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INFECTION SPREAD IN RANDOM GEOMETRIC GRAPHS

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Abstract

In this paper we study the speed of infection spread and the survival of the contact process in the random geometric graph $G = G(n, r_n, f)$ of n nodes independently distributed in $S = [-\frac{1}{2}, \frac{1}{2}]^2$ according to a certain density $f(\cdot)$. In the first part of the paper we assume that infection spreads from one node to another at unit rate and that infected nodes stay in the same state forever. We provide an explicit lower bound on the speed of infection spread and prove that infection spreads in G with speed at least $D_1 n r_n^2$. In the second part of the paper we consider the contact process ξ_t on G where infection spreads at rate $\lambda > 0$ from one node to another and each node independently recovers at unit rate. We prove that, for every $\lambda > 0$, with high probability, the contact process on G survives for an exponentially long time; there exist positive constants c_1 and c_2 such that, with probability at least $1 - c_1/n^4$, the contact process ξ_t^1 starting with all nodes infected survives up to time $t_n = \exp(c_2 n/\log n)$ for all n.

Keywords: Random geometric graph; speed of infection spread; survival time of contact process

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1. Introduction

Consider n nodes independently distributed in the unit square $S = [-\frac{1}{2}, \frac{1}{2}]^2$ according to a certain density f satisfying

$$0 < \inf_{x \in S} f(x) \leq \sup_{x \in S} f(x) < \infty. \tag{1.1}$$

Connect two nodes u and v by an edge e if the Euclidean distance $d(u, v)$ between them is less than r_n. The resulting random geometric graph (RGG) is denoted by $G = G(n, r_n, f)$. The giant component regime ($nr_n^2 = c_1$) and connectivity regime ($nr_n^2 = c_2 \log n$) of RGGs and their applications have been studied in [3], [5], [9], and [10]. In [4] we studied the size of the giant component in the regime $nr_n^2 \to \infty$. In what follows, we assume that $nr_n^2 \to \infty$ as $n \to \infty$ and

$$c_1 \leq nr_n^2 \leq c_2 \log n \tag{1.2}$$

for some positive constants c_1 and c_2. Also, all constants mentioned throughout the paper are independent of n. We mention here that the techniques in [4] could also be used in analyzing other random graphs. In [11], we also estimate the giant component size of Erdős–Rényi graphs using analogous methods.

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1.1. Infection spread in G

In this section we study infection spread in the graph G described above. For each edge e in G, let $t(e)$ denote its associated (random) passage time. The random variables $\{t(e)\}_{\mathrm{e}}$ are independent random variables, each exponentially distributed with unit mean [6]. At time $t = 0$, the node x_0 closest to the origin in S is infected. Any node x_1 that shares an edge e with x_0 is infected after time $t(e)$ and these infected nodes stay in that state forever. What is the minimum time elapsed after which no new nodes are infected? How many nodes are ultimately infected by the above process? In this section we provide sharp bounds for these two questions. The main tool we use to describe our results is the speed of infection spread. We remark that in a more general setup, passage times are not necessarily identically distributed. In [12], we have also studied first passage percolation with nonidentical passage times.

Following an analogous construction as in Chapter 1 of [7], we define the infection process on the probability space $(\Theta, \mathcal{H}, \mathbb{P})$ (see Section 2 for further details). For a reasonable definition of the speed of infection spread, we would first like to ensure that x_0 is close to the origin and an infection starting from x_0 reaches close to the boundary of S. We say that $\Gamma(x_0)$ occurs if $d(0, x_0) \le r_n/2$ and there exists a path of edges $(e_1, \dots, e_f)$ in G such that

(i) e_1 contains x_0 as one of its endvertices,

(ii) $d(e_f, \partial S) \le r_n/2$.

Here and henceforth we adopt the following notation. For $x, y \in \mathbb{R}^2$, we let $d(x, y)$ be the Euclidean distance between x and y. For measurable sets $A, B \in \mathbb{R}^2$, we define $d(A, B) := \inf\{d(a, b) \colon a \in A,\ b \in B\}$. Finally, ∂A refers to the boundary of A. The following result ensures that $\Gamma(x_0)$ occurs with high probability.

Proposition 1.1. *There exists a constant* $\theta > 0$ *such that*

$$\mathbb{P}(\Gamma(x_0)) \ge 1 - \mathrm{e}^{-\theta n r_n^2} \tag{1.3}$$

for all sufficiently large n.

We provide the proof in Section 2.

For any set $A \subseteq \mathbb{R}^2$ and $\alpha > 0$, define $\alpha A = \bigcup_{x \in A}\{\alpha x\}$ to be the dilation of A by the factor α. Let $G(x_0)$ denote the connected cluster of nodes in G containing x_0, and let $I(t)$ be the set of nodes of $G(x_0)$ infected up to time t. We say that the infection spreads at a speed of at least $v_{n,\mathrm{low}}$ if there exists $1 \le a_n = o(r_n^{-1})$ and $0 \le b_n = o(r_n^{-1})$ such that

$$\mathbb{P}\bigg(\cap_{a_n \le m \le r_n^{-1} - b_n}\bigg\{\bigg(G(x_0) \setminus I\bigg(\frac{m}{v_{n,\mathrm{low}}}\bigg)\bigg) \cap m r_n S = \phi\bigg\} \,\bigg|\, \Gamma(x_0)\bigg) = 1 - o(1),$$

where the index m runs through integers between 1 and r_n^{-1} in the intersection. In other words, conditioned on the occurrence of $\Gamma(x_0)$, we want all nodes of $G(x_0)$ contained in $m r_n S$ to be infected within time $m/v_{n,\mathrm{low}}$. This must happen for 'nearly all' indices m, i.e. for all indices between 1 and r_n^{-1} excepting possibly $o(r_n^{-1})$ indices. Unless mentioned otherwise, we use the standard terminology $o(\cdot)$ and $O(\cdot)$ in the regime $n \to \infty$. Analogously, we say that the speed is at most $v_{n,\mathrm{up}}$ if there exists $1 \le c_n = o(r_n^{-1})$ and $0 \le d_n = o(r_n^{-1})$ such that

$$\mathbb{P}\bigg(\cap_{c_n \le m \le r_n^{-1} - d_n}\bigg\{I\bigg(\frac{m}{v_{n,\mathrm{up}}}\bigg) \subseteq m r_n S\bigg\} \,\bigg|\, \Gamma(x_0)\bigg) = 1 - o(1),$$

where, as before, the index m runs through integers between 1 and r_n^{-1} in the intersection. The following is the main result of this section.

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Theorem 1.1. *There exist positive constants* D_1 *and* D_2 *such that*

$$D_1 n r_n^2 \le v_{n,\text{low}} \le v_{n,\text{up}} \le D_2 n\sqrt{n}\log n.$$

Since RGGs satisfying (1.2) are dense graphs, we expect the speed of infection spread to grow with n. In the above result, we provide an explicit lower bound on the speed.

Let T_{elap} denote the time taken to infect all nodes of $G(x_0)$, and let $N_{\text{inf}} = \#G(x_0)$ denote the number of nodes that remain infected in S after time T_{elap}. We have the following corollary regarding T_{elap} and N_{inf}.

Corollary 1.1. *We have*

$$\mathbb{P}\bigg(\frac{r_n^{-1}}{D_3 n\sqrt{n}\log n} \le T_{\text{elap}} \le \frac{r_n^{-1}}{D_4 n r_n^2}\bigg) = 1 - o(1) \tag{1.4}$$

as $n \to \infty$ *and*

$$\frac{r_n^{-1}}{D_3 n\sqrt{n}\log n} \le \mathbb{E}T_{\text{elap}} \le \frac{r_n^{-1}}{D_4 n r_n^2} \tag{1.5}$$

for some positive constants D_3 *and* D_4 *and all sufficiently large* n*. Also,*

$$\mathbb{P}(N_{\text{inf}} \ge n - n\mathrm{e}^{-\theta n r_n^2}) = 1 - o(1) \tag{1.6}$$

for some positive constant θ*, as* $n \to \infty$*.*

Thus, with high probability, infection starting from the node closest to the origin eventually spreads to nearly all nodes.

The study of infection spread in RGGs is important in practical applications. For example, in wireless networks an RGG is used to model the underlying connectivity structure (see, e.g. [5]) and is necessary to estimate the time taken for a virus to spread through the network. Since RGGs are dense graphs, traditional subadditive techniques developed for first passage percolation in regular graphs like $\mathbb{Z}^2$ (see, e.g. [13]) are not directly applicable here. In our analysis, we manually construct backbones to trace the infection spread (see Section 3 for details).

1.2. Contact process on *G*

The contact process is a generalization of infection spread where nodes are capable of recovering. In this subsection we assume that nodes independently recover at unit rate and become infected at rate λ multiplied by the number of infected neighbours. We denote ξ_t^1 to be the contact process on G starting with all nodes infected. For a fixed configuration of nodes, the process ξ_t^1 is a Markov chain whose state space is the set of all subsets of $\{1, \dots, n\}$ and whose transition rates (or Q-matrix) are given by $q(A, A \setminus \{x\}) = 1$ if $x \in A$ and $q(A, A \cup \{x\}) = \lambda\#(A \cap N_G(x))$ if $x \notin A$. Here A is a subset of $\{1, \dots, n\}$ and $N_G(x)$ denotes the set of neighbours of the node x in G.

Since the graph G is finite, the process ξ_t^1 dies out almost surely [6]. Let τ_n denote the extinction time for ξ_t^1, i.e.

$$\tau_n = \inf\{t \ge 0 \colon \xi_t^1 = \phi\}.$$

Again, following a construction analogous to [7], we define the contact process on the random graph G in the probability space $(\Theta, \mathcal{H}, \mathbb{P})$. We state the main result of this subsection.

Theorem 1.2. *Fix* $\lambda > 0$. *There exist positive constants* $\delta_i = \delta_i(\lambda)$, $i = 1, 2, 3$, *such that*

$$\mathbb{P}\bigg(\exp\bigg(\delta_1 n \frac{nr_n^2}{\log n}\bigg) \le \tau_n \le \exp(\delta_2 n \log n)\bigg) \ge 1 - \frac{\delta_3}{n^4} \quad \textit{for all } n \ge 1.$$

For every $\lambda > 0$, with high probability, the contact process survives for a nearly exponential time because of (1.2). In this sense, we could say that the critical value of the contact process on G is 0.

