

# Index

*References in italics are main references or definitions.*

- acausal set, 211  
partial Cauchy surface, 204  
acceleration vector, 70, 72, 79, 84, 107  
relative acceleration of world lines,  
78–80  
achronal boundary, 187, 312  
achronal set, 186, 187, 202, 203, 209, 211,  
266, 267: edge, 202  
affine parameter, 33, 86  
generalized, 259, 278, 291  
Alexandrov topology, 196  
anti-de Sitter space, 131–4, 188, 206, 218  
apparent horizon, 320, 321–3, 324  
area law for black holes, 318, 332, 333  
asymptotic flatness, 221–5  
asymptotically simple spaces, 222:  
empty and simple spaces, 222  
weakly asymptotically simple and  
empty spaces, 225, 310: asymptoti-  
cally predictable spaces, 310, 311,  
312  
strongly future asymptotically pre-  
dictable, 313, 315, 317: regular  
predictable space, 318, 319, 320;  
static, 325, 326; stationary, 324,  
325, 327–31, 334–47  
asymptotically simple past, 316  
atlas, 11, 12, 14  
axisymmetric stationary space-times,  
161–70  
black holes, 329, 331, 341–7
- b-boundary, 283, 289  
b-bounded, 292, 293  
b-completeness, 259, 277, 278  
bases of vectors, one-forms, tensors,  
16–18, 51  
change of basis, 19, 21  
coordinate basis, 21  
orthonormal basis, 38, 52  
pseudo-orthonormal basis, 86  
beginning of universe, 3, 8, 358–9, 363  
in Robertson–Walker models, 137–42  
in spatially homogeneous models, 144–9
- Bianchi's identities, 36, 42, 43, 85  
bifurcation  
of black holes, 315–16  
of event horizons, 326  
Birkhoff's theorem, 372  
black-body radiation in universe, 348–  
50, 354–5, 357, 363  
black holes, 308–23, 315  
final state of, 323–47  
rotating black hole, 329  
boundary  
of manifold, 12  
of future set, 187  
of space-time: c-boundary, 217–21,  
222–5; b-boundary, 276–84, 289–91  
Brans–Dicke scalar field, 59, 64, 71, 77,  
362  
energy inequalities, 90, 95  
bundle, 50, 174  
of linear frames, 51, 53, 54, 174,  
292–4  
of orthonormal frames, 52, 54, 276–83,  
289: metric on, 278  
of tensors, 51, 54, 198  
tangent bundle, 51, 54
- c-boundary, 217–21, 224–5  
canonical form, 48  
Carter's theorem, 331  
Cartesian product, 15  
Cauchy data, 147, 231–3, 254  
Cauchy development, 6, 94, 119, 147,  
201–6, 209–11, 217, 228  
local existence, 248, 255  
global existence, 251, 255  
stability, 253, 255, 301, 310  
Cauchy horizon, 202–4, 265, 287, 362  
examples, 120, 133, 159, 178, 203, 205,  
287  
Cauchy problem, 60, 226–55  
Cauchy sequences, 257, 282  
Cauchy surface, 205, 211, 212, 263, 265,  
274, 287, 313  
examples, 119, 125, 142, 154

- Cauchy surface (*cont.*)  
 lack of, 133, 159, 178, 205, 206  
 partial Cauchy surface, 204, 217,  
 301–2, 310–20, 323
- causal boundary of space-time, 217–21,  
 221–5; *see also* conformal structure
- causal future (past),  $J^+(J^-)$ , 183  
 causal structure, 6, 127–30, 180–225  
 causally simple set, 188, 206, 207, 223  
 local causality neighbourhood, 195
- causality conditions  
 local causality, 60  
 chronology condition, 189  
 causality condition, 190  
 future, past distinguishing conditions,  
 192  
 strong causality condition, 192  
 stable causality condition, 198
- causality violations, 6, 162, 164, 170, 175,  
 189, 492, 197  
 and singularity theorems, 272
- caustics, 120, 132–3, 170; *see also*  
 conjugate points
- charged scalar field, 68
- chart, 11
- Christoffel relations, 40
- chronological future (past),  $I^+(I^-)$ , 182,  
 217
