Can. J. Math., Vol. XXXVIII, No. 6, 1986, pp. 1380-1458

HOPF ALGEBRAS AND PROJECTIVE REPRESENTATIONS OF $G \wr S_n$ AND $G \wr A_n$

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In 1911, Schur published a rather formidable paper **[9]** in which he determined all the complex projective characters for the symmetric group (denoted Σ_n here, despite the title), and for the alternating group A_n (A pronounced "alpha"). As far as we know, the construction of the modules involved is still an unsolved problem. The results of Schur can be expressed in terms of certain induced representations whose characters form a basis for the group of virtual characters, plus formulae expressing the irreducible characters in terms of these induced characters. Here we give a new formulation of the above induced characters in the spirit of the well known "induction algebra" approach to the linear representations of Σ_n. We use some Hopf algebra techniques inspired by **[5]** to give new proofs of Schur's results, and to determine the extra structure which we define.

The main object is a $\mathbf{Z}/2 \times \mathbf{N}$-graded ring

$$\{\bar{H}_{i,j}:\quad i \in \mathbf{Z}/2, j \in \mathbf{N}\}$$

such that in the simplest case:

$\bar{H}_{1,j}$ = group generated by the projective Σ_j-representations with the unique non-trivial cocycle (for $j \geqq 4$);

$\bar{H}_{0,j}$ = group generated by the projective A_j-representations corresponding to the restriction to A_j of the above cocycle for $j \geqq 4$. (This cocycle is also non-zero and is the only one in the Schur multiplier except when $j = 6$ and 7.)

Our results are much more general than stated above in that this machinery works for all monomial groups (i.e., wreath products) $\Gamma \wr \Sigma_j$ and "even monomial" groups $\Gamma \wr A_j$, again for the projective representations corresponding to the cocycle pulled back from the non-trivial cocycle above. Here Γ may be any finite group. We compute $\bar{H}$ as a function of the representation ring of Γ. At this point, we note that such projective representations bijectively correspond in a natural way to "negative" linear representations of covering groups $\tilde{\Sigma}_j\langle\Gamma\rangle$ and $\tilde{A}_j\langle\Gamma\rangle$ defined in Section 1. Thus, in the body of the paper, we deal only with linear representations. The word "negative" above refers to the fact that

Received January 7, 1985.

the central involution z in the kernel of the covering projections

$$\tilde{\Sigma}_j\langle\Gamma\rangle \to \Gamma \wr \Sigma_j$$

should act as -1 on the modules. This emphasis on linear representations of covering groups also eliminates anomalies corresponding to the cases $j < 4$. Here they are treated just as for $j \geqq 4$, but are not interpretable as projective representations.

A second source of anomalies corresponding to the cases $j = 0, 1$ where $\Sigma_j = A_j$ is eliminated by use of $\mathbf{Z}/2$-graded representations, studied in Section 2. Applied to $\tilde{\Sigma}_j\langle\Gamma\rangle$, these yield the representations of $\tilde{A}_j\langle\Gamma\rangle$ for $j > 1$. A more important reason for them is to simplify the definitions and properties of the four binary operations which give the multiplication in $\bar{H}$. Any group equipped with a sign homomorphism $x \mapsto (-1)^{s(x)}$, where $s(x) \in \mathbf{Z}/2$, has $\mathbf{Z}/2$-graded representations. A general technical result, 2.24 below, is the determination of the irreducible negative $\mathbf{Z}/2$-graded and ungraded representations for $\Gamma \hat{\times} \Lambda$ in terms of those for Γ and Λ, using certain twisted tensor products. Here Γ and Λ are groups equipped with "s" and "z", and $\hat{\times}$ is a twisted product of such objects from Section 1.

In Section 3 the main result is stated as two theorems. In 3.4, we list the extra structure and its properties on the $\mathbf{Z}/2 \times \mathbf{N}$-graded group $\bar{H}$, where

$$\bar{H}_{0,j} = GR^-(\tilde{\Sigma}_j\langle\Gamma\rangle) \quad \text{(negative graded representations)}$$

$$\bar{H}_{1,j} = R^-(\tilde{\Sigma}_j\langle\Gamma\rangle) \quad \text{(negative representations).}$$

In 3.3, whose proof occupies all of Section 4, we state a formal result about how a graded ring as in 3.4 must have a certain structure. Sections 5, 6 and 7 give the proof of 3.4. In 5, we state facts about Clifford modules [**1**] needed for the basic representations which generate $\bar{H}$ as a ring. In 6, we give a slightly abstract version of how a Hopf algebra structure, central to the method of proof of 3.3, can arise in this context. Finally in 7 we verify the properties claimed for $\bar{H}$ in 3.4. The last section shows how to get the irreducible representations as linear combinations of the natural basis for $\bar{H}$ of induced representations. The case Γ trivial gives Schur's results [**9**]. Our method relies on an easy generalization of a very difficult identity in [**9**]. We use and do not reprove the special case given there.

Here we have not produced the character formulae implicit in our results, nor have we reformulated our general constructions in terms of matrices and characters. This can be done. We hope in the future to produce an expository tract on this subject, hopefully including reformulations of some of the fairly small number of post-Schur papers in this field, representations over fields other than $\mathbf{C}$, and other questions. It follows, for example, from these results that the splitting field for the collection $\{\tilde{\Sigma}_j\}$ is the field of straight-edge and compass constructible numbers (for characteristic zero representations).

We wish to thank David Jackson, Adelbert Kerber and Alun Morris for helpful comments on parts of this material.

Although not referred to in the body of the text, we have found very useful the papers [7] and [**8**], among others.

1. Groups enriched with sign and involution. Let $\mathcal{G}$ denote the category with objects (Γ, z, s), where Γ is a group, z is an element of order 2 in the centre of Γ, and $s:\Gamma \to \mathbf{Z}/2$ is a homomorphism with $s(z) = 0$. Often Ker s will be denoted Γ_0, and $\Gamma - \Gamma_0$ denoted Γ_1. Morphisms in $\mathcal{G}$ will be group homomorphisms preserving z and commuting with s. Objects will often be ambiguously denoted simply Γ, with z_Γ and s_Γ specifying the other structure.

Definition 1.1. For objects Γ and Γ' in $\mathcal{G}$, $\Gamma \tilde{\times} \Gamma'$ will denote the Cartesian product $\Gamma \times \Gamma'$ together with the "twisted" multiplication

$$(m, m')(l, l') = (z^{s(m')s(l)}ml, m'l').$$

A straightforward calculation yields

PROPOSITION 1.2. *$\Gamma \tilde{\times} \Gamma'$ is a group, and has subgroup* $\{ (1, 1'), (z, z') \}$ *contained in its centre.*

Definition 1.3. As a group, define

$$\Gamma \hat{\times} \Gamma' = (\Gamma \tilde{\times} \Gamma')/\{ (1, 1'), (z, z') \}.$$

Note that a group canonically isomorphic to $\Gamma \hat{\times} \Gamma'$ would have been obtained had we chosen the twisted multiplication with factor $z'^{s(m')s(l)}$ in the second component. Elements of $\Gamma \hat{\times} \Gamma'$ will be denoted simply as ordered pairs (g, g'). To make $\Gamma \hat{\times} \Gamma'$ into an object in $\mathcal{G}$, use the element $(z, 1')$ (which equals $(1, z')$ in $\Gamma \hat{\times} \Gamma'$), and the map

$$s_{\Gamma \hat{\times} \Gamma'}:\Gamma \hat{\times} \Gamma' \to \mathbf{Z}/2$$
$$(g, g') \mapsto s(g) + s(g').$$

The definitions are easily checked, yielding

PROPOSITION 1.4. *$\Gamma \hat{\times} \Gamma'$ is a $\mathcal{G}$-object; the maps*

$$\begin{array}{ccc} \Gamma \to \Gamma \hat{\times} \Gamma' & & \Gamma' \to \Gamma \hat{\times} \Gamma' \\ & \textit{and} & \\ g \mapsto (g, 1') & & g' \mapsto (1, g') \end{array}$$

are $\mathcal{G}$-morphisms mapping onto normal subgroups.