The study of the contact process on RGGs is important in practical applications. For example, as mentioned before, RGGs are often used to model wireless networks (see, e.g. [5]) and in such cases, it is required to know the time for which a virus or malware persists in the presence of recovery agents. Contact processes on finite subsets of regular graphs like $\mathbb{Z}^d$ have been studied before (see [6] for a survey). The survival of contact processes on finite random graphs containing n nodes with power law degree distribution has been studied in [1] and [8]. The proofs therein exploit the fact that, with high probability, there are a sufficiently large number of nodes having degree at least n^{ε} for some constant $\varepsilon > 0$. In contrast, RGGs are 'mildly dense' in the sense that the degree of every node does not exceed $O(\log n)$ with high probability. Consequently, we construct a long path of sufficiently 'dense' squares containing a total of αn nodes for some constant $\alpha > 0$ to study contact processes on RGGs.

The paper is organized as follows. In Section 2 we state and prove the geometric results regarding RGGs that are needed for the analysis of infection spread. In Section 3 we prove the lower and upper bounds on the speed from Theorem 1.1. In Section 4 we prove Corollary 1.1. Finally, in Section 5, we prove Theorem 1.2.

2. Preliminary result for infection spread

We briefly describe the probability space in a little more detail. We define the point process on the probability space $(\Omega, \mathcal{F}, \mu)$. Let $\{Z_i\}_{i\ge1}$ be a countable set of independent random variables exponentially distributed with unit mean, defined on the probability space $(\Xi, \mathcal{G}, \nu)$. Following a construction analogous to Chapter 1 of [7] we define the infection process on the probability space $(\Theta, \mathcal{H}, \mathbb{P})$, where $\Theta = \Omega \times \Xi$, $\mathcal{H} = \mathcal{F} \times \mathcal{G}$, and $\mathbb{P} = \mu \times \nu$. For any event $A \in \mathcal{H}$, we have

$$\mathbb{P}(A) = \int_{\Omega} \mathbb{P}_{\omega}(A)\mu(d\omega), \tag{2.1}$$

where $\mathbb{P}_{\omega}(A) := \nu(\{\xi \in \Xi \colon (\omega, \xi) \in A\})$ is the probability that A occurs for a fixed configuration of points ω.

We now prove Proposition 1.1. The proof is analogous to Lemma 3 of [4] and we provide a brief sketch here. Divide S into small $r_n/\Delta \times r_n/\Delta$ squares $\{S_k\}_{k\ge1}$, where $\Delta = \Delta_n \in [4, 5]$ is such that Δ/r_n is an integer. We choose Δ such that the nodes in adjacent squares are joined together by an edge. Throughout, a rectangle of size $a \times b$ is a translate of $\{0 \le x \le a,\ 0 \le y \le b\}$. We repeatedly use the term denseness of squares for the following concept. For a fixed i, let $10\sigma_i$ be the mean number of nodes in the $r_n/\Delta \times r_n/\Delta$ square S_i. Using (1.1) and $\Delta \in [4, 5]$, we have

$$10\beta_1 nr_n^2 \le 10\sigma_i \le 10\beta_2 nr_n^2, \tag{2.2}$$

where $\beta_1 = \frac{1}{25}\inf_{x\in S} f(x) > 0$ and $\beta_2 = \frac{1}{16}\sup_{x\in S} f(x) < \infty$. Define S_i to be *dense* if it has at least $\lceil\sigma_i\rceil$ nodes and *sparse* otherwise. Here and henceforth, $\lceil x\rceil$ refers to the smallest integer greater than or equal to x. We assume that $\beta_1 nr_n^2$ and $\beta_2 nr_n^2$ are integers; otherwise, the argument below holds by considering $\lceil\beta_i nr_n^2\rceil$ for $i = 1, 2$.

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Proof of Proposition 1.1. A set of squares $\mathcal{C} \subseteq \{S_i\}_i$ is said to be a cluster if the squares form a connected set in $\mathbb{R}^2$. Define $\mathcal{C}$ to be dense if each square in $\mathcal{C}$ is dense.

Let S_{or} denote the $r_n/\Delta \times r_n/\Delta$ square containing the origin. Using standard binomial distribution estimates (see Chapter 1 of [9]) and (2.2), we have

$$\mathbb{P}(S_{\mathrm{or}} \text{ is dense}) \geq 1 - \mathrm{e}^{-\theta_2 n r_n^2} \tag{2.3}$$

for some constant $\theta_2 > 0$ and all sufficiently large n. Let $\Gamma_0(x_0)$ denote the event that there is a path $L_1 = (A_1, \dots, A_t)$ of distinct $r_n/\Delta \times r_n/\Delta$ dense squares such that $A_1 = S_{\mathrm{or}}$ and A_t intersects the boundary of S. Here A_i and A_{i+1} share a corner for every i. Since $\Delta \geq 4$, every node in A_i is joined to every node in A_{i+1} and, hence, $\Gamma_0(x_0) \subseteq \Gamma(x_0)$. Suppose that S_{or} is dense, and denote by $\mathcal{C}_{\mathrm{or}}$ the maximal dense cluster containing S_{or}. If S_{or} is dense and $\Gamma_0(x_0)$ does not occur then $\mathcal{C}_{\mathrm{or}}$ must necessarily be surrounded by a circuit of sparse squares contained in S. Such an event must be very unlikely because sparse squares occur with probability at most $\mathrm{e}^{-\theta_2 n r_n^2}$.

Indeed, letting $A = \Gamma_0^c(x_0) \cap \{S_{\mathrm{or}} \text{ is dense}\}$ and applying a contour argument analysis analogous to that used in the proof of Lemma 3 of [4], it follows that, for integer $k \geq 1$, $\mathbb{P}(\{\#\mathcal{C}_{\mathrm{or}} = k\} \cap A) \leq k\mathrm{e}^{-2\theta_0 n r_n^2 \sqrt{k}}$ for a fixed positive constant θ_0 and all $n \geq N_0$, where N_0 is a constant that does not depend on k. Summing over k, we find that $\mathbb{P}(A) \leq \sum_{k\geq 1} k\mathrm{e}^{-2\theta_0 n r_n^2 \sqrt{k}} \leq \mathrm{e}^{-\theta_0 n r_n^2}$ for all sufficiently large n. From the estimate (2.3), we then obtain (1.3). This completes the proof of the proposition.

To estimate the time taken for the infection to cross the boundary of $m r_n S$, as m runs through 1 to r_n^{-1}, we need to find paths whose edges have low passage time. Oriented left–right crossings described below are useful in that respect. Let $K_n = (\log n)/n r_n^2$. For positive integers m and M, let R_1 be an $m r_n/\Delta \times M K_n r_n/\Delta$ rectangle containing exactly $m M K_n$ squares from $\{S_k\}_k$. Without loss of generality we allow K_n to be an integer throughout and the argument presented below holds; otherwise, $K_n = \lceil (\log n)/n r_n^2 \rceil$. We define an oriented left–right crossing in R to be any sequence of distinct rectangles $L = (Y_1, Y_2, \dots, Y_t)$ such that $\{Y_i\}_i \subset \{S_k\}_k$ and the following conditions hold.

(a) For every i, the rectangles Y_i and Y_{i+1} share only a corner.

(b) For every i, the x-coordinate of the centre of Y_{i+1} is larger than that of Y_i.

(c) Y_1 intersects the left side of R and Y_t intersects the right side.

If every square in a left–right crossing L is dense, we define L to be a dense left–right crossing. Here, our definition for oriented left–right crossings is slightly different from the definition for (unoriented) left–right crossings in [4]. The concept of left–right crossings, with varying definitions in different contexts, was used in [3], [9], and [10]. Let $E_n(R_1)$ denote the event that R_1 has a dense oriented left–right crossing. We have the following result.

Proposition 2.1. *There exist positive constants C_1 and M such that, for all sufficiently large n and $n^{1/9} \leq m \leq \Delta/r_n$, we have*

$$\mathbb{P}(E_n(R_1)) \geq 1 - \frac{C_1}{n^9}.$$

Proof. To prove the result, we employ Poissonization and assume that the nodes are distributed according to a Poisson process with an intensity function $n f(\cdot)$. Defining $\mathbb{P}_o$ to be

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the probability measure under the Poissonized system, we prove that $E_n(R_1)$ occurs with $\mathbb{P}_o$-probability at least $1 - 1/n^{10}$. We then translate to the original probability measure $\mathbb{P}$ using

$$\mathbb{P}(E_n(R_1)) \geq 1 - C_1\sqrt{n}(1 - \mathbb{P}_o(E_n(R_1)) \tag{2.4}$$

for some absolute constant C_1, to prove the proposition. To prove (2.4), we note that in the Poisson case the number of nodes N in the unit square S is a Poisson random variable with mean n. Since $\mathbb{P}_o(A^c) \geq \mathbb{P}(A^c)\,\mathrm{Pr}(N = n)$ and $\mathrm{Pr}(N = n) = \mathrm{e}^{-n}n^n/n! \geq C_1/\sqrt{n}$ for some positive constant C_1 (by Stirling's formula), we obtain (2.4).

For the rest of this proof, we work in the Poissonized system. Our first step is to translate the problem to $\mathbb{Z}^2$. We identify each $r_n/\Delta \times r_n/\Delta$ square S_i with a vertex $z_i \in \mathbb{Z}^2$ in the natural way such that the rectangle R_1 corresponds to an $m \times MK_n$ rectangle R_1^{int} in $\mathbb{Z}^2$. We construct an oriented percolation model on R_1^{int} in the following way. We draw an arrow from z_i to z_j if the corresponding squares S_i and S_j share exactly one corner and both are dense. From the estimate for dense squares in (2.3) we find that an oriented arrow occurs with probability at least $1 - \mathrm{e}^{-2\theta_1 n r_n^2}$ for some constant $\theta_1 > 0$. Let $\mathbb{P}_{\mathrm{or}}$ denote the measure corresponding to the oriented percolation model, and let $E_n(R_1^{\mathrm{int}})$ denote the event that R_1^{int} contains an oriented left–right crossing. Following a contour argument as in [2], we have

$$\mathbb{P}_{\mathrm{or}}(E_n(R_1^{\mathrm{int}})) \geq 1 - m(\mathrm{e}^{-\theta_1 n r_n^2})^{MK_n} \geq 1 - \frac{1}{n^{10}}$$

if $M \geq 1$ is sufficiently large (for further details, we refer the reader to the proof of Proposition 5.2 below where an analogous analysis is performed). If an oriented left–right crossing occurs in R_1^{int} then there is a oriented dense left–right crossing in R_1. Thus, $\mathbb{P}_o(E_n(R_1)) \geq 1 - 1/n^{10}$ for all sufficiently large n and from (2.4) we get the proposition.