- chronology condition, 189, 192, 194, 266  
 violating set, 189
- cigar singularity, 144
- closed trapped surface, 2, 262, 263, 266  
 examples, 155, 161  
 in asymptotically flat spaces, 311, 319  
 outer trapped surface, 319; marginally  
 outer trapped surface, 321  
 outside collapsing star, 301, 308  
 in expanding universe, 353–8
- Codacci's equation, 47, 232, 352
- collapse of star, 3, 8, 300–23, 360
- compact space-time, 40, 189
- compact space sections, 272–5
- completeness conditions  
 inextendibility, 58  
 metric completeness, 257  
 geodesic completeness, 257  
 b-completeness, 259, 278–283
- completion by Cauchy sequences, 282,  
 283
- components of connection, 31
- components of tensor, 19  
 of  $p$ -form, 21
- conformal curvature tensor, 41, 85; *see*  
 Weyl tensor
- conformal metrics, 42, 60, 63, 180, 222
- conformal structure of infinity and  
 singularities
- c-boundary, 217–21
- examples, 122, 127, 132, 141, 145, 154,  
 158, 160, 165, 177  
 in asymptotically flat spaces, 222–4  
 horizons, 128–30
- conformally flat theory, 75–6
- congruence of curves, 69
- conjugate points, 4, 5, 267  
 on timelike geodesics, 97, 98, 111, 100,  
 112, 217  
 on null geodesics, 100, 101, 115, 102,  
 116
- connection, 30, 31, 34, 40, 41, 59, 63  
 and bundles over  $\mathcal{M}$ , 53–5, 277  
 on hypersurface, 46
- conservation  
 of energy and momentum, 61, 62, 67, 73  
 of matter, theorem, 94, 298  
 of vorticity, 83–4
- constraint equations, 232
- continuity conditions  
 for map, 11  
 of space-time, 57, 284
- contraction of tensor, 19
- contracted Bianchi identities, 43
- convergence of curves, *see* expansion
- convergence of fields  
 weak, 243  
 strong, 243
- convex normal neighbourhood, 34, 60,  
 103, 105, 184  
 local causality neighbourhood, 195
- coordinates, 12  
 normal coordinates, 34, 41  
 coordinate singularities, 118, 133, 150,  
 156, 163, 171,
- Copernican principle, 134, 135, 142,  
 350, 356, 358
- cosmological constant, 73, 95, 124, 137,  
 139, 168, 362
- cosmological models  
 isotropic, 134–42  
 spatially homogeneous, 142–9
- covariant derivatives, 31–5, 40, 59
- covering spaces, 181, 204–5, 273, 293
- cross-section of a bundle, 52
- curvature tensor, 35, 36, 41  
 identities, 36, 42, 43  
 of hypersurface, 47  
 physical significance, 78–116
- curve, 15  
 geodesic, 33, 63, 103–16, 213–17  
 non-spacelike, 105, 112, 184, 185, 207,  
 213  
 null, 86–8  
 timelike, 78–86, 103, 182, 184, 213–17
- de Sitter space-time, 124–31
- density of matter in universe, 137, 357

- development, 228, 248, 251, 253  
 existence, 246–9
- deviation equation  
 timelike curves, 80  
 null geodesics, 87
- diffeomorphism, 22, 56, 74, 227
- differentiability conditions, 11, 12  
 and singularities, 284–7
- of initial data, 251  
 of space-time, 57–8
- differential of function, 17
- distance from point, 103–5
- distance function, 215
- distributional solution of field equations, 286
- domain of dependence, *see* Cauchy development, 201
- dominant energy condition, 91, 92, 94, 237, 293, 323
- edge of achronal set, 202
- Einstein's field equations, 74, 75, 77, 95, 227–55  
 constraint equations, 232  
 distributional solutions, 286  
 exact solutions, 117–79  
 existence and uniqueness of solutions, 248, 251, 255  
 initial data, 231–3  
 reduced equations, 230  
 stability of solutions, 253, 255
- Einstein static universe, 139  