Note that cokernels of these maps are isomorphic to $\Gamma'/\{1, z'\}$ and $\Gamma/\{1, z\}$ respectively, and are not objects in $\mathcal{G}$; that the usual projections are not well defined on $\Gamma \hat{\times} \Gamma'$; and that

$$\#(\Gamma \hat{\times} \Gamma') = \frac{1}{2}(\#\Gamma)(\#\Gamma').$$

PROPOSITION 1.5. *There are* $\mathcal{G}$*-isomorphisms*

i) $\Gamma \hat{\times} \Gamma' \xrightarrow{\tau} \Gamma' \hat{\times} \Gamma$ *given by*

$(g, g') \mapsto (g', z^{sg \cdot sg'} g)$;

ii) $(\Gamma \hat{\times} \Gamma') \hat{\times} \Gamma'' \xrightarrow{\alpha} \Gamma \hat{\times} (\Gamma' \hat{\times} \Gamma'')$ *given by*

$[(g, g'), g''] \mapsto [g, (g', g'')]$.

Proof. It is easy to check that the maps are well defined and are $\mathcal{G}$-morphisms. Their inverses have obvious formulae, and are also in $\mathcal{G}$.

Note that in i), $(g, g') \mapsto (g', g)$ will not work, and that in ii), both groups are isomorphic to $\Gamma \times \Gamma' \times \Gamma''/N$, where $\Gamma \times \Gamma' \times \Gamma''$ has a suitable twisted multiplication and

$$N = \{(1, 1', 1''), (1, z', z''), (z, 1', z''), (z, z', 1'')\}.$$

Iterating, there is a $\mathcal{G}$-object $\Gamma_1 \hat{\times} \Gamma_2 \hat{\times} \dots \hat{\times} \Gamma_n$ determined up to a unique isomorphism by $\mathcal{G}$-objects Γ_i. One makes $\hat{\times}$ into a covariant functor in the obvious way. Then τ and α are natural transformations.

Example 1.6. Recall that the symmetric group Σ_n can be presented with generators $\{(12), (23), \dots, (n-1\ n)\}$ and relations

$$(i\ i+1)^2 = [(i\ i+1) \cdot (i+1\ i+2)]^3 = 1 \quad \text{and}$$

$$(i\ i+1) \cdot (j\ j+1) = (j\ j+1) \cdot (i\ i+1) \quad \text{for } j > i+1.$$

Define $\tilde{\Sigma}_n$ to be the group with generators $\{z, t_1, t_2, \dots, t_{n-1}\}$ subject to relations $z^2 = 1$; $t_i^2 = (t_i t_{i+1})^3 = z$; and $t_i t_j = z t_j t_i$ for $j > i + 1$. We get a $\mathcal{G}$-object using the given z and determining s by $s(z) = 0$ and $s(t_i) = 1$. There is an epimorphism

$$\tilde{\Sigma}_n \xrightarrow{\theta_n} \Sigma_n$$

determined by $\theta_n(t_i) = (i\ i+1)$. Then

$$\operatorname{Ker} \theta_n = \{1, z\},$$

so $\#\tilde{\Sigma}_n = n! \cdot 2$, and s is the composition of θ_n with the parity homomorphism.

We now generalize the covering $\tilde{\Sigma}_n$ of Σ_n to a covering $\tilde{\Sigma}_n\langle\Gamma\rangle$ of the monomial group $\Sigma_n\langle\Gamma\rangle$ (see **[3]**). The latter is more often denoted as a wreath product $\Gamma \wr \Sigma_n$. It has elements $(g_1, \dots, g_n; u)$ with $g_i \in \Gamma$, $u \in \Sigma_n$. We define

$$\tilde{\Sigma}_n\langle\Gamma\rangle = \{ (g_1, \ldots, g_n; \nu):g_i \in \Gamma, \nu \in \tilde{\Sigma}_n\}$$

with multiplication

$$(g_1, \ldots, g_n; \nu) \cdot (g'_1, \ldots, g'_n; \nu')$$
$$= (g_1 g'_{\theta(\nu)^{-1}(1)}, \ldots, g_n g'_{\theta(\nu)^{-1}(n)}; \nu\nu').$$

It is easily checked that this gives a group of order $(\#\Gamma)^n \cdot n! \cdot 2$ which double covers $\Sigma_n\langle\Gamma\rangle$ using

$$(g_1, \ldots, g_n; \nu) \mapsto (g_1, \ldots, g_n; \theta_n(\nu)).$$

This covering epimorphism will also be denoted θ_n or just θ. There is a normal subgroup Γ^n of $\tilde{\Sigma}_n\langle\Gamma\rangle$, and a split epimorphism $\tilde{\Sigma}_n\langle\Gamma\rangle \to \tilde{\Sigma}_n$, given using respectively the first "n" coordinates and the last one. We make $\tilde{\Sigma}_n\langle\Gamma\rangle$ into a $\mathcal{G}$-object using the element $(1, \ldots, 1; z)$ and the map

$$s_{\tilde{\Sigma}_n\langle\Gamma\rangle}:\tilde{\Sigma}_n\langle\Gamma\rangle \to \tilde{\Sigma}_n \xrightarrow{s_{\tilde{\Sigma}_n}} \mathbf{Z}/2.$$

Now we generalize the well known Young subgroups as follows.

Definition 1.7. Define $\phi = \phi_{a,b}\langle\Gamma\rangle$ as follows:

$$\phi_{a,b}\langle\Gamma\rangle:\tilde{\Sigma}_a\langle\Gamma\rangle \hat{\times} \tilde{\Sigma}_b\langle\Gamma\rangle \to \tilde{\Sigma}_{a+b}\langle\Gamma\rangle$$
$$[(g_1, \ldots, g_a; 1), (1, \ldots, 1; 1)] \mapsto (g_1, \ldots, g_a 1, 1, \ldots, 1; 1)$$
$$[(1, 1, \ldots, 1; 1), (g_1, \ldots, g_b; 1)] \mapsto (1, \ldots, 1, g_1, \ldots, g_b; 1).$$

For $1 \leqq i < a$

$$[(1, 1, \ldots, 1; t_i); (1, 1, \ldots, 1; 1)] \mapsto (1, \ldots, 1, \ldots, 1; t_i).$$

For $1 \leqq j < b$

$$[(1, \ldots, 1; 1); (1, \ldots, 1; t_i)] \mapsto (1, \ldots, 1; t_{a+i}).$$

PROPOSITION 1.8. i) *There is a unique homomorphism satisfying the formulae in* 1.7, *and it is a* $\mathcal{G}$*-map.*

ii) *Letting*

$$u_{a,b} = (t_1 t_2 \ldots t_{a+b-1})^b \in \tilde{\Sigma}_{a+b} \quad \textit{and}$$
$$\zeta_{a,b}(x) = z^{abs(x)}\iota[(1, 1, \ldots, 1; u_{a,b})](x),$$

where $\iota(y)(x) = yxy^{-1}$, *we have a commutative diagram*

$$\begin{array}{ccc} \tilde{\Sigma}_a\langle\Gamma\rangle \hat{\times} \tilde{\Sigma}_b\langle\Gamma\rangle & \xrightarrow{\tau} & \tilde{\Sigma}_b\langle\Gamma\rangle \hat{\times} \tilde{\Sigma}_a\langle\Gamma\rangle \\ \downarrow \phi_{a,b} & & \downarrow \phi_{b,a} \\ \tilde{\Sigma}_{a+b}\langle\Gamma\rangle & \xrightarrow{\zeta_{a,b}} & \tilde{\Sigma}_{b+a}\langle\Gamma\rangle \end{array}$$

https://doi.org/10.4153/CJM-1986-070-1 Published online by Cambridge University Press

iii) *The following commutes*

$$\begin{tikzcd}
(\tilde{\Sigma}_a\langle\Gamma\rangle \hat{\times} \tilde{\Sigma}_b\langle\Gamma\rangle) \hat{\times} \tilde{\Sigma}_c\langle\Gamma\rangle \arrow[r, "\phi_{a,b} \hat{\times} 1"] \arrow[d, "\alpha"', "\cong"] & \tilde{\Sigma}_{a+b}\langle\Gamma\rangle \hat{\times} \tilde{\Sigma}_c\langle\Gamma\rangle \arrow[dd, "\phi_{a+b,c}"] \\
\tilde{\Sigma}_a\langle\Gamma\rangle \hat{\times} (\tilde{\Sigma}_b\langle\Gamma\rangle \hat{\times} \tilde{\Sigma}_c\langle\Gamma\rangle) \arrow[d, "1 \hat{\times} \phi_{b,c}"'] & \\
\tilde{\Sigma}_a\langle\Gamma\rangle \hat{\times} \tilde{\Sigma}_{b+c}\langle\Gamma\rangle \arrow[r, "\phi_{a,b+c}"] & \tilde{\Sigma}_{a+b+c}\langle\Gamma\rangle
\end{tikzcd}$$

iv) *If* $\phi_{a,b}$ *denotes the usual embedding of generalized Young subgroups,* (*see* [3] *where these are denoted* ϕ), *then the diagram below is commutative.*

$$\begin{tikzcd}
\tilde{\Sigma}_a\langle\Gamma\rangle \tilde{\times} \tilde{\Sigma}_b\langle\Gamma\rangle \arrow[r] \arrow[d, "\theta \times \theta"'] & \tilde{\Sigma}_a\langle\Gamma\rangle \hat{\times} \tilde{\Sigma}_b\langle\Gamma\rangle \arrow[r, "\phi_{a,b}"] \arrow[dl, dashed] & \tilde{\Sigma}_{a+b}\langle\Gamma\rangle \arrow[d, "\theta"] \\
\Sigma_a\langle\Gamma\rangle \times \Sigma_b\langle\Gamma\rangle \arrow[rr, "\underset{\sim}{\phi}_{a,b}"] & & \Sigma_{a+b}\langle\Gamma\rangle
\end{tikzcd}$$