3. Proof of Theorem 1.1

We first prove the lower bound on the speed. We choose $a_n = n^{1/9}$ to be the starting index from which we trace the infection spread (see the definition prior to Theorem 1.1). This suffices since $n^{1/9} = o(r_n^{-1})$ by (1.2). (In fact, any $\alpha < \frac{1}{2}$ suffices since $n^\alpha = o(r_n^{-1})$ by (1.2).)

Fix integer $n^{1/9} \leq m \leq \Delta/r_n$ and tile mr_nS/Δ horizontally into a set $\mathcal{R}_{\mathrm{H}}$ of $mr_n/\Delta \times MK_nr_n/\Delta$ rectangles and also vertically into a set $\mathcal{R}_{\mathrm{V}}$ of disjoint rectangles each of size $MK_nr_n/\Delta \times mr_n/\Delta$. Here and henceforth, we fix the constant M such that Proposition 2.1 holds. For now, we allow m to be a multiple of MK_n and extend it to the general case at the end.

The strategy of the proof is as follows. The first step is to construct a backbone of low passage time paths in each rectangle of $\mathcal{R}_{\mathrm{H}}$ and $\mathcal{R}_{\mathrm{V}}$; we obtain an explicit upper bound on the passage time of each path of the backbone. We then estimate the time taken for the infection to reach some node of this backbone starting from x_0, the node of G closest to the origin. Finally, we show that our estimates hold for each integer m between 1 and r_n^{-1} except perhaps $o(r_n^{-1})$ indices, resulting in the lower bound on the speed.

For a vertical rectangle R in $\mathcal{R}_{\mathrm{V}}$, we define $E_n(R)$ to be the event that R contains a oriented dense top–bottom crossing. Again, Proposition 2.1 is applicable to each rectangle R in $\mathcal{R}_{\mathrm{V}}$ with the left–right crossing replaced by the top–bottom crossing. Defining $E_{n,\mathrm{tot}} := \bigcap_{R\in\mathcal{R}_{\mathrm{H}}\cup\mathcal{R}_{\mathrm{V}}} E_n(R)$ and using the fact that the number of rectangles in the set $\mathcal{R}_{\mathrm{V}} \cup \mathcal{R}_{\mathrm{H}}$ is $O(\Delta/r_n) = O(\sqrt{n})$ (by (1.2)), we then have that

$$\mathbb{P}(E_{n,\mathrm{tot}}) \geq 1 - O(\sqrt{n})\frac{1}{n^9} \geq 1 - \frac{1}{n^8} \tag{3.1}$$

for all large enough n.

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We henceforth assume that $E_{n,\text{tot}}$ occurs and let m be a multiple of MK_n. We perform the following analysis for m that is an odd multiple of MK_n and show at the end how to extend it for even multiples. Now consider the lowermost rectangle $R_2 \in \mathcal{R}_{\text{H}}$ and let $L(R_2) = (J_1, J_2, \dots, J_m)$ be the bottommost oriented dense left–right crossing of R_2. Every oriented crossing in R_2 has exactly m squares. Let u_1 be the node that is closest to the centre of J_1. Since $\Delta \geq 4$, every node in J_2 is connected to u_1. For $1 \leq i \leq r-1$, we perform the following iteratively. Consider the set of all edges from u_i that have an endvertex in J_{i+1} and choose that edge h_i with the minimal passage time. The endvertex of h_i distinct from u_i is set to be u_{i+1}. Let $L_h(R_2) = (h_1, \dots, h_{r-1})$ be the resulting path of edges.

Since every J_i is dense, at each iteration in the above procedure, we have chosen the minimum among at least $\beta_1 n r_n^2$ edges, where $\beta_1 > 0$ is the constant in (2.2). We therefore expect the passage time of each edge of $L_h(R_2)$ and the sum total of passage times to be low. Defining $T(R_2) = \sum_{i=1}^{r-1} t(h_i)$ if $E_{n,\text{tot}}$ occurs and $T(R_2) = \infty$ if $E_{n,\text{tot}}$ does not occur, we have the following result.

Lemma 3.1. *There exist positive constants D_1 and δ_1 (independent of the choice of n and m) such that*

$$\mathbb{P}\left(\left\{T(R_2) \geq \frac{D_1 m}{n r_n^2}\right\} \cap E_{n,\text{tot}}\right) \leq \mathrm{e}^{-\delta_1 m}.$$

We prove the above lemma at the end of this section. The term $T(R_2)$ can be thought of as an upper bound on the time taken for the infection to spread from one end of R_2 to the other. Recall that we continue to assume that $E_{n,\text{tot}}$ holds and, therefore, the path $L_h(R_2)$ is well defined. Now, to determine the time taken for the infection to spread to the 'top' of $mr_n S/\Delta$, we grow low passage time paths from $L_h(R_2)$ in the vertical direction. This is possible because the horizontal rectangle R_2 intersects every vertical rectangle $R \in \mathcal{R}_{\text{V}}$ and each of these rectangles has a dense top–bottom crossing (due to the occurrence of the event $E_{n,\text{tot}}$).

Fix the leftmost vertical rectangle $R_l \in \mathcal{R}_{\text{V}}$, and consider the leftmost oriented dense top–bottom crossing $TB(R_l) = (A_1, \dots, A_m)$ of R_l. Here, A_1 touches the bottom of $mr_n S/\Delta$. The dense left–right crossing $L(R_2)$ obtained above and the dense top–bottom crossing $TB(R_l)$ intersect in the sense that there exist squares A_{l_0} and J_{i_0} that share a side.

By previous construction, there exists an edge h_{i_0} of $L_h(R_2)$ that has an endvertex u_{i_0} in J_{i_0}. Every node in A_{l_0} is connected to u_{i_0}. We now start from u_{i_0} and perform the same iterative edge searching procedure that was used to obtain $L_h(R_2)$ above, on the latter part $(A_{l_0}, \dots, A_m)$ of $TB(R_l)$. Set $u'_{l_0} = u_{i_0}$. For each $l_0 \leq i \leq m-1$, we iteratively choose the edge h'_i with minimal passage time that has one endvertex as u'_i and one endvertex in A_{i+1}, and denote the resulting path of edges by $(h'_{l_0}, h'_{l_0+1}, \dots, h'_s)$. Similarly, we construct a 'downward' path of edges $(h'_{l_0-1}, \dots, h'_1)$, starting from u_{i_0} and ending at some node in the square A_1 using the same iterative procedure as above. Finally, we define $TB_h(R_l)$ to be the concatenation $TB_h(R_l) = (h'_1, \dots, h'_{l_0-1}, h'_{l_0}, h'_{l_0+1}, \dots, h'_s)$ of the two paths and define the passage time of the rectangle R_l to be $T(R_l) = \sum_{i=1}^{s} t(h'_i)$. The path $TB_h(R_l)$ contains $s \leq m+2$ edges. We illustrate the above procedure in Figure 1, where the path $L_h(R_2)$ is shown as a wavy horizontal line.

Now repeat the above procedure for each $R \in \mathcal{R}_{\text{V}}$ to obtain corresponding paths $TB_h(R)$. This results in a connected set of edges $\mathcal{P}_{\text{e}}$ that form a comb-like backbone; see Figure 2. The advantage of working with $\mathcal{P}_{\text{e}}$ is that we have an explicit bound on the passage time of each of its paths via Lemma 3.1. This is because even if the passage times of two distinct paths in $\mathcal{P}_{\text{e}}$ are not necessarily independent, Lemma 3.1 holds for each of their passage times individually

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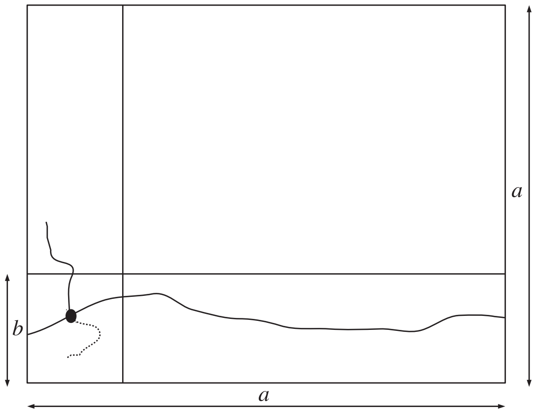

FIGURE 1: Construction of backbones in the rectangles R_2 and R_l. Here $a = mr_n/\Delta$, $b = MK_nr_n/\Delta$, and the black circle represents u_{i_0}.

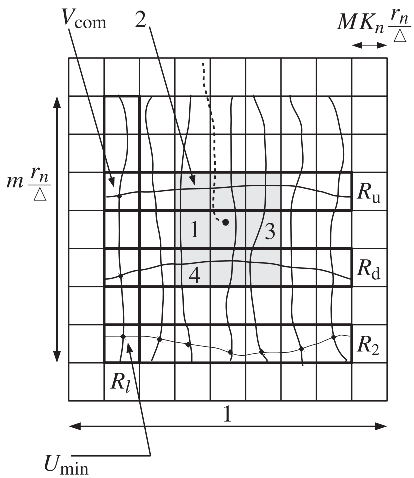

FIGURE 2: The path π_1 from x_0 necessarily intersects the super backbone. Here the node u_{i_0} is labelled as u_{min}

with the *same* constants D_1 and δ_1. This can then be used to estimate the time taken for the infection to spread from some node of a path in $\mathscr{P}_e$ to the boundary.

