Can. J. Math., Vol. XXXI, No. 2, 1979, pp. 354–362

FACTORIZATION OF AFFINITIES

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1. Introduction. The decomposition of mappings into a minimal number of simple mappings is a common sight in geometry. One well-known instance is the representation of a plane motion by three reflections (see e.g. H. S M. Coxeter [**3**]) or the representation of equiaffinities by a minimal number of shears or reflections ([**14**], [**5**], [**7**], [**8**]). Theorems of this nature not only give valuable insight into the nature of the mapping, but they are also often used as a base for characterization theories (see e.g. F. Bachmann [**2**], M. Götzky [**10**]). A more abstract version of the same type of results is the famous Cartan-Dieudonné theorem. Its usefulness is indisputable. P. Scherk [**13**] gave a refined version of this theorem.

J. Dieudonné has shown [**6**] that every element in the general linear group $GL(V)$ can be written as a product of transvections and one simple transformation. He also determined the minimal number of factors necessary in such a product. J. Ch. Fisher [**9**] considers a different kind of decomposition. He requires that all but one of the factors are reflections. He also assumes commutativity of the scalars.

From a geometer's point of view it is more satisfactory to obtain information on affine transformations. The affine counterparts of the two situations mentioned in the preceding paragraph will be decompositions of an affinity into shears and one axial affinity or into affine reflections and one axial affinity. In each case we shall determine the shortest possible factorization of an affinity. This way we shall get a considerably shorter decomposition than Fisher's.

In fact we shall not only deal with the two special decompositions. We shall solve the length problem in such a way that we can prescribe the type of all but one factor individually.

As a consequence, we shall also be able to investigate factorizations in affine hyperreflection groups. The most important among these groups is perhaps the group generated by all affine reflections.

In dealing with affinities it is necessary to cope with translations. We shall do that by firstly solving the length problem for a certain subgroup of the general linear group. Then we shall see that this group is isomorphic to the affine group if the scalars form a commutative field. We shall be able to use many results of two earlier papers, [**7**] and [**8**].

Received November 23, 1977 and in revised form May 31, 1978. This research was supported in part by the National Research Council of Canada under grant no. A7251.

Finally, we shall introduce hyperreflection groups even if the field of scalars is not commutative. For these groups as well as for the affine hyperreflection groups many results remain valid in this more general situation.

It may be of interest to mention that no restriction on the dimension of the geometry has been made. In case of infinite dimension we can of course only decompose elements whose paths are finite-dimensional. Also, in § 6, we have to extend the definition of determinants. Other than that the inclusion of infinite-dimensional spaces goes unnoticed.

2. The length in the affine subgroup of $GL(V)$. We shall introduce a number of notions and prove some results for certain subgroups of the general linear group $GL(V)$. The names will indicate their geometric significance. This will become clear in the next section where we establish the connection to the affine geometry and where we apply the results obtained here.

Certainly, Theorems 1 and 3 are of independent interest since they solve length problems in a certain subgroup of the general linear group over any (not necessarily commutative) field K and of any not necessarily finite dimension.

Let V be a left vector space over the field K. The dual space of V will be denoted by V^*. A subspace W of V with codim $W = 1$ is called a *hyperplane*.

For our investigations, it will often be important to consider the following two subspaces which are attached to each $\pi \in GL(V)$, namely $B(\pi) = \{\mathbf{x}^\pi - \mathbf{x};\ \mathbf{x} \in V\}$, called the *path* of π, and $F(\pi) = \{\mathbf{x} \in V; \mathbf{x}^\pi = \mathbf{x}\}$, called the *fix* of π.

A mapping $\pi \in GL(V)$ is called *simple* if $\dim B(\pi) = (= \operatorname{codim} F(\pi)) = 1$.

A simple mapping π with $B(\pi) \not\subset F(\pi)$ is called a *dilatation*, a simple mapping π with $B(\pi) \subset F(\pi)$ is called a *transvection*.

If $\lambda \neq 0$ is an element in the center of the field K, then the mapping $\mathbf{x} \to \lambda\mathbf{x}$ for all $\mathbf{x} \in V$ is called a *homothety*.