Proof. Elements $(g_1, \ldots, g_n;\ 1)$ and $(1, \ldots, 1;\ t_i)$ generate $\tilde{\Sigma}_n\langle\Gamma\rangle$ subject to obvious relations, so the formulae certainly determine at most one homomorphism. Checking the relations shows that there is such a homomorphism, and it sends z to z. Analysis of each diagram is done by checking on generators as given in the definition. Only ii) requires some extra comment. We use the following calculations in $\tilde{\Sigma}_{a+b}$: Let

$$r = t_1 t_2 \ldots t_{a+b-1}, \quad \text{and}$$

$$v = t_1 t_2 \ldots t_{a+b-2} t_{a+b-1} t_{a+b-2} \ldots t_2 t_1.$$

Then

$$r t_i r^{-1} = z^{a+b+1} t_{i+1} \quad \text{for } 1 \leqq i < a + b - 1$$

$$r t_{a+b-1} r^{-1} = z^{a+b} v$$

$$r v r^{-1} = z^{a+b} t_1.$$

It follows easily, since $u_{a,b} = r^b$, that

$$u t_i u^{-1} = z^{ab} t_{b+i} \quad \text{for } 1 \leqq i < a$$

$$u t_{a+j} u^{-1} = z^{ab} t_j \quad \text{for } 1 \leqq j < b,$$

as required.

https://doi.org/10.4153/CJM-1986-070-1 Published online by Cambridge University Press

Note that

$$(-1)^{s(u_{a,b})} = [(-1)^{a+b-1}]^b = (-1)^{ab}.$$

Also using iii), if we iterate we get maps

$$\phi_{n_1,n_2,\dots} : \tilde{\Sigma}_{n_1}\langle\Gamma\rangle \hat{\times} \tilde{\Sigma}_{n_2}\langle\Gamma\rangle \hat{\times} \dots \to \tilde{\Sigma}_{n_1+n_2+\dots}\langle\Gamma\rangle$$

which are well defined modulo composing with the unique natural isomorphisms between different choices of "bracketing" the domain. By an iteration of iv), these maps have images which are the liftings of the generalized Young subgroups of $\Sigma_{n_1+n_2\dots}\langle\Gamma\rangle$. In [**9**, VII] the group $\mathcal{L}_{\nu_1,\nu_2,\dots}$ is the image of $\phi_{\nu_1,\nu_2,\dots}$ above (for Γ trivial).

PROPOSITION 1.9. *Given non-negative integers i_1, i_2, j_1, j_2, set $a = i_1 + i_2$, $b = j_1 + j_2$, $c = i_1 + j_1$, $d = i_2 + j_2$ and $n = a + b = c + d$. Let*

$$w = w(i_1, i_2, j_1, j_2) = (t_{i_1+1}t_{i_1+2} \dots t_{a+j_1-1})^{i_2} \in \tilde{\Sigma}_n.$$

Then the following diagram commutes

$$\begin{tikzcd}
 & \tilde{\Sigma}_{i_1}\langle\Gamma\rangle \hat{\times} \tilde{\Sigma}_{i_2}\langle\Gamma\rangle \hat{\times} \tilde{\Sigma}_{j_1}\langle\Gamma\rangle \hat{\times} \tilde{\Sigma}_{j_2}\langle\Gamma\rangle \arrow[dl, "\phi_{i_1,i_2} \hat{\times} \phi_{j_1,j_2}"'] \arrow[dr, "(\phi_{i_1,j_1} \hat{\times} \phi_{i_2,j_2})(1 \hat{\times} \tau \hat{\times} 1)"] & \\
\tilde{\Sigma}_a\langle\Gamma\rangle \hat{\times} \tilde{\Sigma}_b\langle\Gamma\rangle \arrow[d, "\phi_{a,b}"'] & & \tilde{\Sigma}_c\langle\Gamma\rangle \hat{\times} \tilde{\Sigma}_d\langle\Gamma\rangle \arrow[d, "\phi_{c,d}"] \\
\tilde{\Sigma}_n\langle\Gamma\rangle \arrow[rr] & & \tilde{\Sigma}_n\langle\Gamma\rangle \\
x \arrow[rr] & & z^{j_1 i_2 s(x)}[\iota(w)](x)
\end{tikzcd}$$

where w is identified with $(1, \dots, 1; w)$.

Furthermore, for a given a, b, c, d with $a + b = c + d$, the elements $w(i_1, i_2, j_1, j_2)$, as we vary over all matrices

$$\begin{bmatrix} i_1 & i_2 \\ j_1 & j_2 \end{bmatrix}$$

with row sums $\begin{pmatrix} a \\ b \end{pmatrix}$ and column sums (c, d), are a complete set of double coset representatives for $\tilde{\Sigma}_n\langle\Gamma\rangle$ with respect to the images of $\phi_{a,b}$ and $\phi_{c,d}$, i.e., $\tilde{\Sigma}_n\langle\Gamma\rangle$ is the disjoint union

$$\coprod_w (\operatorname{Im} \phi_{c,d})w(\operatorname{Im} \phi_{a,b}).$$

Furthermore

$$\operatorname{Im}[\phi_{a,b} \circ (\phi_{i_1,i_2} \hat{\times} \phi_{j_1,j_2})] = [w^{-1}(\operatorname{Im} \phi_{c,d})w] \cap [\operatorname{Im} \phi_{a,b}].$$

Proof. The diagram is checked by calculating with generators. On generators $(g_1, \dots, g_k; 1)$ for $k = i_1, i_2, j_1$ or j_2, this is trivial. On generators of the form $(1, \dots, 1; t_l)$, the calculation is very similar to that for 1.8 ii). The second half of 1.9 is immediate from the fact that the corresponding result holds before passing to covering groups [**3**], using 1.8 iv) and the lemma below, whose proof is a straightforward manipulation with the definitions.

LEMMA 1.10. *If $\theta:\widetilde{\Gamma} \to \Gamma$ is an epimorphism of groups and $\{\beta_I\}$ is a complete set of (Ω, Λ) double cosets for Γ, then $\{\widetilde{\beta}_I\}$ is a complete set of $(\widetilde{\Omega}, \widetilde{\Lambda})$ double cosets for $\widetilde{\Gamma}$, where $\widetilde{\Omega} = \theta^{-1}\Omega$, $\widetilde{\Lambda} = \theta^{-1}\Lambda$ and $\widetilde{\beta}_I$ is any choice of element in $\theta^{-1}\beta_I$. Furthermore,*

$$(\widetilde{\beta}_I\widetilde{\Lambda}\widetilde{\beta}_I^{-1}) \cap \widetilde{\Omega} = \theta^{-1}[(\beta_I\Lambda\beta_I^{-1}) \cap \Omega].$$

Definition 1.11. The alternating group $A_n \subset \Sigma_n$ gives subgroups

$$A_n\langle\Gamma\rangle = \{(g_1, \dots, g_n; u):u \in A_n\} \subset \Sigma_n\langle\Gamma\rangle$$

$$\widetilde{A}_n\langle\Gamma\rangle = \{(g_1, \dots, g_n; v):v \in \widetilde{A}_n = \theta^{-1}(A_n)\} \subset \widetilde{\Sigma}_n\langle\Gamma\rangle.$$

Then $\widetilde{A}_n\langle\Gamma\rangle = \theta^{-1}A_n\langle\Gamma\rangle$ double covers $A_n\langle\Gamma\rangle$, and is a $\mathscr{G}$-object with trivial homomorphism s.

To count conjugacy classes we need the following definitions related to partitions.

