-k)$, and the result follows.

Proof of the theorem. Let Π meet supporting lines $y = 0$, $y = p$ in segments of length h, k (possibly zero) respectively (Figure 2).

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Since Π contains interior points, $p \geq 2$.

Because Π is convex, each horizontal line between $y = 0$ and $y = p$ cuts the boundary of Π in two points. We deduce that

$$b(\Pi) \leq h + k + 2p .$$

We now distinguish between several different cases.

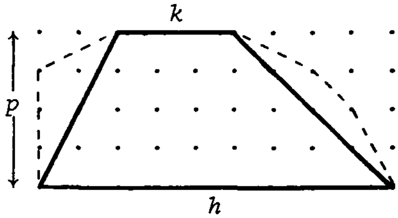

FIGURE 2

Case 1. $p = 2$, or $h + k \geq 4$, or $p = h + k = 3$. Since Π is convex, Π contains the convex hull of the two given segments. Hence

$$A(\Pi) \geq \tfrac{1}{2}p(h+k)$$

and

$$\begin{aligned} f(\Pi) &= 2b(\Pi) - 2A(\Pi) - 2 \\ &\leq 2(h+k+2p) - p(h+k) - 2 \\ &= (h+k-4)(2-p) + 6 \\ &\leq 7 . \end{aligned}$$

Case 2. $p = 3$ and $h + k \leq 2$. Now $b(\Pi) \leq h + k + 2p \leq 8$, and since $c(\Pi) \geq 1$, $f(\Pi) = b(\Pi) - 2c(\Pi) \leq 6$.

Case 3. $p \geq 4$ and $h + k \leq 3$. Let Π meet supporting lines $y = 0$, $y = p$ in points P, R respectively, and supporting lines $x = 0$, $x = p'$ $(p' \geq p)$ in points Q, S respectively.

As before, $b(\Pi) \leq h + k + 2p$. Consider now the effect of transforming Π using an integral, unimodular shear having the x-axis as invariant line. This transformation leaves $A(\Pi)$, $b(\Pi)$, p, $h + k$ unchanged, and preserves the convexity of Π . It may decrease p' to a value less than p ; if this happens, we simply interchange the roles of p and p' . (There can be at most a finite number of such interchanges, since at each step the positive integer $p + p'$ is reduced by at least one.) A further effect of this shear is that all points on the line $y = p$ are translated through some multiple of p . Hence by the lemma, it is possible to shear Π and choose the points P, R so that the x-coordinates of these points differ by u , where

$$0 \leq u \leq \tfrac{1}{2}(p-h-k) .$$

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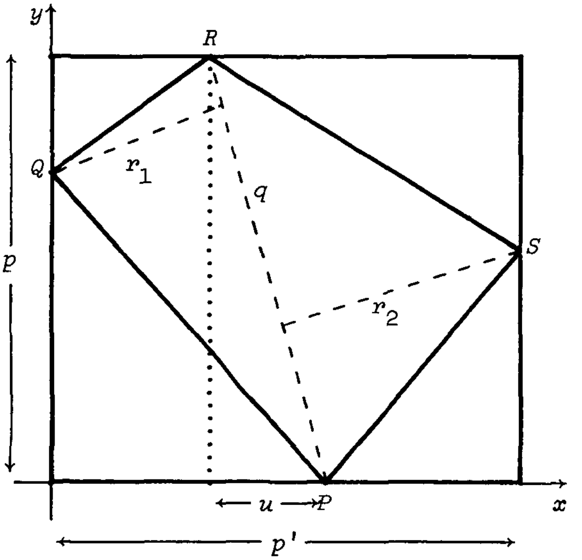

FIGURE 3

Now since Π is convex,

$$\begin{aligned} A(\Pi) &\ge A(PQRS) \\ &= \tfrac{1}{2}q\left(r_1+r_2\right) \quad \text{(see Figure 3)} \\ &\ge \tfrac{1}{2}p(p'-u) \\ &\ge \tfrac{1}{2}p(p-u) \quad \text{since } p' \ge p \\ &\ge \tfrac{1}{4}p(p+h+k)\ , \end{aligned}$$

substituting the upper bound for u. Hence

$$\begin{aligned} f(\Pi) &= 2b(\Pi) - 2A(\Pi) - 2 \\ &\le 2(h+k+2p) - \tfrac{1}{2}p(p+h+k) - 2 \\ &= \tfrac{1}{2}(h+k)(4-p) + \tfrac{1}{2}p(8-p) - 2 \\ &\le 6 \end{aligned}$$

since $p \ge 4$ and $p(8-p)$ assumes its maximum value of 8 for $p = 4$.

Hence in all cases $f(\Pi) \le 7$. For equality here we require $p = 3$, $h + k = 3$, $b(\Pi) = 9$, and $A(\Pi) = \frac{9}{2}$; it is easily verified that Π is equivalent to Δ. The lower value $f(\Pi) = 6$ is attained for a number of lattice polygons Π, for example lattice rectangles with $p = 2$.

This completes the proof of the theorem.

Finally, we observe that if $c(\Pi) = 0$, then $f(\Pi)$ is unbounded. This is illustrated by the triangle with vertices $(0, 1)$, $(1, 1)$, and $(n, 0)$ (n integral), for which $f(\Pi) = n + 1$.

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