 spaces conformal to part of, 121, 126, 131, 139
- Einstein-de Sitter universe, 138
- electromagnetic field, 68
- energy conditions  
 weak energy condition, 89  
 dominant energy condition, 91  
 null convergence condition, 95  
 timelike convergence condition, 95  
 strong energy condition, 95
- energy extraction from black holes, 327–8, 332–3
- energy-momentum tensor of matter fields, 61, 66–71, 88–96, 255
- equation of state of cold matter, 303–7
- ergosphere, 327–31
- Euler-Lagrange equations, 65
- event horizon, 129, 140, 165  
 in asymptotically flat spaces, 312, 315–20, 324–47
- existence of solutions  
 Einstein equations with matter, 255  
 empty space Einstein equations, 248, 251  
 second order linear equations, 243
- exp, exponential map, 33, 103, 119  
 generalized, 292
- expansion  
 of null geodesics, 88, 101, 312, 319, 321, 324, 354  
 of timelike curves, 82–4, 97, 271, 356  
 of universe, 137, 273, 348–59
- extension  
 of development, 228, 249  
 of manifold, 58: locally inextendible, 59  
 of space-time, 145, 150–5, 156–9, 163–4, 171, 175: inextendible, 58, 141; inequivalent extensions, 171–2
- exterior derivative, 25, 35
- Fermi derivative, 80–1
- fibre bundles, *see* bundles
- field equations  
 for matter fields, 65  
 for metric tensor, 71–7  
 for Weyl tensor, 85
- fluid, 69; *see also* perfect fluid
- focal points, *see* conjugate points
- forms  
 one-forms, 16, 44–5  
 $q$ -forms, 21, 47–9
- Friedmann equation, 138
- Friedmann space-times, 135
- function, 14
- fundamental forms of surfaces  
 first, 44, 99, 231  
 second, 46, 99, 100, 102, 110, 232, 262, 273, 274
- future  
 causal,  $J^+$ , 183  
 chronological,  $I^+$ , 182
- future asymptotically predictable, 310
- future Cauchy development,  $D^+$ , 201  
 horizon,  $H^+$ , 202
- future directed non-spacelike curve, 184  
 inextendible, 184, 194, 268
- future distinguishing condition, 192, 195
- future event horizon, 129, 312
- future horismos,  $E^+$ , 184
- future set, 186, 187
- future trapped set, 267, 268
- $g$ -completeness, 257, 258
- gauge conditions, 230, 247
- Gauss' equation, 47, 336, 352
- Gauss' theorem, 49–50
- General Relativity, 56–77, 363  
 postulates, (a), 60, (b), 61, (c), 77  
 breakdown of, 362–3
- generalized affine parameter, 259, 278, 291
- generic condition, 101, 192, 194, 266

- geodesics, 33, 55, 63, 217, 284–5  
 as extremum, 107, 108, 213  
*see also* null geodesics and timelike  
 geodesics
- geodesically complete, 33, 257  
 examples, 119, 126, 133, 170
- geodesically incomplete, 258, 287–9  
 examples, 141–2, 155, 159, 163, 176,  
 190  
*see also* singularities
- globally hyperbolic, 206–12, 213, 215,  
 223
- Gödel's universe, 168–70
- gravitational radiation from black holes,  
 313, 329, 333
- harmonic gauge condition, 230, 247
- Hausdorff spaces, 13, 56, 221, 283  
 non-Hausdorff b-boundary, 283, 289–  
 92  
 non-Hausdorff spaces, 13, 173, 177
- homogeneity  
 homogeneous space-time, 168  
 spatial homogeneity, 134, 142–9, 371
- homomorphisms,  $E^+$ , 184
- horizons  
 apparent horizon, 320–3, 324  
 event horizon, 129, 312, 315, 319,  
 324–33  
 particle horizon, 128
- horizontal subspace (in bundle), 53–5,  
 277–82  
 lift, 54, 277
- Hoyle and Narlikar's  $C$ -field, 90, 126
- Hubble constant, 137, 355
- Hubble radius, 351
- IF, indecomposable future set, 218
- imbedding, 23, 44, 228  
 induced maps of tensors, 45
- immersion, 23
- imprisoned curves, 194–6, 261, 289–  