Definition 1.12. A partition $\alpha = (a_1, \dots, a_l)$ is a finite non-increasing sequence of positive integers. Define $|\alpha| = \Sigma a_i$ and $l(\alpha) = l$. Let $\mathscr{P}$ denote the set of all partitions (including the empty one), so

$$\mathscr{P} = \bigcup_{n=0}^{\infty} \mathscr{P}_n, \quad \text{where } \mathscr{P}_n = \{\alpha \in \mathscr{P}: |\alpha| = n\}.$$

The integers a_i are the *parts* of α. Subsets of $\mathscr{P}_n$ are defined as follows:

$$\mathscr{P}_n' = \{\alpha: \quad \#\{i:a_i \text{ is even}\} \text{ is odd}\}$$

$$\mathscr{P}_n'' = \{\alpha: \quad \#\{i:a_i \text{ is even}\} \text{ is even}\} = \mathscr{P}_n - \mathscr{P}_n'$$

$$\mathscr{P}_n^{\text{odd}} = \{\alpha: \quad \text{all } a_i \text{ are odd}\}$$

$$\mathscr{D}_n = \{\alpha: \quad a_i \neq a_j \text{ if } i \neq j\}; \ \mathscr{D} = \bigcup_{n=0}^{\infty} \mathscr{D}_n.$$

For $* = {}'$ or $''$ or odd, let $\mathscr{D}_n^* = \mathscr{P}_n^* \cap \mathscr{D}_n$. For any set S (finite in the applications) define the sets

$$\mathscr{D}(n, S) = \left\{\phi:S \to \mathscr{D}: \sum_{s \in S} |\phi(s)| = n\right\}$$

$$\mathscr{D}'(n, S) = \left\{\phi \in \mathscr{D}(n, S): \bigcup_{s \in S} \phi(s) \in \mathscr{P}_n'\right\}$$

$$\mathscr{D}''(n, S) = \left\{\phi \in \mathscr{D}(n, S) : \bigcup_{s \in S} \phi(s) \in \mathscr{P}_n''\right\} = \mathscr{D}(n, S) - \mathscr{D}'(n, S)$$

$$\mathscr{P}^{\text{odd}}(n, S) = \left\{\phi : S \to \mathscr{P}^{\text{odd}} : \sum_{s \in S} |\phi(s)| = n\right\}.$$

We define

$$|\phi| = \sum_s |\phi(s)| \quad \text{and} \quad l(\phi) = \sum_s l(\phi(s)).$$

Several times we shall use the well known fact that

$$\#\mathscr{D}_n = \#\mathscr{P}_n^{\text{odd}}.$$

A proof in the spirit of later sections of this paper is discovered by calculating the rank in dimension n of the graded ring (with $\dim x_i = i$)

$$\frac{\mathbf{Z}[x_1, x_2, x_3, x_4, \ldots]}{\langle x_i^2 = x_{2i}\ \forall i\rangle} \cong \mathbf{Z}[x_1, x_3, x_5, \ldots]$$

using the obvious bases suggested by the two ways of writing the ring. It follows immediately that

$$\#\mathscr{D}(n, S) = \#\mathscr{P}^{\text{odd}}(n, S) \quad \text{for all } S.$$

Denote the set of conjugacy classes of a group Γ by Con Γ.

THEOREM 1.13.

$$\#\text{Con } \widetilde{\Sigma}_n\langle\Gamma\rangle - \#\text{Con } \Sigma_n\langle\Gamma\rangle$$
$$= 2\#\mathscr{D}'(n, \text{Con } \Gamma) + \#\mathscr{D}''(n, \text{Con } \Gamma)$$
$$\#\text{Con } \widetilde{A}_n\langle\Gamma\rangle - \#\text{Con } A_n\langle\Gamma\rangle$$
$$= \#\mathscr{D}'(n, \text{Con } \Gamma) + 2\#\mathscr{D}''(n, \text{Con } \Gamma).$$

See [**4**] for the proof.

2. Products of Z/2-graded representations. Below we study representations of $\mathscr{G}$-objects (Ω, z, s) as vector spaces V with an action of Ω. We restrict to "negative" representations, i.e., those where z acts as multiplication by -1. This is essentially irrelevant up to 2.13. All the propositions up to that point have (if anything, simpler) analogues which apply to the category of objects (Ω, s) with no restriction on the representation. Recall the notation $\Omega_0 := \ker s$ and $\Omega_1 = \Omega - \Omega_0$.

Definition 2.1. A *negative* $\mathbf{Z}/2$-graded representation of Ω is a pair $\{V_0, V_1\}$ of complex vector spaces, plus an action

$$\phi : \Omega \times (V_0 \oplus V_1) \to V_0 \oplus V_1$$

making $V_0 \oplus V_1$ into a (finite dimensional) representation of Ω, in which z acts as -1, and such that

$$\phi(\Omega_i \times V_j) \subset V_{i+j},$$

subscripts added mod 2. The letter V will sometimes denote just the space $V_0 \oplus V_1$, and ambiguously sometimes the representation $(V_0 \oplus V_1, \phi)$, and sometimes the $\mathbf{Z}/2$-graded representation $(\{V_0, V_1\}, \phi)$. Isomorphism and direct sums are defined as follows:

$$\{V_0, V_1\} \cong \{W_0, W_1\} \Leftrightarrow \exists \text{ an } \Omega\text{-isomorphism } V_0 \oplus V_1 \xrightarrow{f} W_0 \oplus W_1$$

such that $f(V_i) \subset W_i$ for $i = 0, 1$;

$$\{V_0, V_1\} \oplus \{W_0, W_1\} := \{V_0 \oplus W_0, V_1 \oplus W_1\}$$

with the obvious action.

It is straightforward to check that $\oplus$ passes to isomorphism classes, and, up to isomorphism, is associative, commutative with zero $(= \{\{0\}, \{0\}\})$. Thus:

PROPOSITION 2.2. *The set of isomorphism classes of negative $\mathbf{Z}/2$-graded representations of Ω, which we denote* $\mathrm{GREP}^-(\Omega)$, *is an abelian semigroup under $\oplus$.*

We shall call by $\mathrm{REP}^-(\Omega)$ the analogous semigroup of negative (ungraded) representations. The next few definitions and propositions are the analogues for GREP^- of standard facts for REP^- and REP, and their proofs are the standard ones with minor embellishments to take account of the grading. At this point we should inform the reader that later we'll see

$$\mathrm{GREP}^-(\Omega) \cong \begin{cases} \mathrm{REP}^-(\Omega_0), & \text{if } \Omega_0 \neq \Omega \\ \mathrm{REP}^-(\Omega) \oplus \mathrm{REP}^-(\Omega), & \text{if } \Omega_0 = \Omega. \end{cases}$$

So these results could also be derived by referring to the analogous ungraded results, but no real economy would result.

Definition 2.3. Let $\mathrm{GR}^-(\Omega)$ denote the group completion of $\mathrm{GREP}^-(\Omega)$. Subrepresentations and irreducibility are defined as follows:

$$(\{V_0, V_1\}, \phi) \subset (\{W_0, W_1\}, \psi) \Leftrightarrow V_i \subset W_i \text{ and } \phi = \psi|V.$$

$$\{W_0, W_1\} \text{ is irreducible} \Leftrightarrow [(\{V_0, V_1\} \subsetneq \{W_0, W_1\} \Rightarrow$$
$$V_0 = V_1 = 0) \text{ and } \{W_0, W_1\} \neq \{0, 0\}].$$

PROPOSITION 2.4. GREP^- *and* GR^- *are free on the set* $\mathrm{GIRREP}^-(\Omega)$ *of isomorphism classes of irreducibles.*

Proof. We need only prove it for GREP^-. To write a graded

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representation as a sum of irreducibles, we need to show that a subrepresentation $\{V_0, V_1\}$ of $\{W_0, W_1\}$ always has a complement. Do this by constructing an Ω-invariant positive definite inner product in $W_0 \oplus W_1$ for which $W_i^{\perp} = W_{i+1}$. This is done by the usual averaging trick starting from any inner product with $W_i^{\perp} = W_{i+1}$. Then we have

$$\{V_0, V_1\} \oplus \{V_0^{\perp} \cap W_0, V_1^{\perp} \cap W_1\} \cong \{W_0, W_1\}$$

as required. To prove freeness, i.e., uniqueness of decomposition, apply the usual argument using the inner product constructed below.