If $\pi \in GL(V)$, then we can define $\bar{\pi}$ on $V/F(\pi)$ by $(\mathbf{x} + F(\pi))^{\bar{\pi}} = \mathbf{x}^\pi + F(\pi)$ for all $\mathbf{x} \in V$. An element $\pi \in GL(V)$ is called a *big dilatation* if $\bar{\pi}$ is a homothety that is distinct from the identity.

The use of homotheties is similar to that of homogeneous coordinates which is widely known in the literature. Many authors use homothety as a synonym for dilatation. It would be misleading in our context to interpret a homothety as a dilatation.

In order to describe an affine geometry, it will be necessary to designate a certain hyperplane to be the hyperplane at infinity. Since it turns out to be advantageous to consider groups of transformations whose elements π have their paths $B(\pi)$ in that hyperplane, it seems natural to denote this designated hyperplane by B.

We call the group $M = \{\pi \in GL(V);\ B^\pi = B\}$ the *preaffine subgroup* of $GL(V)$. The group $N = \{\pi \in GL(V); B(\pi) \subset B\}$ is called the *affine subgroup* of $GL(V)$. By $H(V)$ we denote the group of all central homotheties of $GL(V)$.

Clearly, N is a subgroup of M, and N is even normal in M; namely, if $\nu \in N$

and $\mu \in M$, then $B(\nu^\mu) = B(\nu)^\mu \subset B$. Also, $H(V)$ is a normal subgroup of M since $H(V)$ is even in the center of $GL(V)$.

We define an *affine dilatation* π by the following properties: $\pi \in GL(V)$ with $\dim F(\pi) = 1$, $F(\pi) \not\subset B$, and the restriction of π to B is a homothety. Clearly, every affine dilatation is contained in N.

The simple transformations in N are classified as follows:

(*i*) $F(\pi) \neq B$: *axial affinity a*) *shear*: $B(\pi) \subset F(\pi)$ *b*) *strain*: $B(\pi) \not\subset F(\pi)$,

(*ii*) $F(\pi) = B$: *translation*.

In our context, it seems to be convenient not to consider the identity as a translation.

It is well known that a translation is a product of two shears (cf. e.g. Lemma 3 in [**7**]). Also, it is clear that an element $\pi \in GL(V)$ with $\dim B(\pi) = d$ cannot be a product of fewer than d simple transformations.

In our first theorem we decompose any $\pi \in N$ into axial affinities. We shall determine the minimal number of factors needed in such a decomposition.

THEOREM 1. *Assume $\pi \in N$ and π is not a translation. If* $\dim B(\pi) = d$, *then π is a product of d axial affinities.*

Proof. If $\dim B(\pi) = 1$, then π is an axial affinity since π is not a translation. Now let $\dim B(\pi) \geqq 2$. We can choose $\mathbf{v} \notin F(\pi) \cup B$; obviously, $F(\pi) \oplus K\mathbf{v} \neq V$. Choose a hyperplane A through $F(\pi)$ such that $\mathbf{v} \notin A \neq B$ and define the linear form ψ through $v^\psi = 1$, $A^\psi = 0$. By our construction, $0 \neq \mathbf{v}^{\pi-1} \in B(\pi)$; let σ: $\mathbf{x} \rightarrow \mathbf{x} + \mathbf{x}^\psi v^{\pi-1}$. Thus $\mathbf{x}^{\sigma-1} = \mathbf{x}^\psi \mathbf{v}^{\pi-1}$. Hence $B(\sigma) = K\mathbf{v}^{\pi-1} \subset B$ and $\sigma \in N$; also, $F(\sigma) = A \neq B$. Thus σ is an axial affinity. Further, $\mathbf{v}^\pi = \mathbf{v}^\sigma$ and therefore $\mathbf{v}^{\pi\sigma^{-1}} = \mathbf{v}$. We get $B(\pi\sigma^{-1}) \subset B(\pi) \subset B$, also $F(\pi\sigma^{-1}) \supset F(\pi) + K\mathbf{v}$ and $F(\pi\sigma^{-1}) \neq B$. Therefore, $\pi\sigma^{-1} \in N$ and $\pi\sigma^{-1}$ is not a translation. Now we use induction on $\dim B(\pi)$.