As a final step in the construction, we obtain a path that allows us to estimate the time taken for the infection to travel from x_0 to the backbone. Let R_0 denote the rectangle in $\mathscr{R}_H$ containing the origin, and let R_u and R_d denote the rectangles in $\mathscr{R}_H$ sharing an edge with R_0 and lying above and below R_0, respectively. In an analogous manner, as described in the previous paragraphs, we 'grow' low passage time paths $L_h(R_u)$ and $L_h(R_d)$ that are connected to the path $TB_h(R_l)$, start from close to the left edge of mr_nS/Δ, and end close to the right edge. Also, we analogously define the passage times $T(R_u)$ and $T(R_d)$. Finally, we define

$$\mathscr{P} = L_h(R_2) \cup \bigcup_{R\in\mathscr{R}_V} TB_h(R) \cup L_h(R_u) \cup L_h(R_d) \tag{3.2}$$

to be the super backbone. The super backbone is connected by construction.

We henceforth assume that $E_{n,\text{tot}} \cap \Gamma_0(x_0)$ occurs (recall the event $\Gamma_0(x_0)$ defined in the proof of Proposition 1.1). Let $L_0 = (A'_0, \ldots, A'_g)$ be the path of dense squares present due to the occurrence of $\Gamma_0(x_0)$. Here A'_0 contains the origin and A'_g intersects the boundary of S. Thus, we know that there is a path π of edges in G starting from x_0 and crossing $S(3K_n)$, the $3MK_n(r_n/\Delta) \times 3MK_n(r_n/\Delta)$ square with the origin at its centre. Here, M is as in Proposition 2.1. (If there is more than one such path, we choose the path whose sum of the length of edges is the least.) This is useful in estimating the time taken for the infection to reach $\mathcal{P}$ from x_0, as shown below.

Let R'_0 be the vertical rectangle in $\mathcal{R}_{\text{V}}$ containing the origin, and let R'_l and R'_r be rectangles in $\mathcal{R}_{\text{V}}$ sharing the left and right longer edge, respectively, with R'_0. The leftmost oriented crossings $TB(R'_l)$ and $TB(R'_r)$ were used in the construction of the super backbone. We see that the corresponding oriented dense crossings $TB(R'_l)$, $L(R_{\text{d}})$, $TB(R'_r)$ and $L(R_{\text{u}})$ form a 'circuit' of squares in $S(3K_n)$ (in Figure 2 we illustrate this with the paths labelled 1, 4, 3 and 2, respectively). Therefore, there exist two squares $A'_{i_1} \in L_0$ and $F_{i_2} \in TB(R'_l) \cup L(R_{\text{d}}) \cup TB(R'_r) \cup L(R_{\text{u}})$, both contained in $S(3K_n)$, that share a common side. By construction, F_{i_2} contains a node v_{i_2} of an edge in $\mathcal{P}$ and A'_{i_1} contains a node v'_{i_1} of π. Since $\Delta \geq 4$, the nodes v'_{i_1} and v_{i_2} are joined by an edge of G.

We now trace the infection spread from x_0. First, to estimate the time taken for the infection starting from x_0 to reach v, we need the following result on local passage time estimates. Let m_1 be the smallest integer that is a multiple of MK_n such that $S \subseteq m_1 r_n S/\Delta$. The tiling of $m_1 r_n S/\Delta$ into horizontal $m_1 r_n S/\Delta \times MK_n r_n/\Delta$ rectangles and vertical $MK_n r_n/\Delta \times m_1 r_n S/\Delta$ rectangles also tiles $m_1 r_n S/\Delta$ into $MK_n r_n/\Delta \times MK_n r_n/\Delta$ squares $\{S'_i\}_i$ as seen in Figure 2. Let T_i denote the sum of passage times of the edges that have at least one endvertex in S'_i and let $T_{\max} = \max_i T_i$.

Lemma 3.2. *There exists a constant* $C_1 > 0$ *(independent of* n*) such that*

$$\mathbb{P}(T_{\max} > (\log n)^8) \leq \frac{C_1}{n^9}. \tag{3.3}$$

This lemma is assumed for now and proved later. We henceforth assume that $\{T_{\max} \leq (\log n)^8\} \cap E_{n,\text{tot}} \cap \Gamma(x_0)$ occurs. Thus, within time $(\log n)^8$ all nodes of $G(x_0) \cap S(K_n)$ are infected and within time $2(\log n)^8$ all nodes of $G(x_0) \cap S(3K_n)$ are infected (recall that $G(x_0)$ is the component of G containing x_0). By construction, this necessarily implies that the infection has reached some node of the super backbone within time $2(\log n)^8$. In what follows we trace the infection spread in the super backbone.

Define $V_m = \bigcap_R \{T(R) \leq D_1 m/nr_n^2\} \cap E_{n,\text{tot}}$, where the intersection is taken over all rectangles R present in the expression for $\mathcal{P}$ in (3.2) and $T(R)$ denotes the passage time (see Lemma 3.1) of the rectangle R. To estimate $\mathbb{P}(V_m)$, we recall, as mentioned before, that even if the passage times of two distinct paths are not necessarily independent, Lemma 3.1 holds for each of them individually with the *same* constants D_1 and δ_1. Thus, from (3.1) and Lemma 3.1 we obtain

$$\begin{aligned}
\mathbb{P}(V_m^c) &= \mathbb{P}(E_{n,\text{tot}}^c) + \mathbb{P}\bigg(\bigcup_R \bigg\{T(R) > \frac{D_1 m}{nr_n^2}\bigg\} \cap E_{n,\text{tot}}\bigg) \\
&\leq \mathbb{P}(E_{n,\text{tot}}^c) + \sum_R \mathbb{P}\bigg(\bigg\{T(R) > \frac{D_1 m}{nr_n^2}\bigg\} \cap E_{n,\text{tot}}\bigg) \\
&\leq \frac{1}{n^8} + C_3\sqrt{n}\mathrm{e}^{-\delta_1 m}
\end{aligned}$$

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for some positive constant C_3. In obtaining the final estimate above, we use the fact that the number of rectangles in $\mathcal{R}_{\mathrm{H}} \cup \mathcal{R}_{\mathrm{V}} = O(r_n^{-1}) = O(\sqrt{n})$ by (1.2). Since $m \geq n^{1/9}$, we obtain

$$\mathbb{P}(V_m) \geq 1 - \frac{C_2}{n^8} \tag{3.4}$$

for some constant $C_2 > 0$, independent of n and m.

We henceforth assume that $V_m \cap \{T_{\max} \leq (\log n)^8\} \cap \Gamma_0(x_0)$ occurs. The infection then spreads to all nodes of the super backbone within time $2(\log n)^8 + 4D_1 m/nr_n^2$. Recalling the definition of the squares $\{S_i'\}_i$ prior to Lemma 3.2, it then follows that the infection reaches at least one node of each square S_i' contained in $mr_n S/\Delta$ within time $2(\log n)^8 + 4D_1 m/nr_n^2$. Hence, within time $2(\log n)^8 + 4D_1 m/nr_n^2 + (\log n)^8 \leq 5D_1 m/nr_n^2$, the infection reaches all nodes of $G(x_0)$ in $mr_n S/\Delta$. In the final estimate, we use the fact that $m \geq n^{1/9}$ and, therefore, that $(\log n)^8 = o(m/nr_n^2)$ by virtue of (1.2). In other words, $(G(x_0) \setminus I(5D_1 m/nr_n^2)) \cap mr_n S/\Delta = \phi$, which is nearly what we want to prove. For the indices m that are even multiples of MK_n, the above analysis holds with slightly different definitions for R_{u} and R_{d}: there are $1 \times MK_n r_n/\Delta$ rectangles that are spaced $MK_n r_n/\Delta$ apart and equidistant from the origin. It only remains to consider the indices m that are not multiples of MK_n.

Recalling the definition prior to Theorem 1.1, we set $a_n = n^{1/9}$ and $b_n = 2MK_n$ (both of which are $o(r_n^{-1})$ by (1.2)), and proceed as follows. For a fixed integer $n^{1/9} \leq m_3 \leq r_n^{-1} - 2MK_n$, let m be the smallest integer that is a multiple of MK_n such that $S \supseteq mr_n S/\Delta \supseteq m_3 r_n S$. We choose b_n so that there exists such an m. Since $\Delta \in [4, 5]$, we have $4n^{1/9} \leq 4m_3 \leq m \leq 5m_3 + 5MK_n \leq 6m_3$. Here we used the fact that $K_n = \log(n)/nr_n^2 \leq \log n$. Thus, if $V_m \cap \{T_{\max} \leq (\log n)^8\} \cap \Gamma_0(x_0)$ occurs then, by the discussion in previous paragraph and the above inequalities, the event I_{m_3} occurs, where

$$I_{m_3} := \left\{ \left(G(x_0) \setminus I\left(\frac{30 D_1 m_3}{nr_n^2} \right) \right) \cap m_3 r_n S = \phi \right\}.$$

This conclusion holds for each $n^{1/9} \leq m_3 \leq r_n^{-1} - 2MK_n$. Since $r_n^{-1} \leq C\sqrt{n}$ for some constant $C > 0$ (see (1.2)), from (3.4) we have $\mathbb{P}(W_n) \geq 1 - C_2 C\sqrt{n}/n^8 \geq 1 - 1/n^7$ for all sufficiently large n, where $W_n = \bigcap_{n^{1/9} \leq m_3 \leq r_n^{-1} - 2MK_n} V_m$. Therefore, from Lemma 3.2 and Proposition 1.1, and the fact that $\Gamma_0(x_0) \subseteq \Gamma(x_0)$, we have

$$\mathbb{P}(A \cap \Gamma(x_0)) \geq \mathbb{P}(W_n \cap \{T_{\max} \leq (\log n)^8\} \cap \Gamma_0(x_0)) \geq 1 - \frac{1}{n^7} - \frac{C_1}{n^9} - \mathrm{e}^{-\theta_1 nr_n^2} \tag{3.5}$$

for all sufficiently large n, where $A = \bigcap_{n^{1/9} \leq m_3 \leq r_n^{-1} - 2MK_n} I_{m_3}$ and $C_1 > 0$ as in (3.3). Since $\mathbb{P}(A \mid \Gamma(x_0)) \geq \mathbb{P}(A \cap \Gamma(x_0))$, this proves the lower bound in Theorem 1.1.