Definition 2.5. Denote by $\mathrm{GHOM}_\Omega[\{V_0, V_1\}, \{W_0, W_1\}]$ the set of Ω-invariant linear maps

$$f:V_0 \oplus V_1 \to W_0 \oplus W_1$$

for which $f(V_i) \subset W_i$. Evidently GHOM_Ω is a subspace of

$$\mathrm{HOM}_{\mathbf{C}}(V_0 \oplus V_1, W_0 \oplus W_1)$$

whose dimension depends only on the isomorphism classes of $\{V_0, V_1\}$ and $\{W_0, W_1\}$, and is bi-additive with respect to $\oplus$. Hence this dimension yields an inner product

$$\langle\ ,\ \rangle:\mathrm{GR}^-(\Omega) \times \mathrm{GR}^-(\Omega) \to \mathbf{Z}$$

which is positive definite in that

$$\langle x, x\rangle > 0 \quad \text{for all } x \neq 0.$$

PROPOSITION 2.6. $\mathrm{GIRREP}^-(\Omega)$ *is an orthonormal set with respect to* $\langle\ ,\ \rangle$. *Hence it is an orthonormal basis for* $\mathrm{GR}^-(\Omega)$, *and, in particular,* $\langle\ ,\ \rangle$ *is symmetric.*

Proof. The usual argument for Schur's lemma, with a few checks on the grading, gives the required result for irreducibles V, W:

$$\mathrm{GHOM}_\Omega[\{V_0, V_1\}, \{W_0, W_1\}] \cong \begin{cases} \mathbf{C} & \text{if } \{V_0, V_1\} \cong \{W_0, W_1\} \\ 0 & \text{if not.} \end{cases}$$

Below we use the notation R^- and IRREP^- in the obvious way applying to ungraded representations. We define restricting, inducing, and certain maps π, η, rev and ass on representations, graded and ungraded as appropriate. In each case it is easy to verify that the construction is invariant up to isomorphism and additive with respect to $\oplus$. Thus in each case we get a homomorphism between the appropriate group completions which sends representations to representations.

Definition 2.7. Let $\theta:\Omega \to \Omega'$ be a $\mathcal{G}$-map. Define *restriction*

$$\theta^*:\mathrm{GR}^-(\Omega') \to \mathrm{GR}^-(\Omega)$$

on graded representations by

$$\theta^*(\{V_0, V_1\}, \phi) = (\{V_0, V_1\}, \psi)$$

with

$$\psi:(g, v) \mapsto \theta(g) \cdot v = \phi(\theta(g), v).$$

Clearly $1^* = 1$ and $(\theta_1 \circ \theta_2)^* = \theta_2^* \circ \theta_1^*$, so GR^- has been made into a contravariant functor.

Now assume θ is injective. Then we define inducing

$$\theta_*:\mathrm{GR}^-(\Omega) \to \mathrm{GR}^-(\Omega')$$

on graded representations as follows: If

$$[\{W_0, W_1\}] \in \mathrm{GREP}^-(\Omega') \quad \text{and} \quad [\{V_0, V_1\}] \in \mathrm{GREP}^-(\Omega),$$

we say $\{W_0, W_1\}$ is induced from $\{V_0, V_1\}$ via θ if and only if $V_0 \subset W_0$; $V_1 \subset W_1$; $\theta(g) \cdot v = g \cdot v$ for all $g \in \Omega$, $v \in V_0 \oplus V_1$; and

$$W_0 \oplus W_1 = \bigoplus_i h_i \cdot (V_0 \oplus V_1),$$

where h_i ranges over some complete set of left coset representatives of $\theta(\Omega)$ in Ω'. The usual arguments show that such a $\{W_0, W_1\}$ depends up to isomorphism only on the isomorphism class of $\{V_0, V_1\}$, and that such a $\{W_0, W_1\}$ exists for each $\{V_0, V_1\}$; and of course we write

$$[\{W_0, W_1\}] = \theta_*[\{V_0, V_1\}].$$

Clearly

$$1_* = 1 \quad \text{and} \quad (\theta_1 \circ \theta_2)_* = \theta_{1*} \circ \theta_{2*},$$

so GR^- becomes also a covariant functor when we restrict to injective $\mathcal{G}$-maps.

Define $\pi:\mathrm{GR}^-(\Omega) \to \mathrm{R}^-(\Omega)$ on representations by

$$\pi[\{V_0, V_1\}] = [V_0 \oplus V_1].$$

Define $\mathrm{rev}:\mathrm{GR}^-(\Omega) \to \mathrm{GR}^-(\Omega)$ on representations by

$$\mathrm{rev}[\{V_0, V_1\}] = [\{V_1, V_0\}],$$

using the "same" action on $V_1 \oplus V_0 \cong V_0 \oplus V_1$. Sometimes $\mathrm{rev}(x)$ will be denoted by x^{rev}.

Define ass: $\mathrm{R}^-(\Omega) \to \mathrm{R}^-(\Omega)$ on representations by

$$\mathrm{ass}[(V, \phi)] = (V, \psi)$$

where

$$\psi(g, v) = (-1)^{s(g)}\phi(g, v),$$

i.e., the new action $*$ on V^{ass} is given in terms of the old action $\cdot$ on V by

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$$g * v = (-1)^{sg} g \cdot v.$$

Sometimes $\mathrm{ass}(x)$ is denoted x^{ass}.

Define $\eta:\mathrm{R}^-(\Omega) \to \mathrm{GR}^-(\Omega)$ on representations by

$$\eta[V] = [\{D_V, A_V\}]$$

where the spaces are $D_V := \{(v, v):v \in V\}$ and $A_V := \{(v, -v):v \in V\}$ and the action $\times$ is

$$g \times (v_1, v_2) = (g \cdot v_1, g * v_2),$$

where $\cdot$ is the given action, and $*$ the associate action as above. Thus

$$\pi\eta(V) \cong V \oplus V^{\mathrm{ass}}.$$

PROPOSITION 2.8. i) *All the above maps are well defined and homomorphisms.*

ii) *The latter four maps are natural with respect to both inducing and restricting.*

iii) $\mathrm{rev} \circ \mathrm{rev} = 1_{\mathrm{GR}}$; $\mathrm{ass} \circ \mathrm{ass} = 1_{\mathrm{R}}$

iv) $\pi \circ \mathrm{rev} = \pi = \mathrm{ass} \circ \pi$

v) $\eta \circ \mathrm{ass} = \eta = \mathrm{rev} \circ \eta$

vi) $\pi \circ \eta = 1_{\mathrm{R}} + \mathrm{ass}$

vii) $\eta \circ \pi = 1_{\mathrm{GR}} + \mathrm{rev}$.

Proof. i) The only points needing comment are that careful calculation shows that the action is well defined for V^{ass} and $\eta(V)$; that D_V and A_V are preserved (resp. inter-changed) by g, if $g \in \Omega_0$ (resp. $g \in \Omega_1$); and that η is additive. The last point comes from the isomorphism

$$\{D_{V\oplus W}, A_{V\oplus W}\} \xrightarrow{\cong} \{D_V, A_V\} \oplus \{D_W, A_W\}$$

$$(v_1, w_1, v_2, w_2) \mapsto (v_1, v_2, w_1, w_2).$$

ii) Naturality with respect to restricting is in all cases straightforward. As for inducing: For rev, if $\{W_0, W_1\}$ is induced by $\{V_0, V_1\}$ just interchange the subscripts and use the same coset representatives to see that $\{W_1, W_0\}$ is induced by $\{V_1, V_0\}$. For ass, W being induced by V implies W^{ass} is induced by V^{ass}, simply noting that $g * V = g \cdot V$ for the coset representatives or any other g. For π, it comes virtually by definition that $V_0 \oplus V_1$ is induced from $W_0 \oplus W_1$ if $\{V_0, V_1\}$ is induced from $\{W_0, W_1\}$. Finally, for η, suppose W is induced from V. Then $V \subset W$ so $\{D_V, A_V\} \subset \{D_W, A_W\}$. Furthermore, with $W = \oplus\, g \cdot V$ for coset representatives g, we find

$$D_W \oplus A_W \cong W \oplus W^{\mathrm{ass}} = \bigoplus_g (g \cdot V \oplus g * V)$$

$$= \bigoplus_g g \cdot (V \oplus V^{\mathrm{ass}}) \cong \bigoplus_g g \cdot (D_V \oplus A_V).$$

iii) It is obvious that rev and ass are involutions.