 98
- inequalities for energy-momentum ten-  
 sor, 89–96  
 and second order differential equations,  
 237, 240, 241
- inextensible curve, 184, 218, 280
- inextensible manifold, 58, 59, 141–2
- infinity, *see* conformal structure of  
 infinity
- initial data, 233, 252, 254
- injective map, 23
- int, interior of set, 209
- integral curves of vector field, 27
- integration of forms, 26, 49
- intersection of geodesics, *see* conjugate  
 points
- IP, indecomposable past set, 218
- isometry, 43, 56, 135–6, 142, 164, 168,  
 174, 323, 326, 329, 330, 334, 340–6,  
 369–70
- isotropy of observations, 134–5, 349, 358  
 and universe, 351, 354
- Israel's theorem, 326
- Jacobi equation, 80, 96
- Jacobi field, 96, 97, 99, 100
- Kerr solution, 161–8, 225, 301, 310, 327,  
 332  
 as final state of black hole, 325–33  
 global uniqueness, 331
- Killing vector field, 43, 62, 164, 167, 300,  
 323, 325, 327, 330, 339  
 bivector, 167, 330, 331
- Kruskal extension of Schwarzschild  
 solution, 153–5
- Lagrangian, 64–7  
 for matter fields, 67–70  
 for Einstein's equations, 75
- Laplace, 2, 364, 365–8
- length of curve, 37  
 generalized, 259, 280  
 non-spacelike curve, 105, 213, 214, 215:  
 longest curve, 5, 105, 107–8, 120, 213
- Lie derivative, 27–30, 34–5, 43, 79, 87,  
 341–6
- light cone, *see* null cone
- limit of non-spacelike curves, 184–5
- limiting mass of star, 304–7
- Lipschitz condition, 11
- local Cauchy development theorem, 248
- local causality assumption, 60
- local causality neighbourhoods, 195
- local conservation of energy and momen-  
 tum, 61
- local coordinate neighbourhood, 12
- locally inextensible manifold, 59
- Lorentz metric, 38, 39, 44, 56, 190,  
 252
- Lorentz group, 52, 62, 173, 277–80
- Lorentz transformation, 279, 290–1
- m-completeness, 257, 278
- manifold, 11, 14  
 as space-time model, 56, 57, 363
- map of manifold, 22, 23  
 induced tensor maps, 22–4
- marginally outer trapped surface, 321
- matter equations, 59–71, 88–96, 117,  
 254
- maximal development, 251–252
- maximal timelike curve, 110–12
- Maxwell's equations, 68, 85, 156, 179

- metric tensor, 36–44, 61, 63–4  
 covariant derivative, 40, 41  
 Lorentz, 38, 39, 44, 56, 57, 190, 237  
 on hypersurface, 44–6, 231  
 positive definite, 38, 45, 126, 257, 259,  
 278, 282, 283  
 space of metrics, 198, 252  
 microwave background radiation, 139,  
 348–50, 354, 356  
 isotropy, 348–53, 358  
 Minkowski space-time, 118–24, 205, 218,  
 222, 274, 275, 310  
 Misner's two-dimensional space-time,  
 171–4
- naked singularities, 311  
 Newman–Penrose formalism, 344  
 Newtonian gravitational theory, 71–4,  
 76, 80, 201, 303–5  
 non-spacelike curve, 60, 112, 184, 185,  
 207  
 geodesic, 105, 213  
 Nordström theory, 76  
 normal coordinates, 34, 41, 63  
 normal neighbourhood, 34, 280; *see also*  
 convex normal neighbourhood  
 null vector, 38, 57  
 cone, 38, 42, 60, 103–5, 184, 198:  
 reconverging, 266, 354  
 convergence condition, 95, 192, 263,  
 265, 311, 318, 320  
 geodesics, 86–8, 103, 105, 116, 133,  
 171, 184, 188, 203, 204, 258, 312,  
 319, 354: reconverging, 267, 271,  
 354, 356; closed null geodesics,  
 190–1, 290  
 hypersurface, 45
- optical depth, 355, 357, 359  
 orientable manifold, 13  
 time orientable, 181, 182  
 space orientable, 181, 182  
 orientation  
 of boundary, 27  
 of hypersurface, 44  
 orthogonal group  $O(p, q)$ , 52, 277–83  