We shall obtain a more interesting decomposition of elements in N in Theorem 3. As a preparation, we have to procure some technical information first.

LEMMA 2. *Assume $\pi \in N$*, codim $F(\pi) = 2$, *and $F(\pi) \subset B$. Then the following two statements are equivalent.*

a) *There is some $\mathbf{v} \in V\backslash B$ such that $\mathbf{v}^\pi - \mathbf{v} \in F(\pi)\backslash\{0\}$.*

b) *$\pi = \sigma\tau$ where σ is an axial affinity and τ a shear.*

Proof. Assume *a*. Let A denote a hyperplane with $\mathbf{v} \notin A \neq B$. Let ψ be the linear form with $\mathbf{v}^\psi = 1$, $A^\psi = 0$. Put τ: $\mathbf{x} \rightarrow \mathbf{x} + \mathbf{x}^\psi(\mathbf{v}^\pi - \mathbf{v})$. Then $F(\tau) \not\subset B$ and $\mathbf{v}^\tau = \mathbf{v}^\pi$. Therefore, $F(\pi\tau^{-1}) = F(\pi) + K\mathbf{v} \neq B$ and $\sigma = \pi\tau^{-1}$ is an axial affinity.

Assume *b*. Since $B(\pi) \subset B(\sigma) + B(\tau)$ and $F(\sigma) \cap F(\tau) \subset F(\pi)$, our assumptions yield $B(\pi) = B(\sigma) \oplus B(\tau)$ and $F(\pi) = F(\sigma) \cap F(\tau)$. Obviously, $F(\pi) \subset F(\tau) \cap B$. The hyperplanes $F(\tau)$ and B being distinct, we have codim $(F(\tau) \cap B) = 2$. Hence $F(\pi) = F(\tau) \cap B$. From our assumption, $B(\tau) \subset F(\tau) \cap B$. Thus $B(\tau) \subset F(\pi)$. Hence $B(\tau) = K\mathbf{r} \subset B(\pi) \cap F(\pi)$ for

https://doi.org/10.4153/CJM-1979-040-4 Published online by Cambridge University Press

some $\mathbf{r} \neq 0$. Consequently, there is some $\mathbf{v} \in V \backslash F(\pi)$ such that $\mathbf{r} = \mathbf{v}^\pi - \mathbf{v}$. We have $\tau: \mathbf{x} \to \mathbf{x} + \mathbf{x}^\psi(\mathbf{v}^\pi - \mathbf{v})$ for some $\psi \in V^*$ and $\tau^{-1}: \mathbf{x} \to \mathbf{x} - \mathbf{x}^\psi(\mathbf{v}^\pi - \mathbf{v})$. For $\mathbf{x} = \mathbf{v}^\pi$ we get $\mathbf{v}^{\pi\tau^{-1}} - \mathbf{v} = \mathbf{v}^\pi - \mathbf{v} - \mathbf{v}^{\pi\psi}(\mathbf{v}^\pi - \mathbf{v}) = (1 - \mathbf{v}^{\pi\psi})(\mathbf{v}^\pi - \mathbf{v}) \in B(\sigma) \cap B(\tau) = \{0\}$, hence $\mathbf{v}^\sigma - \mathbf{v} = 0$. Thus $\mathbf{v} \in F(\sigma) \backslash F(\pi)$ and $F(\sigma) = F(\pi\tau^{-1}) = F(\pi) \oplus K\mathbf{v}$. Since σ is an axial affinity, we have $F(\sigma) \neq B$. As $F(\pi) \subset B$, this yields $\mathbf{v} \notin B$.

We are going to express every $\pi \in N$ as a product of shears and at most one axial affinity, and we shall determine for each $\pi \in N$ the minimal number of factors required for a decomposition of that form. This minimal number will be called the *shear length* $\mathrm{sl}(\pi)$ of π.