iv) That $\pi \circ \text{rev} = \pi$ is obvious. The map

$$V_0 \oplus V_1 \to (V_0 \oplus V_1)^{\text{ass}}$$

given by $(v_0, v_1) \mapsto (v_0, -v_1)$ is an Ω-isomorphism, so

$$(\text{ass} \circ \pi)\{V_0, V_1\} = (V_0 \oplus V_1)^{\text{ass}} \cong V_0 \oplus V_1 = \pi\{V_0, V_1\}.$$

v) For this we use isomorphisms as follows

$$\begin{array}{ll} \{D_{V^{\text{ass}}}, A_{V^{\text{ass}}}\} \cong \{D_V, A_V\} & \{D_V, A_V\} \cong \{A_V, D_V\} \\ \quad D_{V^{\text{ass}}} \to D_V & \quad D_V \to A_V \\ \quad (v, v) \mapsto (v, v) & \quad (v, v) \mapsto (v, -v) \\ \quad A_{V^{\text{ass}}} \to A_V & \quad A_V \to D_V \\ \quad (v, -v) \mapsto (-v, v) & \quad (v, -v) \mapsto (v, v). \end{array}$$

vi) The map

$$[(v_1, v_1), (v_2, -v_2)] \mapsto (v_1 + v_2, v_1 - v_2)$$

provides the required isomorphism $D_V \oplus A_V \cong V \oplus V^{\text{ass}}$.

vii) The maps

$$\begin{array}{ccc} V_0 \oplus V_1 \to D_{V_0 \oplus V_1} & & V_1 \oplus V_0 \to A_{V_0 \oplus V_1} \\ & \text{and} & \\ (v_0, v_1) \mapsto [(v_0, v_1), (v_0, v_1)] & & (v_1, v_0) \mapsto [(v_0, v_1), -(v_0, v_1)] \end{array}$$

give the required isomorphism

$$\{D_{V_0 \oplus V_1}, A_{V_0 \oplus V_1}\} \cong \{V_0 \oplus V_1, V_1 \oplus V_0\}.$$

PROPOSITION 2.9.

i) $\pi\ \text{GREP}^-(\Omega) = \{x \in \text{REP}^-\Omega : x^{\text{ass}} = x\}$.

ii) $\eta\ \text{REP}^-(\Omega) = \{y \in \text{GREP}^-\Omega : y^{\text{rev}} = y\}$.

Similarly with GR^- *and* R^- *replacing* GREP^- *and* REP^-, *respectively.*

Proof. The last sentence clearly follows from i) and ii). The inclusions $\subset$ are iv) and v) of 2.8, so we prove

i) $V \cong V^{\text{ass}} \Rightarrow \exists\{V_0, V_1\}$ with $V_0 \oplus V_1 \cong V$.

ii) $\{V_0, V_1\} \cong \{V_1, V_0\} \Rightarrow \exists V$ with $\{D_V, A_V\} \cong \{V_0, V_1\}$.

i) Write $V \cong \bigoplus_\alpha V_\alpha$ with $[V_\alpha] \in \text{IRREP}^-\Omega$. Then

$$V^{\text{ass}} \cong \bigoplus_\alpha V_\alpha^{\text{ass}}.$$

Since IRREP^- is clearly invariant under ass, there is a bijective map $\alpha \mapsto \tilde{\alpha}$ with $V_{\tilde{\alpha}} \cong V_\alpha^{\text{ass}}$. When $\alpha \neq \tilde{\alpha}$, we have

$$V_\alpha \oplus V_{\tilde{\alpha}} \cong V_\alpha \oplus V_\alpha^{\text{ass}} \cong \eta V_\alpha.$$

It remains only to consider the cases where $\alpha = \tilde{\alpha}$, that is

$$V_\alpha \cong V_\alpha^{\text{ass}}.$$

This reduces the problem to the case where V is irreducible. Let $f:V \to V^{\text{ass}}$ be an Ω-isomorphism. Then $f \circ f$ is multiplication by a non-zero complex number with square roots $\pm w$. Let V_i be the $(-1)^i w$-eigenspace of f. Then certainly $V = V_0 \oplus V_1$, and a mechanical check shows that $g \cdot V_j \subset V_{s(g)+j}$, as required.

ii) Here exactly as in the first part of i), but using rev in place of ass, we reduce to the case of irreducible $\{V_0, V_1\}$. Let

$$f:V_0 \oplus V_1 \to V_1 \oplus V_0$$

realize the isomorphism $\{V_0, V_1\} \cong \{V_1, V_0\}$. Since

$$f \circ f \in \text{GHOM}_\Omega[\{V_0, V_1\}, \{V_0, V_1\}],$$

by 2.6 we can alter f by a scalar and assume $f \circ f = 1$. Let V be the $+1$-eigenspace of the endomorphism f of $V_0 \oplus V_1$. Then the maps

$$\begin{array}{ccccccc} D_V & \longrightarrow & V_0 & \qquad & A_V & \longrightarrow & V_1 \\ [(v_0, v_1), (v_0, v_1)] & \mapsto & v_0 & & [(v_0, v_1), -(v_0, v_1)] & \mapsto & v_1 \end{array}$$

yield the required isomorphism $\{D_V, A_V\} \cong \{V_0, V_1\}$.

THEOREM 2.10. *For any $\mathcal{G}$-object Ω, there exist integers μ, ν and elements a_i, b_i, c_i, d_i so that*

$$\text{GIRREP}^-(\Omega) = \{a_1, a_1^{\text{rev}}, a_2, a_2^{\text{rev}}, \dots, a_\mu, a_\mu^{\text{rev}}, b_1, b_2, \dots, b_\nu\}$$

$$\text{IRREP}^-(\Omega) = \{c_1, \dots, c_\mu, d_1, d_1^{\text{ass}}, d_2, d_2^{\text{ass}}, \dots, d_\nu, d_\nu^{\text{ass}}\}$$

with

$$a_i \neq a_i^{\text{rev}};\ b_i = b_i^{\text{rev}};\ c_i = c_i^{\text{ass}};\ d_i \neq d_i^{\text{ass}};$$

$$\pi a_i = \pi a_i^{\text{rev}} = c_i;\ \pi b_i = d_i + d_i^{\text{ass}};$$

$$\eta c_i = a_i + a_i^{\text{rev}};\ \eta d_i = \eta d_i^{\text{ass}} = b_i.$$

Proof. List $\text{GIRREP}^-\Omega$ as given, with $a_i \neq a_i^{\text{rev}}$ but $b_i = b_i^{\text{rev}}$. Then $\pi a_i = \pi a_i^{\text{rev}}$ by 2.8 iv). Call this c_i. Then $c_i = c_i^{\text{ass}}$. Since $b_i = b_i^{\text{rev}}$, by 2.9 ii) we have an element $d_i \in \text{REP}^-\Omega$ with $\eta d_i = b_i$. Then using 2.8 we get

$$\eta d_i^{\text{ass}} = b_i \quad \text{and}$$

$$\eta c_i = \eta \pi a_i = a_i + a_i^{\text{rev}} \quad \text{and}$$

$$\pi b_i = \pi \eta d_i = d_i + d_i^{\text{ass}}.$$

What remains is to prove:

i) c_i and d_i are irreducible
ii) $d_i \neq d_i^{\text{ass}}$ and
iii) $\text{IRREP}^-\Omega$ has no elements other than the c_i, d_i and d_i^{ass}.

i) Let $c_i = [V]$ and suppose $V \cong U \oplus W$ with both U and W non-zero. Then $\eta[U] \neq 0 \neq \eta[W]$, so (re-naming if necessary) we have

$$\eta[U] = a_i \text{ and } \eta[W] = a_i^{\text{rev}},$$

since

$$\eta[U \oplus W] = a_i + a_i^{\text{rev}}.$$

But now

$$a_i^{\text{rev}} = \eta[U]^{\text{rev}} = \eta[U] = a_i,$$

a contradiction. Thus c_i is irreducible. To show d_i and d_i^{ass} are irreducible is easier since they map to the irreducible b_i under η.

ii) Suppose $d_i = d_i^{\text{ass}}$. Then $\pi e = d_i$ for some $e \in \text{GREP}^-(\Omega)$. Hence

$$b_i = \eta d_i = \eta\pi e = e + e^{\text{rev}}.$$

But $e \neq 0$ since $d_i \neq 0$. Thus b_i is not irreducible, a contradiction.

iii) Let $f \in \text{IRREP}^-\Omega$ and write

$$\eta f = \sum \alpha_i a_i + \sum \alpha_i' a_i^{\text{rev}} + \sum \beta_i b_i$$

for non-negative integers α_i, α_i' and β_i. Then $\alpha_i = \alpha_i'$ since $\eta f = (\eta f)^{\text{rev}}$. Now

$$f + f^{\text{ass}} = \pi\eta f = 2 \sum \alpha_i c_i + \sum \beta_i(d_i + d_i^{\text{ass}}).$$

Since f is irreducible, $2 \sum \alpha_i + 2 \sum \beta_i = 2$. Thus all but one coefficient is zero, and $\eta f =$ either $a_i + a_i^{\text{rev}}$ or b_i. In the first case

$$f + f^{\text{ass}} = 2c_i,$$

so $f = c_i$ by uniqueness. In the second case

$$f + f^{\text{ass}} = d_i + d_i^{\text{ass}},$$

so f is either d_i or d_i^{ass}, as required.