 orthogonal vectors, 36  
 orthonormal basis, 38, 52, 54, 80–2,  
 276–83, 291  
 pseudo-orthonormal basis, 86–7, 344  
 outer trapped surface, 319, 320
- pancake singularity, 144  
 paracompact manifold, 14, 34, 38, 57  
 parallel transport, 32, 40, 277  
 non-integrability, 35, 36  
 p.p. singularity, 260, 290, 291  
 parallelizable manifold, 52, 182
- partially imprisoned non-spacelike curve,  
 194, 289–92  
 partial Cauchy surface, 204, 217, 265,  
 274, 295, 301  
 and black holes, 310–24  
 particle horizon, 128, 140, 144  
 past, *dual of future*, 183; *thus* past set  
*is dual of future set*, 186  
 PIPs, PIFs, 218  
 Penrose collapse theorem, 262  
 Penrose diagram, 123  
 perfect fluid, 69–70, 79, 84, 136, 143, 168,  
 305, 372  
 plane-wave solutions, 178, 188, 206, 260  
 postulates for special and general  
 relativity  
 space-time model, 56  
 local causality, 60  
 conservation of energy and momentum,  
 61  
 metric tensor, 71, 77  
 p.p. curvature singularity, 260, 289–92  
 prediction in General Relativity, 205–6  
 product bundle, 50  
 propagation equations  
 expansion, 84, 88  
 shear, 85, 88  
 vorticity, 83, 88  
 properly discontinuous group, 173  
 pseudo-orthonormal basis, 86–7, 102,  
 114, 271, 290, 344
- rank of map, 23  
 Raychaudhuri equation, 84, 97, 136, 275,  
 286, 352  
 redshift, 129, 139, 144, 161, 309, 355, 358  
 regular predictable space, 318, 323  
 Reissner–Nordström solution, 156–61,  
 188, 206, 225, 310, 360–1  
 global uniqueness, 326  
 Ricci tensor, 36, 41, 72–5, 85, 88, 95, 290,  
 352  
 Riemann tensor, 35, 36, 41, 85, 290, 352  
 Robertson–Walker spaces, 134–42, 276,  
 352–7
- scalar field, 67, 68, 95; *see also* Brans–  
 Dicke  
 scalar polynomial curvature singularities,  
 141–2, 146, 151, 260, 289  
 Schwarzschild solution, 149–56, 225, 262,  
 310, 316, 326  
 local uniqueness, 371  
 global uniqueness, 326  
 outside star, 299, 306, 308–9, 316, 360  
 Schwarzschild radius, 299, 300, 307–8, 353  
 mass, 306, 309  
 length, 353, 358

- second fundamental form of hypersurface, 46, 47  
   of 3-surface, 99, 273, 274  
   of 2-surface, 102, 262
- second order hyperbolic equation, 233–43
- second variation, 108, 110, 114, 296
- semispacelike set, *see* achronal set, 186
- separation of timelike curves, 79, 96, 99  
   of null geodesics, 86–7, 102
- shear tensor, 82, 85, 88, 97, 324, 351
- singularity, 3, 256–61, 360–4  
   s.p. singularity, 260, 289  
   p.p. singularity, 260, 290–2  
   examples, 137–42, 144–6, 150–1, 159,  
     162, 171–4, 177  
   theorems, 7, 147, 263, 266, 271, 272,  
     274, 285, 288, 292  
   description, 276–84  
   nature, 284–9, 360–1, 363  
   in collapsing stars, 308, 310, 311, 360–1  
   in universe, 355, 358–9
- singularity-free space-times, 258, 260  
   examples, 119, 126, 133, 139, 170,  
     305–6
- skew symmetry, 20–1
- Sobolev spaces, 234
- s.p. curvature singularity, 141–2, 146,  
   151, 260, 289
- spacelike hypersurface, 45
- spacelike three-surface, 99, 170, 201, 204,  
   313
- spacelike two-surface, 101, 262
- spacelike vector, 38, 57
- space-orientable, 181
- space-time manifold, 4, 14, 56, 57  
   breakdown, 363  
   connection, 41, 59, 63  
   differentiability, 57, 58, 284–7  
   inextensible, 58  
   metric, 56, 60, 227  
   non-compact, 190  
   space and time orientable, 181–2  
   topology, 197
- spatially homogeneous, 134, 142–9,  
   371
- Special Relativity, 60, 62, 71, 118