A product of a translation τ and a shear σ is called a *parabolic rotation* if $B(\tau) \not\subset F(\sigma)$.

THEOREM 3. *Assume* $\pi \in N$ *and* $\dim B(\pi) = d < \infty$. *If* π *is a translation, a parabolic rotation, or a big dilatation with* $d > 1$, *then* $\mathrm{sl}(\pi) = d + 1$, *and in all other cases* $\mathrm{sl}(\pi) = d$. *For translations and parabolic rotations there are decompositions into* $d + 1$ *factors such that all factors are shears.*

Proof. As we remarked earlier, a translation is a product of two shears. Therefore, we can assume from now on that π is not a translation. If $\bar{\pi}$ is the identity, use Lemma 1 of [**7**]. If π is a big dilatation with $d > 1$, use Lemma 6 of [**7**].

From now on we assume that $\bar{\pi}$ is not a homothety. If $F(\pi) \not\subset B$ or $\operatorname{codim} F(\pi) > 2$, use Lemma 2 of [**7**].

Finally, we deal with the only remaining case, $F(\pi) \subset B$ and $\operatorname{codim} F(\pi) = 2$. By Lemma 2 of [**7**], we have $\pi = \sigma\tau$ where σ is simple and τ is a transvection with $B(\sigma), B(\tau) \subset B(\pi) \subset B$ and $F(\pi) \subset F(\sigma), F(\tau)$. (This means that τ is a translation or a shear.) If $F(\sigma), F(\tau) \neq B$, then we are finished. If $F(\sigma) = B$ and $F(\tau) \neq B$, then σ is a translation and π is a parabolic rotation; namely, if $B(\sigma) \subset F(\tau)$, then $B(\pi) = B(\sigma) + B(\tau) \subset F(\tau)$. Also, $B(\pi) \subset B = F(\sigma)$, hence $B(\pi) \subset F(\tau) \cap F(\sigma) = F(\pi)$. Hence $\bar{\pi}$ is a homothety, which contradicts the above assumption.

If $F(\sigma) \neq B$ and $F(\tau) = B$, we have to distinguish two cases. Firstly, if σ is a transvection, then π is a parabolic rotation; namely, if $B(\tau) \subset F(\sigma)$, then we obtain $B(\pi) \subset F(\pi)$, analogous to the case $F(\sigma) = B$ and $F(\tau) \neq B$. Secondly, σ is not a transvection. Then for all $\mathbf{v} \in B$ we get $\mathbf{v}^\sigma \in B$ and consequently $\mathbf{v}^{\sigma\tau} - \mathbf{v} = \mathbf{v}^\sigma - \mathbf{v} \in B(\sigma)$, but $B(\sigma) \not\subset F(\pi) = F(\tau) \cap F(\sigma)$ since $B(\sigma) \not\subset F(\sigma)$. Therefore, there is some $\mathbf{v} \in V \backslash B$ with $\mathbf{v}^\pi - \mathbf{v} \in F(\pi) \backslash \{0\}$. By Lemma 2 we get that π is a product of an axial affinity and a shear.

Now let π be a parabolic rotation, $\pi = \sigma\tau$, where σ is a shear and τ a translation. Let $\mathbf{v} \in B \backslash F(\pi)$, then $0 \neq \mathbf{v}^{\pi-1} = \mathbf{v}^{\sigma\tau-1} = \mathbf{v}^{\sigma-1} \in B(\sigma) \subset F(\sigma) \cap F(\tau) = F(\pi)$. Also, $B(\sigma) \subset B(\pi) = B(\sigma) + B(\tau)$, hence $B(\sigma) = B(\pi) \cap F(\pi)$. Consequently, if $\mathbf{w} \in V$ and $\mathbf{w}^\pi - \mathbf{w} \in F(\pi)$, then $\mathbf{w}^\pi - \mathbf{w} = \lambda(\mathbf{v}^\pi - \mathbf{v}) \in B(\pi) \cap F(\pi)$ with $\lambda \in K$. Hence $(\mathbf{w} - \lambda\mathbf{v})^\pi = \mathbf{w} - \lambda\mathbf{v} \in F(\pi) \subset B$ and

https://doi.org/10.4153/CJM-1979-040-4 Published online by Cambridge University Press

therefore $\mathbf{w} \in B$. Consequently, there is no $\mathbf{w} \in V \backslash B$ with $\mathbf{w}^{\pi} - \mathbf{w} \in F(\pi) \backslash \{0\}$. This implies that a parabolic rotation cannot be written as a product of an axial affinity and a shear. Using Lemma 3 in [**7**], we get that every parabolic rotation can be written as a product of three shears.