PROPOSITION 2.11. *Let* $x \in \text{GREP}^-\Omega$. *With notation as in* 2.10, *we have*

i) *if* $\pi x = c_j$ *then* $x = a_j$ *or* a_j^{rev}
ii) *if* $\pi x = d_j$ *then* $x = b_j$.

Proof. Simply write x as a non-negative linear combination from $\text{GIRREP}^-\Omega$ and use uniqueness.

Note that 2.11 requires more than just that $x \in \text{GR}^-\Omega$. The reader may also have noticed the similarity between the relation of IRREP^- to

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GIRREP^- and the relation of the irreducible representations of a group to those of a subgroup of index 2. By 2.12 below, this is no accident in the more interesting case when $\Omega_0 \neq \Omega$.

PROPOSITION 2.12. *If* $\Omega_0 \neq \Omega$, *the formula*

$$\{V_0, V_1\} \mapsto V_0$$

induces a natural isomorphism

$$\mathrm{GR}^-\Omega \xrightarrow{\Phi} \mathrm{R}^-\Omega_0$$

such that the following diagrams commute:

$$\begin{tikzcd}
\mathrm{GR}^-\Omega \arrow[r, "\Phi"] \arrow[dr, "\pi"'] & \mathrm{R}^-\Omega_0 \arrow[d, "induce"] \\
& \mathrm{R}^-\Omega
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{GR}^-\Omega \arrow[r, "\Phi"] & \mathrm{R}^-\Omega_0 \\
& \mathrm{R}^-\Omega \arrow[u, "restrict"'] \arrow[ul, "\eta"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{GR}^-\Omega \arrow[r, "\Phi"] \arrow[d, leftrightarrow, "\mathrm{rev}"'] & \mathrm{R}^-\Omega_0 \arrow[d, shift left, "\iota_*"] \\
\mathrm{GR}^-\Omega \arrow[r, "\Phi"] & \mathrm{R}^-\Omega_0 \arrow[u, shift left, "\iota^*"]
\end{tikzcd}$$

where $\iota:\Omega_0 \rightarrow \Omega_0$ *is conjugation by any element of* Ω_1.

Proof. Choose an element $g_1 \in \Omega_1$ and define a map

$$\zeta:\mathrm{R}^-\Omega_0 \rightarrow \mathrm{GR}^-\Omega$$

by $\zeta[V] = [\{V, V\}]$ where the action on $\zeta[V]$ is as follows

$$g \cdot (v, v') = \begin{cases} (gv, g_1^{-1}gg_1v') & \text{if } g \in \Omega_0 \\ (gg_1v', g_1^{-1}gv) & \text{if } g \in \Omega_1. \end{cases}$$

This is easily seen to be a $\mathbf{Z}/2$-graded representation. Also

$$\Phi \circ \zeta[V] = [V].$$

Now $\zeta \circ \Phi\{V_0, V_1\} = \{V_0, V_0\}$ with action as above. But $\{V_0, V_1\} \cong \{V_0, V_0\}$ using

$$\begin{array}{ll} V_0 \rightarrow V_0 & \text{and} \quad V_1 \rightarrow V_0 \\ v_0 \mapsto v_0 & \qquad\quad v_1 \mapsto g_1^{-1}v_1. \end{array}$$

Thus Φ is an isomorphism with inverse ζ. Since $V_1 = g_1V_0$, we see that $V_0 \oplus V_1$ is induced from V_0 by the inclusion $\Omega_0 \hookrightarrow \Omega$, so the first diagram commutes. The second one requires an Ω_0-isomorphism $D_V \rightarrow V$

for all $[V] \in \mathrm{REP}^{-}\Omega$. The map $(v, v) \mapsto v$ yields one. The third diagram requires, for each $[\{V_0, V_1\}] \in \mathrm{GREP}^{-}\Omega$, and Ω_0-isomorphism $\iota^* V_0 \to V_1$. Such a map is given by $v_0 \mapsto g_1 v_0$.

PROPOSITION 2.13. *If* $\iota_h:\Omega \to \Omega$ *is the inner automorphism* $g \mapsto hgh^{-1}$, then

$$\iota_h{}^* = \iota_{h_*} = \mathrm{rev}^{s(h)}:\mathrm{GREP}^{-}\Omega \to \mathrm{GREP}^{-}\Omega.$$

Proof. We have

$$\iota_h{}^*(\{V_0, V_1\}, \phi) \cong (\{V_0, V_1\}, \psi)$$

where $\psi(g, v) = \phi(h^{-1}gh, v)$. The map $v \mapsto h^{-1} \cdot v$ is an isomorphism of representations

$$\pi\{V_0, V_1\} \cong \pi\iota_h{}^*\{V_0, V_1\}.$$

If $s(h) = 0$, it maps V_i to V_i. If $s(h) = 1$, it maps V_i to V_{i+1}, as required. A similar proof works for ι_{h_*}, but alternatively note that

$$\iota_{h_*} = (\iota_h{}^*)^{-1} = \iota_{h^{-1}}{}^*.$$

PROPOSITION 2.14. *For the map* $\zeta_{a,b}$ *in* 1.8 i), *we have*

$$\zeta_{a,b_*} = \mathrm{rev}^{ab}:\mathrm{GR}^{-}(\widetilde{\Sigma}_{a+b}\langle\Gamma\rangle) \to \mathrm{GR}^{-}(\widetilde{\Sigma}_{a+b}\langle\Gamma\rangle)$$

and

$$\zeta_{a,b_*} = \mathrm{ass}^{ab}:\mathrm{R}^{-}(\widetilde{\Sigma}_{a+b}\langle\Gamma\rangle) \to \mathrm{R}^{-}(\widetilde{\Sigma}_{a+b}\langle\Gamma\rangle).$$

Proof. Recall that $\zeta_{a,b} = \alpha \circ \beta$ where

$$\alpha(x) = z^{abs(x)}x \quad \text{and} \quad \beta(x) = u_{a,b}xu_{a,b}^{-1}.$$

Now $\beta_* = 1$ on R^{-} since β is an inner automorphism and clearly

$$[x \mapsto z^{s(x)}x]_* = \mathrm{ass},$$

so $\alpha_* = \mathrm{ass}^{ab}$ on R^{-}. Thus

$$\zeta_* = \alpha_* \circ \beta_* = \mathrm{ass}^{ab}.$$

On GR^{-}, $\alpha_* = 1$ since $\mathbf{Z}/2$-gradable representations are self associate by 2.9 i), but

$$\beta_* = \mathrm{rev}^{s(u_{a,b})}$$

by 2.13. But, as noted after 1.8,

$$s(u_{a,b}) \equiv ab \bmod 2,$$

as required.

PROPOSITION 2.15. *Let* $X = \{x_1, x_2, \ldots, x_\lambda\} \subset \mathrm{REP}^{-}\Omega$. *In order that* $X = \mathrm{IRREP}^{-}\Omega$, *it suffices that*

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$$\Sigma(\dim x_i)^2 = \frac{1}{2}\#\Omega \quad \textit{and} \quad \langle x_i, x_j \rangle = \delta_{ij}.$$

Proof. Since $\langle x_i, x_i \rangle = 1$ and $x_i \in \mathrm{REP}^-\Omega$, we have

$$x_i \in \mathrm{IRREP}^-\Omega.$$

Also $x_i \neq x_j$ for $i \neq j$ since $\langle x_i, x_j \rangle = 0$. But, using a superscript $+$ to refer to representations in which z acts as multiplication by $+1$, we clearly have

$$\mathrm{IRREP}\ \Omega = \mathrm{IRREP}^+\Omega \cup \mathrm{IRREP}^-\Omega \quad \text{(disjoint)}$$

and

$$\mathrm{IRREP}^+\Omega \cong \mathrm{IRREP}(\Omega/\{1, z\}).$$

That X is all of $\mathrm{IRREP}^-\Omega$ now follows, since

$$\sum_{y \in \mathrm{IRREP}^+\Omega} (\dim y)^2 = \#(\Omega/\{1, z\}) = \frac{1}{2}\#\Omega$$

and

$$\sum_{u \in \mathrm{IRREP}\Omega} (\dim u)^2 = \#\Omega.$$

For later purposes, it is convenient to give an "external" version of Mackey's theorem. The word external indicates that we consider monomorphisms which are not necessarily inclusions of subgroups. This is only superficially more general than the usual version. Our version below for GR differs from that for R only by the appearance of the terms $\mathrm{rev}^{s(g_\Delta)}$ (which cannot in general be eliminated by rechoosing double coset representatives). The proof differs from that for R only by a few embellishments to take account of gradings.