- speed of light, 60, 61, 94
- spinors, 52, 59, 182
- spherically symmetric solutions, 135,  
   149–61, 299, 305–6, 369–72
- stable causality, 198
- stability  
   of Einstein's equations, 253, 255, 301  
   of singularity, 273, 360
- star, 299–308  
   white dwarfs, neutron stars, 304, 307  
   life history, 301, 307–8
- static space-times, 72, 73  
   spherically symmetric, 149–61, 305–6,  
     371
- regular predictable space-times, 325–9
- stationary axisymmetric solutions, 161–70
- stationary regular predictable space-times, 323–47
- stationary limit surface, 165–167, 328, 331
- steady-state universe, 90, 126
- Stokes theorem, 27
- strong causality condition, 192, 194, 195,  
   208, 209, 217, 222, 261, 267, 271
- strong energy condition, 95
- strongly future asymptotically predictable, 313, 317, 318
- summation convention, 15
- symmetric and skew-symmetric tensors,  
   20–1
- symmetries of space-time, 44  
   axial symmetry, 329  
   homogeneity, 168  
   spatial homogeneity, 135, 142  
   spherical symmetry, 369  
   static spaces, 72, 325  
   stationary spaces, 323  
   time-symmetry, 326
- tangent bundle, 51, 53–4, 292, 351
- tangent vector space, 16, 51  
   dual space, 17
- Taub-NUT space, 170–8, 206, 261, 289–92
- tensor of type  $(r, s)$ , 17  
   field of type  $(r, s)$ , 21  
   bundle of tensors of type  $(r, s)$ , 51
- tensor product, 18
- theorems  
   conservation theorem, 94  
   singularities in homogeneous cosmologies, 147  
   local Cauchy development, 248  
   global Cauchy development, 251  
   Cauchy stability theorem, 253  
   singularity theorems: theorem 1, 263;  
     theorem 2, 266; theorem 3, 271;  
     theorem 4, 272; theorem 5, 292;  
     weakened conditions, 285, 288
- tidal force, 80
- TIFs, TIPs, 218
- time coordinates, 170, 198
- time orientable, 131, 181, 182
- time symmetric, 326, 328  
   black hole, 330
- timelike convergence conditions, 95, 265,  
   266, 271, 272, 285, 363
- timelike curves, 69, 79–85, 103, 184,  
   213–15, 218

- timelike geodesics, 63, 96–100, 103, 111–12, 133, 159, 170, 217, 258, 288  
timelike hypersurface, 44  
timelike singularity, 159, 360–1  
timelike vector, 38, 57  
topology of manifold, 12–14  
    Alexandrov topology, 196, 197  
topology of set of Lorentz metrics, 198, 252  
topology of space of curves, 208, 214  
torsion tensor, 34, 41  
totally imprisoned curves, 194, 195, 289–98  
trapped region, 319–20  
trapped set, 267  
trapped surface, *see* closed trapped surface  
uniqueness of solutions  
    of Einstein's equations: locally, 246, 255; globally, 251, 255  
    of second order linear equations, 239, 243  
universe, 3, 348–59, 360, 362, 364  
spatially homogeneous universe models  
    anisotropic, 142–9; isotropic, 134–42, 351–3, 356–7  
vacuum solutions of field equations, 118, 150, 161, 170, 178, 244–54  
variation  
    of fields in Lagrangian, 65  
    of timelike curve, 106–10, 295  
    of non-spacelike curves, 112–16, 191  
vector, 15, 16, 38, 57  
    field, 21, 27, 51, 52, 54, 55, 277, 278  
    variation vector, 107–16, 191, 275, 295  
    *see also* Killing vector  
vertical subspaces in bundles, 53, 277  
volume, 48, 49  
vorticity  
    of Jacobi fields, 97  
    of null geodesics, 88  
    of timelike curves, 82–4, 352  
weak energy condition, 89, 94  
weakly asymptotically simple and empty spaces, 225, 310  
Weyl tensor, 41, 42, 85, 88, 101, 224, 344