3. The affine group. We shall introduce the projective and the affine group. Our main observation is contained in Theorem 5. It states that for Pappian geometries the affine group is isomorphic to the affine subgroup N of the general linear group. Analytically, the Pappian geometries are those with a commutative field of coordinates. Since we have solved the length problem for this group N in the preceding section, we now also know the answer to the length problem for the affine group.

The *projective general linear group* P is the quotient group of the general linear group modulo its center: $P = PGL(V) = GL(V)/H(V)$.

Every Desarguesian affine geometry can be described by a vector space V together with a distinguished hyperplane B representing the hyperplane at infinity. Then the *affine group* A is the quotient group of the preaffine subgroup of the general linear group modulo the group of homotheties: $A = M/H(V)$.

We start with an easy but useful observation.

LEMMA 4. *The mapping $\nu \rightarrow \nu \cdot H(V)$ of N into A is injective.*

Proof. Let $\nu \in N$, $\nu \cdot H(V) = H(V)$, thus $\nu \in H(V) \cap N$. Since $B(\nu') = V$ for every $\nu' \in H(V) \backslash \{\mathrm{id}\}$, we have $\nu = \mathrm{id}$.

We can now proceed to the following result.

THEOREM 5. *If K is commutative, then $A \cong N$.*

Proof. It is sufficient to show that the mapping $\nu \rightarrow \nu \cdot H(V)$ of N into A is surjective; cf. Lemma 4. Let $\mu \in M$. We shall find some $\nu \in N$ such that $\mu \in \nu \cdot H(V)$. Let $\mathbf{v} \in V \backslash B$. Then $\mathbf{v}^{\mu} = \alpha \mathbf{v} + \mathbf{b}$ with $\alpha \in K$ and $\mathbf{b} \in B$, where $\alpha \neq 0$ since $\alpha = 0$ together with $B^{\mu} \subset B$ implies $V^{\mu} \subset B$. Since K is commutative, η: $\mathbf{x} \rightarrow \alpha^{-1}\mathbf{x}$ for all $\mathbf{x} \in V$ is a homothety. Let $\nu = \mu\eta$; thus $\mu \in \nu \cdot H(V)$. Obviously, $\mathbf{b}'^{\nu} - \mathbf{b}' \in B$ for every $\mathbf{b}' \in B$. Also $\mathbf{v}^{\nu} - \mathbf{v} = \alpha^{-1}\mathbf{b} \in B$. Hence $B(\nu) \subset B$ and $\nu \in N$.

With the aid of Theorem 5 we can interpret Theorem 3 immediately as a length theorem for a Pappian affine space. It only remains to establish a bijection of the axial affinities in A and the axial affinities in N. This will in retrospect justify the geometric names for certain simple transformations in N.

Let $P(V)$ denote the projective space over V and $P(B)$ the infinite hyperplane determined by B. It determines the affine space $P(V) \backslash P(B)$. Let $\pi \cdot H(V) \in A$ denote an "axial affinity" of that space and $P(F)$ for some hyperplane F of V the axis of $\pi \cdot H(V)$. Then $P(F) \not\subset P(B)$ and hence $F \not\subset B$. By Theorem 5, there is exactly one $\sigma \in \pi \cdot H(V) \cap N$. Since $\sigma \in \pi \cdot H(V)$,

https://doi.org/10.4153/CJM-1979-040-4 Published online by Cambridge University Press

there is a $\lambda \in K\backslash\{0\}$ such that $\mathbf{y}^\sigma = \lambda\mathbf{y}$ or $\mathbf{y}^\sigma - \mathbf{y} = (\lambda - 1)\mathbf{y}$ for all $\mathbf{y} \in F$. Since $\sigma \in N$, we have $B(\sigma) \subset B$. As $\lambda \neq 1$ would imply $F \subset B(\sigma) \subset B$, we have $\lambda = 1$ and $F = F(\sigma)$. Hence σ is simple. Since $F(\sigma) \not\subset B$, σ is an axial affinity.