Proof of Lemma 3.1. Let $B = \{T(R_2) > 2D_2 m/nr_n^2\}$, where $D_2 > 0$ is a constant. As in (2.1), the term $\mathbb{P}_\omega(B)$ denotes the probability that event B occurs for a fixed configuration of points ω. From the discussion in the paragraph preceding Lemma 3.1, if $\omega \in E_{n,\mathrm{tot}}$ then the passage time $T(R_2)$ of R_2 satisfies $T(R_2) = \sum_{i=1}^{q} t(h_i) \leq \sum_{i=1}^{q} X_i \leq \sum_{i=1}^{m+2} X_i$, where $\{X_i\}_i$ are independent and identically distributed (i.i.d.) with $X_i = \min\{t_{i,j} : j = 1, 2, \ldots, \beta_1 nr_n^2\}$, and the $\{t_{i,j}\}_j$ are a set of independent random variables exponentially distributed with unit mean. Here $\beta_1 > 0$, as in (2.2). If $\omega \in E_{n,\mathrm{tot}}$, we have $\mathbb{P}_\omega(B) \leq \mathbb{P}(\sum_{i=1}^{m+2} X_i > 2D_2 m/nr_n^2)$, where the right-hand side expression does not depend on ω. Integrating over ω, we have

$$\mathbb{P}\left(\left\{ T(R_2) > \frac{2D_2 m}{nr_n^2} \right\} \cap E_{n,\mathrm{tot}} \right) \leq \mathbb{P}\left(\sum_{i=1}^{10Mm} X_i > \frac{2D_2 m}{nr_n^2} \right). \tag{3.6}$$

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Since $\beta_1 n r_n^2 X_i$ is exponentially distributed with mean 1, we use the Chernoff bound to obtain, for $D_2 > 0$ and $s \in (0, 1)$,

$$\mathbb{P}\bigg(\sum_{i=1}^{m+2} X_i > \frac{2D_2 m}{n r_n^2}\bigg) \le (\mathbb{E}\exp(sX_1\beta_1 n r_n^2))^{m+2}\mathrm{e}^{-2s\beta_1 D_2 m} = \bigg(\frac{1}{1-s}\bigg)^{m+2}\mathrm{e}^{-2s\beta_1 D_2 m}.$$

Therefore, fixing $s = \frac{1}{2}$ and choosing the constant $D_2 > 0$ sufficiently large, for all sufficiently large $n \ge N_0$ and all $n^{1/9} \le m \le \Delta/r_n$, the last expression in the above equation is no more than $2^{m+2}\mathrm{e}^{-\beta_1 D_2 m} \le \mathrm{e}^{-\delta_1 m}$ for some positive constant δ_1.

Proof of Lemma 3.2. For a fixed $K \ge 1$, let $E_K(n)$ denote the event that every square in the set of $r_n/\Delta \times r_n/\Delta$ squares $\{S_i\}_i$ contains less than $\lceil K \log n\rceil$ nodes. Using the fact that $n r_n^2 \le c_2 \log n$ (see (1.2)), we have

$$\mathbb{P}(E_K(n)) \ge 1 - \frac{C_1}{n^{10}} \tag{3.7}$$

if K is sufficiently large. Here $C_1 > 0$ is a constant independent of n. Fix such a K and assume that $K \log n$ is an integer; otherwise, an analogous argument holds with $\lceil K \log n\rceil$. Fix a configuration $\omega \in E_K(n)$, and, for a fixed i, let $\mathcal{E}_i$ denote the set of edges with at least one endvertex in the square S_i'. We fix the M in Proposition 2.1 to be larger if necessary so that every edge in $\mathcal{E}_i$ is contained in the $3MK_n r_n/\Delta \times 3MK_n r_n/\Delta$ square S_i'' with the same centre as S_i'. This is possible since $MK_n r_n/\Delta \ge Mr_n/c_2\Delta \ge Mr_n/c_2 5$. The first inequality above follows from (1.2) and the second follows since $\Delta \le 5$.

The square S_i' contains $(MK_n)^2$ squares in $\{S_j\}_j$. Therefore, the number of nodes in S_i'' is less than $(3MK_n)^2 K \log n$. Consequently, the number of edges in $\mathcal{E}_i$ is less than $9M^2K^2K_n^4(\log n)^2 \le C_1(\log n)^6$ for some positive constant C_1. Here we used the fact that $K_n = (\log n)/n r_n^2$ and (1.2). Arguing as in the derivation of (3.6) in the proof of Lemma 3.1, we average over the configurations and obtain

$$\begin{aligned}\mathbb{P}(T_i > (\log n)^8) &\le \mathbb{P}(\{T_i > (\log n)^8\} \cap E_K(n)) + \frac{1}{n^{10}}\\ &\le \mathbb{P}\bigg(\sum_{i=1}^{C_1(\log n)^6} t_i > (\log n)^8\bigg) + \frac{1}{n^{10}},\end{aligned}$$

where t_i are i.i.d. exponential with unit mean. We have

$$\begin{aligned}\mathbb{P}\bigg(\sum_{i=1}^{C_1(\log n)^6} t_i > (\log n)^8\bigg) &\le \mathbb{P}\bigg(\bigcup_{i=1}^{C_1(\log n)^6}\{t_i > C_1^{-1}(\log n)^2\}\bigg)\\ &\le C_1(\log n)^6\mathrm{e}^{-C_1^{-1}(\log n)^2}.\end{aligned}$$

Thus, $\mathbb{P}(T_i > (\log n)^8) \le C_1(\log n)^6\mathrm{e}^{-C_1^{-1}(\log n)^2} + 1/n^{10} \le 2/n^{10}$ for all sufficiently large n. Since the maximum possible number of squares in $\{S_i'\}_i$ is $(\Delta/r_n)^2 = O(n)$ by (1.2), we have $\mathbb{P}(T_{\max} > (\log n)^8) \le \sum_i \mathbb{P}(T_i > (\log n)^8) \le O(n)/n^{10}$, proving (3.3).

3.1. Proof of the upper bound on the speed

At time $t = 0$, the node x_0 of G closest to the origin is infected. Suppose that $\Gamma(x_0)$ occurs (see the definition prior to (1.3)). For a fixed integer $\log n \le m \le r_n^{-1} - 5$, we now examine

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the path π_m through which the infection first reaches the boundary of mr_nS. More precisely, let $\pi = (h_0, \dots, h_b)$ be a self-avoiding path of edges such that

(iii) h_0 contains x_0 as one of its endvertices, exactly one endvertex of h_b lies in $S \setminus mr_nS$, and

(iv) all other endvertices of the edges $\{h_i\}_i$ lie in mr_nS.

Such a path definitely exists because of the occurrence of the event $\Gamma(x_0)$. Define $T(\pi) = \sum_{i=0}^{b} t(h_i)$ to be the passage time of π, and let π_m be that path whose passage time is $T(\pi_m) = \min_\pi T(\pi)$, where the minimum is taken over all paths satisfying (iii) and (iv) above. Such a unique path exists since the passage times are continuous random variables.

To bound $T(\pi_m)$, we recall the event $E_K(n)$ defined prior to (3.7). We fix K such that (3.7) holds and consider a configuration $\omega \in E_K(n) \cap \Gamma(x_0)$. Each node has fewer than $K_1 \log n$ neighbours for some fixed constant $K_1 > 0$. We assume that $K_1 n \log n$ is an integer; an analogous argument holds with $\lceil K_1 n \log n \rceil$ otherwise. The number of edges T of G is less than $K_1 n \log n$. If $e_1, \dots, e_T$ denotes the set of edges, we then have $t(e_i) \geq_{\mathrm{st}} \min_{1 \leq j \leq K_1 n \log n} t_j =: X_0$, where the $\{t_j\}_j$ are i.i.d. exponential with unit mean and '$\geq_{\mathrm{st}}$' denotes stochastic domination. Since π_m contains at least $m/4$ edges, we then have $T(\pi_m) \geq_{\mathrm{st}} mX_0/4$. We note that $\mathbb{P}(X_0 \geq 1/n\sqrt{n}\log n) \geq 1 - C_1/\sqrt{n}$, for some constant $C_1 > 0$ independent of ω. Thus, $\mathbb{P}_\omega(T(\pi_m) \geq m/4n\sqrt{n}\log n) \geq 1 - C_1/\sqrt{n}$, where $\mathbb{P}_\omega(\cdot)$ is as defined in (2.1). This happens for each $\log n \leq m \leq r_n^{-1} - 5$. Thus, for any fixed $\omega \in E_K(n) \cap \Gamma(x_0)$, we have

$$\mathbb{P}_\omega\biggl(\bigcap_{\log n \leq m \leq r_n^{-1}-5} \biggl\{T(\pi_m) \geq \frac{m}{4n\sqrt{n}\log n}\biggr\}\biggr) \geq 1 - \frac{C_1 r_n^{-1}}{\sqrt{n}},$$

and the final expression is $1 - o(1)$ as $n \to \infty$ since $nr_n^2 \to \infty$. Thus, from the above discussion and (2.1), we have

$$\begin{aligned}\mathbb{P}\biggl(\bigcap_{\log n \leq m \leq r_n^{-1}-5} &\biggl\{T(\pi_m) \geq \frac{m}{4n\sqrt{n}\log n}\biggr\} \cap E_K(n) \cap \Gamma(x_0)\biggr) \\ &\geq \int_{E_K(n)\cap\Gamma(x_0)} \biggl(1 - \frac{C_2 r_n^{-1}}{\sqrt{n}}\biggr)\mu(\mathrm{d}\omega) \\ &\geq \biggl(1 - \frac{C_2 r_n^{-1}}{\sqrt{n}}\biggr)\biggl(1 - \frac{O(1)}{n^{10}} - \mathrm{e}^{-\theta_1 n r_n^2}\biggr),\end{aligned}$$

where the final terms in the last estimate follow from (3.7) and (1.3), respectively. Therefore,

$$\mathbb{P}\biggl(\bigcap_{\log n \leq m \leq r_n^{-1}-5} \biggl\{T(\pi_m) \geq \frac{m}{4n\sqrt{n}\log n}\biggr\} \cap E_K(n) \cap \Gamma(x_0)\biggr) = 1 - o(1),$$

and from (3.7) we have

$$\mathbb{P}\biggl(\bigcap_{\log n \leq m \leq r_n^{-1}-5} \biggl\{T(\pi_m) \geq \frac{m}{4n\sqrt{n}\log n}\biggr\} \cap \Gamma(x_0)\biggr) = 1 - o(1).$$

We note that if $T(\pi_m) \geq m/4n\sqrt{n}\log n$ then $I(m/4n\sqrt{n}\log n) \subseteq (m+1)r_nS$. Again, by the same argument following (3.5), this proves the upper bound on the speed in Theorem 1.1.