4. Decomposition with prescribed factors in the affine subgroup of $GL(V)$. We recall a concept that was introduced in [**8**]. Let $\epsilon \in K\backslash\{0\}$. A simple transformation ρ is called an *ϵ-dilatation* if there is some $\mathbf{r} \in V\backslash\{0\}$ and some $\psi \in V^*$ such that ρ: $\mathbf{x} \to \mathbf{x} + \mathbf{x}^\psi\mathbf{r}$ for all $\mathbf{x} \in V$ and $1 + \mathbf{r}^\psi = \epsilon$. Here the field element ϵ is only determined up to conjugates.

Clearly, $\epsilon = 1$ identifies the transvections among all ϵ-dilatations; $\epsilon = -1$ identifies all reflections.

In [**8**], we obtained a decomposition of every $\pi \in GL(V)$ into ϵ_i-dilatations with given ϵ_i. It turned out that we can prescribe all but one ϵ_i of the dilatations that represent π. Let us call $m(\pi)$ the minimal number of factors needed in this decomposition. We get $m(\pi) = \dim B(\pi)$ most of the time. The exception occurs when $\bar{\pi}$ is a homothety: $\mathbf{x} + F(\pi) \to \lambda\mathbf{x} + F(\pi)$ for all $\mathbf{x} \in V$, where one prescribed factor of π is an ϵ_i-dilatation with $\epsilon_i \neq \lambda$. Then $m(\pi) = \dim B(\pi) + 1$.

We shall see in this section that for elements in N we can obtain a decomposition of π into $m(\pi)$ axial affinities. The results in [**8**] are strong enough to provide most of the answers. We only have to learn more about the one ϵ-dilatation in the decomposition whose value ϵ could not be prescribed. The following two lemmas will deal with the described difficulties.

LEMMA 6. *Assume $\pi = \tau\rho$ where τ is a translation and ρ an ϵ-dilatation in N with $\epsilon \neq 1$. If $B(\tau) \neq B(\rho)$, then there is some $\mathbf{v} \in V\backslash B$ such that $\mathbf{v}^\pi - \mathbf{v} \notin K\mathbf{v} + F(\pi)$.*

Proof. Since $F(\tau) = B$ and $F(\rho) \neq B$, we get $\dim B(\pi) = 2$. Consequently, $B(\pi) = B(\tau) + B(\rho)$ and $F(\pi) = F(\tau) \cap F(\rho) \subset B$. We conclude $B(\pi) \not\subset F(\pi)$; namely, $B(\pi) \subset F(\pi)$ implies $B(\rho) \subset F(\rho)$. Now let C be a complement of $F(\pi)$ with $C \cap B = \{0\}$. Then $C^{\pi-1} = B(\pi)$. Consequently, there is some $\mathbf{v} \in V\backslash B$ with $\mathbf{v}^\pi - \mathbf{v} \notin F(\pi)$. But then also $\mathbf{v}^\pi - \mathbf{v} \notin K\mathbf{v} + F(\pi)$ since $\mathbf{v}^\pi - \mathbf{v} \in B(\pi) \subset B$ and the hyperplanes $F(\pi) \oplus K\mathbf{v}$ and B through $F(\pi)$ are distinct.

LEMMA 7. *Let $\pi \in N$, $\dim B(\pi) = 2$, and $\epsilon \in K\backslash\{0\}$. Assume there is some $\mathbf{v} \in V\backslash B$ such that $\mathbf{v}^\pi - \mathbf{v} \notin K\mathbf{v} + F(\pi)$. Then $\pi = \sigma\rho$ where σ is an axial affinity and ρ an ϵ-dilatation in N.*

Proof. Put $\mathbf{r} = \mathbf{v}^\pi - \mathbf{v}$. By our assumptions, $V = F(\pi) \oplus K\mathbf{v} \oplus K\mathbf{r}$. Hence $\mathbf{v}^\psi = 1$, $F(\pi)^\psi = 0$, $\mathbf{r}^\psi = \epsilon - 1$ defines $\psi \in V^*$, and ρ: $\mathbf{x} \to \mathbf{x} + \mathbf{x}^\psi\mathbf{r}$ is an ϵ-dilatation in N. Obviously, $\mathbf{v}^\rho = \mathbf{v}^\pi$. Put $\sigma = \pi\rho^{-1}$. Thus $F(\sigma) = F(\pi) \oplus K\mathbf{v}$, and σ is simple. Also, $F(\sigma) \neq B$ since $\mathbf{v} \notin B$, $B(\sigma) \subset B(\pi) + B(\rho) = B(\pi) \subset B$, therefore $\sigma \in N$, and σ is an axial affinity.

With the help of these two lemmas we shall establish a decomposition for every $\pi \in N$.

THEOREM 8. *Let* $\pi \in N$ *with* $\dim B(\pi) = d < \infty$ *and* $\epsilon_i \in K\backslash\{0, 1\}$, $i = 2, \ldots, d+1$. *Then there are axial affinities* $\rho_1, \ldots, \rho_t$ *such that* ρ_i *are* ϵ_i-*dilatations for* $i = 2, \ldots, t$ *and* $\pi = \rho_1\rho_2 \ldots \rho_t$, *where* $t = m(\pi)$ *if* π *is not a translation and* $t = 2$ *if* π *is a translation.*

Proof. If π is a translation, then we can decompose π into an ϵ_2-dilatation ρ_2 and some transformation ρ_1, where $B(\rho_2) = B(\pi)$ and $F(\rho_2) \neq B$. Clearly, then $\rho_1 = \pi\rho_2^{-1}$ is also simple with $B(\rho_1) = B(\pi)$ and $F(\rho_1) \neq B$.

Now assume π is not a translation. According to Lemmas 1 to 3 of [**8**], there is a decomposition $\pi = \rho_1'\rho_2' \ldots \rho_{m(\pi)}'$ where ρ_i' is an ϵ_i-dilatation for $i \geqq 2$, and $B(\rho_i') \subset B(\pi)$ and $F(\pi) \subset F(\rho_i')$ for all i. Since $\pi \in N$, we have $B(\pi) \subset B$ and therefore $B(\rho_i') \subset B$ for all i. Since $\epsilon_i \neq 1$, we have $F(\rho_i') \neq B$ for $i \geqq 2$. Thus $\rho_2', \ldots, \rho_{m(\pi)}'$ are axial affinities. We put $\rho_3 = \rho_3', \ldots, \rho_{m(\pi)} = \rho_{m(\pi)}'$. If ρ_1' is an axial affinity, we also put $\rho_1 = \rho_1'$ and $\rho_2 = \rho_2'$. Finally, let ρ_1' be a translation. Put $\pi' = \rho_1'\rho_2'$. Since $m(\pi)$ is minimal, we have $\dim B(\rho_1'\rho_2') = 2$, hence $B(\rho_1') \neq B(\rho_2')$. By Lemmas 6 and 7, π' permits a decomposition $\pi' = \rho_1\rho_2$ where ρ_1, ρ_2 are axial affinities and ρ_2 is an ϵ_2-dilatation. This completes the proof.

We note the following special cases. Choosing $\epsilon_i = -1$ for $i = 2, \ldots, m(\pi)$, we obtain a decomposition of π into $m(\pi) - 1$ reflections and one axial affinity. Let us recall from [**8**] that always $m(\pi) = \dim B(\pi)$ or $m(\pi) = \dim B(\pi) + 1$ (cf. paragraph 3 of this section). If we assume that $\dim V = n$, then we learn from our theorem that every $\pi \in N$ yields a decomposition with at most n factors. If we in addition assume that the field K is commutative, then our statement is true for all affinities. Thus we have considerably improved J. Ch. Fisher's result ([**9**], p. I: 4-2, Lemma 4.3). In his decomposition, Fisher needs about twice as many factors as we do. It is obvious from [**8**] that the commutativity of K is not needed if we are only interested in decompositions of elements in the general linear group.