4. Proof of Corollary 1.1

Proof of (1.4). Let m be a multiple of MK_n (where the constant M is as in Proposition 2.1) that satisfies $mr_nS/\Delta \subseteq S \subseteq (m + MK_n)r_nS/\Delta$. Using $\Delta \in [4, 5]$ and $K_n = (\log n)/nr_n^2 = o(r_n^{-1})$ by (1.2), we get $3r_n^{-1} \le m \le 6r_n^{-1}$ for all sufficiently large n. The square mr_nS/Δ is the largest square contained in S to which the tiling argument of the proof of Theorem 1.1 described in Section 3 can be applied. Consequently, there exists a backbone of low passage time connections as described in the paragraph preceding (3.4).

Suppose that the events V_m, defined prior to (3.4), and $\{T_{\max} \le (\log n)^8\}$, defined prior to Lemma 3.2, occur, and let $U_m := V_m \cap \{T_{\max} \le (\log n)^8\}$. Let $\Gamma_0(x_0)$ be as defined in Proposition 1.1 and, for a constant $K \ge 1$, let $E_K(n)$ be as defined prior to (3.7). Fixing K such that (3.7) holds, it follows from (1.3), (3.4), (3.3), and (3.7) that $U_m \cap \Gamma_0(x_0) \cap E_K(n)$ occurs with probability $1 - o(1)$. By the proofs of the lower and upper bounds on the speed in Theorem 1.1, with probability $1-o(1)$, the time T_0 elapsed before all nodes of $G(x_0) \cap mr_nS/\Delta$ are infected therefore satisfies

$$\frac{3D_1 r_n^{-1}}{n\sqrt{n}\log n} \le \frac{D_1 m}{n\sqrt{n}\log n} \le T_0 \le \frac{D_2 m}{nr_n^2} \le \frac{6D_2 r_n^{-1}}{nr_n^2}$$

for some positive constants D_1 and D_2. The first and last inequalities hold by our choice of m. Moreover, since $\{T_{\max} \le (\log n)^8\}$ occurs and $(\log n)^8 = o(r_n^{-1})/nr_n^2$ by (1.2), we know that, by time $6D_2r_n^{-1}/nr_n^2 + (\log n)^8 \le 7D_2r_n^{-1}/nr_n^2$, all nodes of $G(x_0)$ are infected. This proves the upper and lower bounds in (1.4), and the lower bound in (1.5).

Proof of (1.5). The lower bound in (1.5) is proved above. To prove the upper bound in (1.5), we let $\Gamma_1(x_0)$ denote the event that $x_0 \in S(K_n)$ and the component $G(x_0)$ contains at least one node outside $S(3K_n)$. We recall that $S(3K_n)$ is the $3MK_nr_n/\Delta \times 3MK_nr_n/\Delta$ square with its centre as the origin and where M is the constant in Proposition 2.1. Fixing $K \ge 1$, as in the previous paragraph, we now write

$$\begin{aligned}\mathbb{E}T_{\text{elap}} &= \mathbb{E}T_{\text{elap}}\,\mathbf{1}(U_m \cap \Gamma_1(x_0)) + \mathbb{E}T_{\text{elap}}\,\mathbf{1}(U_m \cap \Gamma_1^c(x_0) \cap E_K(n)) \\ &\quad + \mathbb{E}T_{\text{elap}}\,\mathbf{1}(U_m \cap \Gamma_1^c(x_0) \cap E_K^c(n)) + \mathbb{E}T_{\text{elap}}\,\mathbf{1}(U_m^c) \\ &\le \mathbb{E}T_{\text{elap}}\,\mathbf{1}(U_m \cap \Gamma_1(x_0)) + \mathbb{E}T_{\text{elap}}\,\mathbf{1}(\Gamma_1^c(x_0) \cap E_K(n)) \\ &\quad + \mathbb{E}T_{\text{elap}}\,\mathbf{1}(E_K^c(n)) + \mathbb{E}T_{\text{elap}}\,\mathbf{1}(U_m^c),\end{aligned} \tag{4.1}$$

and evaluate each term separately.

For the first term, we note that $\Gamma_1(x_0)$ occurs and, therefore, there is a path, π_1, of edges from $x_0 \in S(K_n)$ that crosses $S(3K_n)$. By an analogous argument, as in the two paragraphs following (3.2), the path π_1 intersects the super backbone $\mathcal{P}$ (present due to the occurrence of U_m). Thus, it follows from the proof of the upper bound of (1.4) above that $\mathbb{E}T_{\text{elap}}\,\mathbf{1}(U_m \cap \Gamma_1(x_0)) \le 7D_2r_n^{-1}/nr_n^2$. We now show that each of the remaining terms in (4.1) is $o(r_n^{-1})/nr_n^2$.

To evaluate the second term, we write $\Gamma_1^c(x_0) = \Gamma_{1,1}(x_0) \cup \Gamma_{1,2}(x_0)$, where $\Gamma_{1,1}(x_0)$ is the event that $x_0 \notin S(K_n)$ and $\Gamma_{1,2}(x_0)$ is the event that $G(x_0)$ is contained in $S(3K_n)$. If $\Gamma_{1,2}(x_0) \cap E_K(n)$ occurs then the component containing x_0 is completely contained in $S(3K_n)$. The time elapsed before no new nodes are infected is bounded above by the sum of the passage times of edges contained in the square $S(3K_n)$. Since $E_K(n)$ occurs, the square $S(3K_n)$ contains fewer than $(3MK_n)^2 \log n \le (\log n)^4$ nodes and, therefore, fewer than $(\log n)^8$ edges if n is sufficiently large. Here we use $K_n = (\log n)/nr_n^2 \le \log n$ for all sufficiently large n since $nr_n^2 \to \infty$. Since the passage time of any edge has unit mean, this implies that

$\mathbb{E}(T_{\text{elap}}\,\mathbf{1}(\Gamma_{1,2}(x_0) \cap E_K(n))) \le \mathbb{E}\sum_{i=1}^{(\log n)^8} t_i = (\log n)^8$ for all sufficiently large n. In the above, the $\{t_i\}_i$ are i.i.d. exponential with unit mean. Using (1.2), it follows that the right-hand side of the above is $o(r_n^{-1})/nr_n^2$.

We estimate $\mathbb{E}(T_{\text{elap}}\,\mathbf{1}(\Gamma_{1,1}(x_0) \cap E_K(n)))$ and the third and fourth terms in (4.1) together. We note that if $\Gamma_{1,1}(x_0)$ occurs then $S(K_n)$ is empty. Again, using standard binomial estimates from Chapter 1 of [9], it follows that $\mathbb{P}(\Gamma_{1,1}(x_0)) \le \mathrm{e}^{-\theta_1 (MK_n)^2 nr_n^2}$ for some constant $\theta_1 > 0$ and all sufficiently large n. Choosing M to be greater, if necessary, it follows that $(MK_n)^2 nr_n^2 = M^2(\log n)^2/nr_n^2 \ge 10\log n/\theta_1$ so that $\mathbb{P}(\Gamma_{1,1}(x_0)) \le 1/n^{10}$. Here we use (1.2) and the fact that $K_n = (\log n)/nr_n^2$. Thus, using the Cauchy–Schwarz inequality, we bound $\mathbb{E}T_{\text{elap}}\,\mathbf{1}(\Gamma_{1,1}(x_0) \cap E_K(n))$ above by

$$\mathbb{E}T_{\text{elap}}\,\mathbf{1}(\Gamma_{1,1}(x_0)) \le (\mathbb{E}T_{\text{elap}}^2)^{1/2}\mathbb{P}(\Gamma_{1,1}(x_0))^{1/2} \le \frac{1}{n^5}(\mathbb{E}T_{\text{elap}}^2)^{1/2}. \tag{4.2}$$

Similarly, it follows that $\mathbb{E}T_{\text{elap}}\,\mathbf{1}(U_m^c) \le (\mathbb{E}T_{\text{elap}}^2)^{1/2}\mathbb{P}(U_m^c)^{1/2}$ for the fourth term in (4.1). From (3.4) and Lemma 3.2, it follows that

$$\mathbb{P}(U_m^c) \le \mathbb{P}(V_m^c) + \mathbb{P}(T_{\max} > (\log n)^8) \le \frac{2}{n^8} + \frac{1}{n^8} \le \frac{3}{n^8}$$

for all sufficiently large n. Thus, the fourth term is bounded above by $C_1(\mathbb{E}T_{\text{elap}}^2)^{1/2}/n^4$ for some positive constant C_1.

Finally, from (3.7) and the Cauchy–Schwarz inequality, it follows that the third term in (4.1) is bounded by $(\mathbb{E}T_{\text{elap}}^2)^{1/2}\mathbb{P}(E_K^c(n))^{1/2} \le (\mathbb{E}T_{\text{elap}}^2)^{1/2}/n^5$. Thus, from (4.2), the sum of $\mathbb{E}(T_{\text{elap}}\,\mathbf{1}(\Gamma_{1,1}(x_0) \cap E_K(n)))$ and the third and fourth terms in (4.1) are bounded above by $C_2(\mathbb{E}T_{\text{elap}}^2)^{1/2}/n^4$ for some positive constant C_2. Since the number of edges in G is at most n^2, it follows that $T_{\text{elap}} \le \sum_{i=1}^{n^2} t_i$, where the t_i are i.i.d. exponential with unit mean. Hence, by the arithmetic mean inequality it follows that $\mathbb{E}T_{\text{elap}}^2 \le \mathbb{E}n^2\sum_{i=1}^{n^2} t^2(e_i) \le C_2 n^4$ for some positive constant C_2. Here we use the fact that $\mathbb{E}X^2 < \infty$ for an exponential random variable X with unit mean. Thus, $(\mathbb{E}T_{\text{elap}}^2)^{1/2}/n^4 \le C_3/n^2 = o(r_n^{-1})/nr_n^2$ for some positive constant C_3 by (1.2).