5. Affine hyperreflection groups. In this section we assume that K is commutative.

If ϵ is a primitive mth root of unity in K, then we call an ϵ-dilatation a *hyperreflection*. In [**8**], we have seen that the hyperreflection group

$$G_m = \{\pi \in GL(V);\ \dim B(\pi) < \infty \text{ and } (\det \pi)^m = 1\}$$

is generated by hyperreflections, and we solved the length problem for hyperreflection groups. Let us denote this length by $\mathrm{hl}(\pi)$.

Now we introduce the affine counterpart of G_m, the *affine hyperreflection group* $G_m' = \{\pi \in G_m;\ B(\pi) \subset B\}$. The solution of the length problem for this group follows immediately from the analogous results on groups G_m.

THEOREM 9. *Let $\pi \in G_m'$ and $m \neq 1$. Then π is a product of* $\mathrm{hl}(\pi)$ *affine hyperreflections in G_m'.*

Proof. We have $\epsilon \neq 1$ since $m \neq 1$. Consequently, all ϵ-dilatations in Theorem 7 in [**8**] are already affinities.

The specialization to the reflection group G_2' is now easily obtained from Corollary 9 in [**8**].

6. Hyperreflection groups over skewfields. In this final section, we shall briefly discuss how hyperreflection groups can be defined over skewfields, and we shall state the solutions of the length problem for both of the groups G_m and G_m'.

We need the following generalization of a well-known lemma; cf. [**11**].

LEMMA 10. *Let V be a vector space of dimension n over the field K and $\pi \in GL(V)$. Then* $\det \pi = \det_{B(\pi)}|\pi$.

Proof. Let $b_1, \ldots, b_k$ be a basis of $B(\pi)$ and $c_{k+1}, \ldots, c_n$ a basis of a complement of $B(\pi)$. For every $x \in V$ we have $x^\pi = x + b$ where $b \in B(\pi)$. With respect to the basis $b_1, \ldots, b_k, c_{k+1}, \ldots, c_n$, the matrix of π has the form

$$\left(\begin{array}{c|c} D & O \\ \hline C & I \end{array}\right)$$

hence $\det \pi = \det DI = \det D$ (cf. [**1**], p. 156).

In the same way as in [**8**], we extend the concept of a determinant to determinants of $\pi \in GL(V)$ with finite dimensional path, even if the dimension of V is infinite. We define $\det \pi = \det_{B(\pi)}|\pi$. This determinant function is a homomorphism [**8**].

We can now introduce hyperreflection groups also over skewfields.

Let $C(K^*)$ be the commutator subgroup of K^* and Γ a cyclic subgroup of $K^*/C(K^*)$. Assume the order of Γ is $m \neq 1$ and γ is a generator of Γ. We define $G_m = \{\pi \in GL(V);\ \dim B(\pi) < \infty \text{ and } \det \pi \in \Gamma\}$ and $G_m' = G_m \cap N$.

We have to make one final definition. *A κ-dilatation* for $\kappa \in K^*/C(K^*)$ is a dilatation ρ with $\det \rho = \kappa$.

If $\epsilon \in K^*$, $\epsilon \cdot C(K^*) = \kappa$, and if ρ is an ϵ-dilatation, then obviously ρ is also a κ-dilatation.

If γ is, as above, the generator of the cyclic group Γ, then γ is clearly distinct from $1 \cdot C(K^*) \in K^*/C(K^*)$ since $m \neq 1$.

With these remarks, it is now possible to solve the length problem for the hyperreflection groups just introduced. The result is a theorem that is analogous to Theorem 7 in [**8**]. A proof along the same line as the proof of that theorem offers no difficulties. In the same way we obtain a length theorem that is analogous to Theorem 9 of the preceding section.

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