Proof of (1.6). To prove (1.6), we note from the proof of Theorem 1.1 that the infection starting from the node x_0 closest to the origin crosses the boundary of $\frac{1}{2}r_n^{-1}S$ with probability $1 - o(1)$. By using the construction of the giant component in the proof of Theorem 1(i) of [4], we know that this path intersects the giant component with probability $1 - o(1)$. From the estimate on the size of the giant component in Theorem 1(ii) [4], we know that the giant component contains at least $n - n\mathrm{e}^{-\theta nr_n^2}$ nodes with probability $1 - o(1)$ for some constant $\theta > 0$. Equation (1.6) then follows.

5. Proof of Theorem 1.2

We construct the probability space as in Section 2. To analyse contact processes on G, we would like to obtain a subgraph of G containing at least θn nodes, each having a sufficiently large number of neighbours, for some constant $\theta > 0$. As in Section 2, we divide S into small $r_n/\Delta \times r_n/\Delta$ disjoint squares $\{S_k\}_{k\ge1}$, where $\Delta = \Delta_n \in [4, 5]$ is such that Δ/r_n is an integer. Recall the definition of dense squares from Section 2.

The first result we obtain is that there exists a sufficiently long path of dense squares with high probability. Let $\{Y_i\}_{1\le i\le J} \subseteq \{S_k\}_k$ be a set of distinct squares. Assume that $\Gamma = (Y_1, \dots, Y_J)$ is a *path* if, for every $1 \le i \le J-1$, the square Y_i shares a corner with Y_{i+1}. We say that Γ

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is dense if every square in Γ is dense. For integer $m \geq 1$, let Z_m denote the event that there exists a path containing at least m dense squares.

Proposition 5.1. *There exist positive constants D_1 and D_2 such that*

$$\mathbb{P}(Z_m) \geq 1 - \frac{D_1}{n^9}$$

for $m = \lceil D_2/K_n r_n^2 \rceil$ and all sufficiently large n.

We prove the above result at the end of this section. Fix a configuration of nodes $\omega \in Z_m$, where $m = \lceil D_2/K_n r_n^2 \rceil$, so that we have a path containing at least $\lceil D_2/K_n r_n^2 \rceil$ dense squares. Since each dense square contains at least $\beta_1 n r_n^2$ nodes (see (2.2)), we have a node path Π containing at least $\lceil D_2 K_n^{-1} r_n^{-2} \rceil \times \beta_1 n r_n^2 =: N \geq \theta_2 n/K_n$ nodes for some constant $\theta_2 > 0$. Moreover, we extract a subgraph that can be identified with the graph $\mathbb{Z}_{N,K}$, defined as follows: fix integer $K \geq 1$ (to be determined later), and let $r = [N/(2K)]$. Divide $[1, 2rK] \cap \mathbb{Z}$ into segments $\{V_i\}_{1 \leq i \leq r}$, where $V_i = [(i-1)2K + 1, 2iK] \cap \mathbb{Z}$ contains $2K$ points. For $2 \leq i \leq r-1$, join every point in V_i with every point in ${V'}_{i+1} = [2iK + 1, 2iK + K] \cap \mathbb{Z}$ by an edge. Also, join every point in V_i to every point in ${V''}_{i-1} = [2(i-2)K + K + 1, 2(i-1)K] \cap \mathbb{Z}$ by an edge. Join every point in V_1 to every point in ${V'}_2$ by an edge, and join every point in V_r to every point in ${V''}_{r-1}$ by an edge.

It suffices to study the contact process on the graph $\mathbb{Z}_{N,K}$. Since the number of neighbours of each node of Π in G is at least $[\theta_1 n r_n^2]$ for some constant $\theta_1 > 0$, and $n r_n^2 \to \infty$, the integer K can be chosen as large as we want provided n is sufficiently large. Here $[x]$ refers to the integer part of x. Let η_t^1 denote the contact process on $\mathbb{Z}_{N,K}$ starting with all nodes infected, defined on the probability space $(\Psi, \mathcal{A}, \mathbb{P}_K)$.

Proposition 5.2. *There exist positive constants K, α_1, α_2, and α_3 such that if $s_N = \exp(\alpha_1 N)$ then*

$$\mathbb{P}_K(\eta_{s_N}^1 \neq \varnothing) \geq 1 - \mathrm{e}^{-\alpha_3 N}$$

for all sufficiently large N.

We henceforth fix K such that the above proposition holds. We remark that the constants in the above result are independent of the choice of $\omega \in Z_m$. In other words, for each configuration $\omega \in Z_m$, the following is true: with $\mathbb{P}_\omega$-probability at least $1 - \mathrm{e}^{-\alpha_3 N}$, the infection survives for at least s_N time units (recall the definition of $\mathbb{P}_\omega$ prior to (2.1)). Since $N \geq \theta_2 n$, the above result together with (2.1) and Proposition 5.1 implies that $\mathbb{P}(\xi_{s_N}^1 \neq \varnothing) \geq (1 - \mathrm{e}^{-\alpha_3 N})(1 - D_1/n^9) \geq 1 - 1/n^4$ for all sufficiently large n. This proves the lower bound on the extinction time in Theorem 1.2. The upper bound is proved in the next subsection. We now prove Propositions 5.2 and 5.1 in that order.

Proof of Proposition 5.2. We compare the process with an oriented bond percolation process. Identify the segment V_i with $i \in \mathbb{Z}$. Divide the time slot into intervals of size ε for some small fixed constant $\varepsilon \in (0, 1)$ to be determined later. The point $(i, j) \in \mathbb{Z}^2$ refers to the state of the segment V_i at time $j\varepsilon$. Oriented bonds are allowed to be drawn from (i, j) to $(i+1, j+1)$ and $(i-1, j+1)$ only. Say that V_i is active at time $j\varepsilon$ if it contains at least one active node. Set all nodes to be active at time $t = 0$. Suppose that V_i is active at time $j\varepsilon$.

Draw an arrow from (i, j) to $(i+1, j+1)$ if there is an active path from some active node in V_i to some node in ${V'}_{i+1}$ in the time interval $T_{j,\varepsilon} := [j\varepsilon, (j+1)\varepsilon)$; here we say that an active path occurs from $x \in V_i$ to $y \in {V'}_{i+1}$ in $T_{j,\varepsilon}$ if (i) the recovery clocks of both x and y

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do not ring in $T_{j,\varepsilon}$, and (ii) the infection clock from x to y rings at least once in $T_{j,\varepsilon}$. If such an active path occurs, we say that y is active. The event that y is active thus depends only on the infection clocks of the nodes in V_i and the recovery clocks of the nodes in V_i and V'_{i+1}, all restricted to the interval $T_{j,\varepsilon}$. This 'local dependence' allows us to apply a contour argument to estimate the survival probability of active nodes.

To calculate the probability of such an active path, let $x_1 \in V_i$ be a node that is active at time $j\varepsilon$. Let $Z(x_1)$ denote the first time the recovery clock of x_1 rings after $j\varepsilon$, and, for $1 \le r \le K$, let Y_r denote the first time the recovery clock of $2iK + r \in V'_{i+1}$ rings after $j\varepsilon$. Finally, for $1 \le r \le K$, let X_r be the first time the infection clock from x_1 to $2iK + r \in V'_{i+1}$ rings after $j\varepsilon$, and let $X_{r_1} := \min_{1\le r\le K} X_r$. An arrow from (i, j) to $(i + 1, j + 1)$ occurs if $\mathcal{A}(x_1) := \{Z(x_1) > \varepsilon\} \cap \{X_{r_1} < \varepsilon\} \cap \{Y_{r_1} > \varepsilon\}$.

By the Poisson property we have that $\mathbb{P}(X_{r_1} < \varepsilon) = 1 - \mathrm{e}^{-\lambda K\varepsilon}$. Moreover, r_1 is uniformly distributed in $\{1, 2, \dots, K\}$, i.e. $\mathbb{P}(r_1 = r) = 1/K$ for $1 \le r \le K$. Thus, $\mathbb{P}(Y_{r_1} > \varepsilon) = \sum_{1\le r\le K} K^{-1}\mathbb{P}(Y_r < \varepsilon) = \mathrm{e}^{-\varepsilon}$. An analogous estimate holds for $Z(x_1)$.

From the definition of $\mathcal{A}(x_1)$ and the calculations above, an arrow from (i, j) to $(i + 1, j + 1)$ occurs with probability at least $\mathrm{e}^{-\varepsilon}(1 - \mathrm{e}^{-\lambda K\varepsilon})\mathrm{e}^{-\varepsilon}$. Choose K large enough so that $(1 - \mathrm{e}^{-\lambda K\varepsilon})\mathrm{e}^{-2\varepsilon} > 1 - 3\varepsilon$. An analogous procedure is employed for drawing an oriented bond from (i, j) to $(i - 1, j + 1)$. Thus, we have an oriented bond percolation process $\mathbb{P}_{\mathrm{or},\varepsilon}$, where each bond is present with probability at least $1 - 3\varepsilon$. Fix an integer $T \ge 1$, and, for $\gamma > 0$, let $TD(r, T)$ denote the event that there exists an oriented top–bottom crossing of the rectangle $R_{r,T} := ([1, r] \times [0, T]) \cap \mathbb{Z}^2$. If an oriented top–bottom crossing occurs in $R_{r,T}$ then some node in $\mathbb{Z}_{N,K}$ has survived up to time $T\varepsilon$. Using a contour argument analogous to [2] for 1-dependent site percolation, we obtain positive constants ε, γ_1, and γ_2 such that

$$\mathbb{P}_{\mathrm{or},\varepsilon}(TD(r, \mathrm{e}^{\gamma_1 r})) \ge 1 - \mathrm{e}^{-\gamma_2 r} \tag{5.1}$$

for all sufficiently large r, provided $\varepsilon > 0$ is sufficiently small. Fix such an $\varepsilon > 0$. Clearly, (5.1) implies the proposition.

To prove estimate (5.1), we apply a contour argument as in [2]. Let $A = TD(r, \mathrm{e}^{\gamma_1 r})$. To estimate $\mathbb{P}(A)$, we grow oriented paths starting from the bottom $([1, r] \times \{0\}) \cap \mathbb{Z}^2$ of $R_{r,T}$, as described above. Let $\mathcal{C}$ denote the maximal collection of oriented paths containing $([1, r]\times\{0\})\cap\mathbb{Z}^2$. If a vertex $(i, j) \in \mathcal{C}$ and if the oriented bond from (i, j) to either $(i+1, j+1)$ or $(i + 1, j - 1)$ is absent in $\mathcal{C}$, we call the nonexistent bond the boundary bond and say that it was terminated. As in [2], with each vertex x of $\mathcal{C}$ as the centre, we draw a square with oriented edges that forms a clockwise contour around x and which has all the neighbours of x in $\mathbb{Z}^2$ as its corners. There is an outermost contour Π_0 that is oriented clockwise and encloses $\mathcal{C} \supseteq [1, r] \times \{0\}$. Suppose that A^c occurs, and let Π denote the part of Π_0 that crosses $R_{r,T}$ from left to right. We then write $A^c = \bigcup_{1\le j\le T} A_j \cap A^c$, where A_j denotes the event that Π cuts the segment $(\{1\} \times [j - 1, j))$ of the left edge of $R_{r,T}$.

Counting the left and right arrows as in [2], we obtain the following: if Π contains $k \ge r$ edges, there exists a subset Π' consisting of at least $k/16$ edges, each of which cut the boundary bonds that were *independently* terminated. The number of choices for Π are at most 8^k and, for each choice of Π, the number of choices for Π' are at most $\binom{k}{k/16} \le (16\mathrm{e})^{k/16}$. Here we use $\binom{n}{m} \le (n\mathrm{e}/m)^m$. Since each boundary bond is terminated with probability at most 3ε, we have

$$\mathbb{P}_{or,\varepsilon}(A_j \cap A^c) \le \sum_{k\ge r} 8^k (16\mathrm{e})^{k/16}(3\varepsilon)^{k/16} \le \mathrm{e}^{-2\varepsilon_1 r}$$

for all sufficiently large r and some constant $\varepsilon_1 > 0$, provided $\varepsilon > 0$ is sufficiently small. Fix such an $\varepsilon > 0$.

Finally, setting $T = \mathrm{e}^{\varepsilon_1 r}$, we obtain

$$\mathbb{P}_{\mathrm{or},\varepsilon}(A^c) = \sum_{1\le j\le T} \mathbb{P}_{\mathrm{or},\varepsilon}(A_j \cap A^c) \le T\mathrm{e}^{-2\varepsilon_1 r} \le \mathrm{e}^{-\varepsilon_1 r},$$

proving estimate (5.1). This completes the proof of Proposition 5.2.

Proof of Proposition 5.1. Oriented dense left–right crossings, defined in Section 2, are useful here. As before, we tile S horizontally into a set $\mathcal{R}_{\mathrm{H}}$ of $1 \times MK_n r_n/\Delta$ disjoint rectangles and vertically into a set $\mathcal{R}_{\mathrm{V}}$ of $MK_n r_n/\Delta \times 1$ disjoint rectangles. Let $E_{n,\mathrm{tot}}$ be the event that each rectangle contains a dense oriented left–right crossing, as defined prior to (3.1). As before, we assume that the tiling is perfect, as in Figure 3(a) of [4]. Otherwise, an analogous analysis with tiling, as in Figure 3(b) of [4], holds.

Suppose now that $E_{n,\mathrm{tot}}$ occurs. Each rectangle in $\mathcal{R}_{\mathrm{H}}$ contains an oriented dense left–right crossing and, therefore, there are at least $\Delta/MK_n r_n =: q$ disjoint left–right dense crossings, each containing $x = \Delta/r_n$ dense squares. If we can concatenate these crossings without losing too many squares, we can then hopefully obtain a path Γ containing at least $C_1/K_n r_n^2$ dense squares for some constant $C_1 > 0$. Let $\{L_i\}_{1\le i\le q} = \{(J_{i,1}, \dots, J_{i,x})\}_i$ be the set containing the lowermost oriented left–right crossing from each rectangle in $\mathcal{R}_{\mathrm{H}}$. Recall that the first square $J_{i,1}$ of each crossing L_i intersects the left face of S. Let $TD_{\mathrm{right}} = (W_1, \dots, W_x)$ be the leftmost oriented top–bottom dense crossing of the rightmost 'vertical' rectangle $R_{\mathrm{right}} \in \mathcal{R}_{\mathrm{V}}$, and let $TD_{\mathrm{left}} = (X_1, \dots, X_x)$ be the leftmost oriented top–bottom dense crossing of the leftmost vertical rectangle $R_{\mathrm{left}} \in \mathcal{R}_{\mathrm{V}}$. Thus, R_{right} intersects the right edge of S and R_{left} intersects the left edge of S. We assume that W_1 and X_1 intersect the bottom edge of S. We use TD_{right} and TD_{left} to 'join' the disjoint dense left–right crossings $\{L_i\}_i$.

A formal iterative procedure for concatenation is described as follows. The crossings L_1 and TD_{left} intersect in the sense that there are squares $J_{1,j}$ and W_i that share a common side. Let $i_0 = \max\{j\colon J_{1,j} \cap TD_{\mathrm{left}} \ne \varnothing\}$ be the 'last time' L_1 intersects the crossing TD_{left}, and let $X_{h_1} = J_{1,i_0}$. Set $TD^{(1)}_{\mathrm{left}} = (X_{h_1}, \dots, X_x)$. Let $j_1 = \min\{j\colon J_{1,j} \cap TD_{\mathrm{right}} \ne \varnothing\}$ be the 'first time' the left–right crossing L_1 intersects the top–bottom crossing TD_{right}, and let $J_{1,j_1} \cap W_{k_1} \ne \varnothing$. Set $\Sigma_1 = (J_{1,i_0}, \dots, J_{1,j_1})$. By construction, the subpath $(W_{k_1}, \dots, W_x)$ intersects L_2. Set $l_1 = \min\{j\colon J_{2,j} \cap (W_{k_1}, \dots, W_x) \ne \varnothing\}$ to be the first time this happens, and let $W_{r_1} \cap J_{2,l_1} \ne \varnothing$. By construction, the concatenation of $(W_{k_1}, \dots, W_{r_1})$ to Σ_1 is a path which we denote by Γ_1. Set $TD^{(1)}_{\mathrm{right}} = (W_{r_1}, \dots, W_x)$. Starting from J_{2,l_1} we now determine the first time (going from right to left) the path L_2 hits the path $TD^{(1)}_{\mathrm{left}}$ as $i_1 = \max\{j\colon J_{2,j} \cap TD^{(1)}_{\mathrm{left}} \ne \varnothing\}$. We concatenate the subpath $(J_{2,l_1}, \dots, J_{2,i_1})$ of L_2 to Γ_1 and denote by Σ_2 the new path. Continuing in this way, the iteration terminates after a finite number of steps to yield the desired final path Γ.

In the above procedure, the total number of dense left–right crossings is $q = \Delta/MK_n r_n$ and, after concatenation, we lose at most $2(MK_n)^2$ dense squares for each rectangle in $\mathcal{R}_{\mathrm{H}}$. Since each dense left–right crossing contains at least Δ/r_n dense squares, the total number of dense squares in Γ is at least $\Delta^2/MK_n r_n r_n - \Delta 2(MK_n)^2/MK_n r_n \ge C_1/K_n r_n^2$ for all sufficiently large n and some constant $C_1 > 0$. Here we use $\Delta \in [4, 5]$, (1.2), and the fact that $K_n = O(\log n)$. This complete the proof of Proposition 5.1.

5.1. Proof of the upper bound on the extinction time

Let $\{S_k\}_k$ denote the set of $r_n/\Delta \times r_n/\Delta$ squares, as described in Section 2, and for integer constant $K \geq 1$, and let $E_K(n)$ denote the event that each square in $\{S_k\}_k$ contains less than $\lceil K \log n \rceil$ nodes. If K is large, from (3.7) we know that $\mathbb{P}(E_K(n)) \geq 1 - 1/n^4$. If $E_K(n)$ occurs, each node in G has less than $K_1 \log n$ neighbours for some sufficiently large constant $K_1 > 0$. Fix a configuration of nodes $\omega \in E_K(n)$.

Divide the time axis into disjoint intervals $\{T_i\}_{i \geq 1}$ of unit length, and let Y_i denote the event that in the time interval T_i the recovery clock of each node rings at least once and none of the infection clocks ring. It follows that $\mathbb{P}_\omega(\xi_i^1 = \varnothing) \geq \mathbb{P}_\omega(\bigcup_{1 \leq j \leq i} Y_j) = 1 - \mathbb{P}_\omega(\bigcap_{1 \leq j \leq i} Y_j^c) = 1 - (1 - \mathbb{P}_\omega(Y_1))^i$, where $\mathbb{P}_\omega(\cdot)$ is as defined in (2.1) and the final expression follows by conditioning and the Markov property. Since $\omega \in E_K(n)$, we have

$$\mathbb{P}_\omega(Y_1) \geq ((\mathrm{e}^{-\lambda})^{K_1 \log n}(1 - \mathrm{e}^{-1}))^n \geq \mathrm{e}^{-3K_2 n \log n}$$

for some constant $K_2 > 0$. Setting $i = \mathrm{e}^{6K_2 n \log n}$, it follows from the above two estimates that $1 - \mathbb{P}_\omega(\xi_i^1 = \varnothing) \leq (1 - 1/\sqrt{i})^i \leq \mathrm{e}^{-\sqrt{i}} \leq \mathrm{e}^{-n \log n}$ for all large n. Here we use the fact that $1 - x \leq \mathrm{e}^{-x}$ for all $x > 0$. Using (2.1) and (3.7), we obtain

$$\mathbb{P}(\xi_i^1 = \varnothing) \geq (1 - \mathrm{e}^{-n \log n})\left(1 - \frac{1}{n^4}\right) \geq 1 - \frac{2}{n^4}$$

for all sufficiently large n.

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