THEOREM 2.16. *Let Λ, Ω and Γ be $\mathcal{G}$-objects, and $\alpha:\Lambda \to \Gamma$, $\beta:\Omega \to \Gamma$ be $\mathcal{G}$-monomorphisms. Let $\{g_\Delta\}$ be a set of representatives for the $(\alpha\Omega, \beta\Lambda)$-double cosets in Γ where Δ denotes the double coset $(\alpha\Omega)g_\Delta(\beta\Lambda)$. Suppose, for each Δ, we have a $\mathcal{G}$-object Ψ_Δ and $\mathcal{G}$-monomorphisms $\alpha_\Delta:\Psi_\Delta \to \Lambda$, $\beta_\Delta:\Psi_\Delta \to \Omega$ such that*

$$\begin{tikzcd}
\Lambda \arrow[d, "\beta"'] & \Psi_\Delta \arrow[l, "\alpha_\Delta"'] \arrow[r, "\beta_\Delta"] & \Omega \arrow[d, "\alpha"] \\
\Gamma \arrow[rr] & & \Gamma \\
g \arrow[rr, mapsto] & & g_\Delta g g_\Delta^{-1}
\end{tikzcd}$$

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is commutative. Assume also that

$$\beta\alpha_\Delta\Psi_\Delta = (g_\Delta^{-1}\alpha\Omega g_\Delta) \cap \beta\Lambda.$$

Then

$$\alpha^*\beta_* x = \sum_\Delta \operatorname{rev}^{s(g_\Delta)}\beta_{\Delta *}\alpha_\Delta{}^* x \quad \textit{for all } x \in \mathrm{GR}^-\Lambda.$$

Proof. We may take

$$x = [\{W_0, W_1\}] \in \mathrm{GREP}^-\Lambda \quad \text{and}$$
$$\beta_* x = [\{V_0, V_1\}] \in \mathrm{GREP}^-\Gamma,$$

where $W_i \subset V_i$; the actions of $\beta(h)$ and h on W agree, for each $h \in \Lambda$; and where

$$V = \bigoplus_{i \in I} g_i \cdot W$$

for any left coset decomposition

$$\Gamma = \coprod_{i \in I} g_i \cdot \beta\Lambda.$$

Define

$$V_\Delta = \sum_{g \in \Delta} g \cdot W = \sum_{h \in \alpha\Omega \cdot g_\Delta} h \cdot W \subset V, \quad \text{and}$$

$$(V_\Delta)_i = V_\Delta \cap V_i \quad \text{for } i = 0, 1.$$

Then V_Δ is invariant under the action of Ω,

$$V = \bigoplus_\Delta V_\Delta \quad \text{and} \quad V_i = \bigoplus_\Delta V_{\Delta_i}.$$

Thus it suffices to prove that

$$\beta_{\Delta_*}\alpha_\Delta{}^* x = \operatorname{rev}^{s(g_\Delta)}[\{V_{\Delta_0}, V_{\Delta_1}\}].$$

By the commutativity of the diagram, the two actions below of Ψ_Δ on $g_\Delta \cdot W$ agree:

$$(h, g_\Delta \cdot w) \mapsto g_\Delta \cdot \beta\alpha_\Delta(h) \cdot w = g_\Delta \cdot \alpha_\Delta(h) \cdot w \quad \text{and}$$

$$(h, g_\Delta \cdot w) \mapsto \alpha\beta_\Delta(h) \cdot g_\Delta \cdot w.$$

The first action gives a graded representation $[g_\Delta \cdot W]$ which agrees with

$$\operatorname{rev}^{s(g_\Delta)}\alpha_\Delta^*(x) \in \mathrm{GREP}^-\Psi_\Delta$$

since the map $w \mapsto g_\Delta \cdot w$ is a Ψ_Δ-isomorphism which either preserves or switches gradings depending on whether $s(g_\Delta) = 0$ or 1. But for any left coset decomposition

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$$\Omega = \coprod_{j \in J} h_j \beta_\Delta \Psi_\Delta,$$

we have

$$V_\Delta = \bigoplus_{j \in J} \alpha(h_j) \cdot g_\Delta \cdot W.$$

Thus

$$[V_\Delta] = \beta_{\Delta_*}[g_\Delta \cdot W] \in \mathrm{GREP}^-\Omega,$$

where $g_\Delta \cdot W$ has the "second action" above. Since the actions agree, this completes the proof.

To describe the representations of $\Omega \hat{\times} \Gamma$ in terms of those of Ω and of Γ, we use certain external products defined below.

Definition 2.17. To avoid cumbersome notation, we often use V to stand for a graded representation $\{V_0, V_1\}$. Given

$$[\{V_0, V_1\}] \in \mathrm{GREP}^-\Omega \quad \text{and} \quad [\{W_0, W_1\}] \in \mathrm{GREP}^-\Lambda,$$

define $[V \boxtimes_1 W] \in \mathrm{GREP}^-(\Omega \hat{\times} \Lambda)$ by setting

$$(V \boxtimes_1 W)_0 = (V_0 \otimes W_0) \oplus (V_1 \otimes W_1)$$
$$(V \boxtimes_1 W)_1 = (V_0 \otimes W_1) \oplus (V_1 \otimes W_0)$$

with action

$$(\omega, \lambda) \cdot (v \otimes w) = (-1)^{s(\lambda)t(v)} \omega v \otimes \lambda w \quad \text{when } v \in V_{t(v)}.$$

For the rest of this section we use lower case Greek for group elements with $\omega \in \Omega$, $\lambda \in \Lambda$, $\gamma \in \Gamma$, etc., to aid readability.

PROPOSITION 2.18. *The operation* $\boxtimes_1$ *is well defined.*

Proof. Firstly (z, z) acts trivially; $(z, 1)$ acts as -1; and the right side of the action formula is linear in v and w. So we get a function

$$\Omega \hat{\times} \Lambda \to \mathrm{END}^-(\{(V \boxtimes_1 W)_0, (V \boxtimes_1 W)_1\}).$$

A mechanical check shows this to be a homomorphism, where verification that

$$(\omega', \lambda')[(\omega, \lambda)(v \otimes w)] = [(\omega', \lambda')(\omega, \lambda)](v \otimes w)$$

uses

$$s(\lambda)t(v) + s(\lambda')t(\omega v) = s(\omega)s(\lambda') + s(\lambda\lambda')t(v).$$

Definition 2.19. If $[\{V_0, V_1\}] \in \mathrm{GREP}^-\Omega$ and $[W] \in \mathrm{REP}^-\Lambda$, define

$$[V \boxtimes_2 W] \in \mathrm{REP}^-(\Omega \hat{\times} \Lambda)$$

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to be $(V_0 \oplus V_1) \otimes W$ with action as in the definition of $\boxtimes_1$. Checking that this is well defined is the same as above. As noted later in 2.22, we have

$$V \boxtimes_2 \pi U \cong \pi(V \boxtimes_1 U),$$

but there is no operation $\diamondtimes$ on REP^- for which

$$V \boxtimes_2 W = (\pi V) \diamondtimes W.$$

Definition 2.20. If $[V] \in \mathrm{REP}^-\Omega$ and $[W] \in \mathrm{REP}^-\Lambda$, let $V \boxtimes\!\boxtimes W$ be the sub-graded representation of $\eta V \boxtimes_1 \eta W$ given by

$$(V \boxtimes\!\boxtimes W)_0 = \mathrm{Span}_{\mathbf{C}}\{ (v, v) \otimes (w, w) + i(v, -v) \otimes (w, -w): \\ v \in V, w \in W\}$$

$$(V \boxtimes\!\boxtimes W)_1 = \mathrm{Span}_{\mathbf{C}}\{ (v, v) \otimes (w, -w) - i(v, -v) \otimes (w, w): \\ v \in V, w \in W\}.$$

Caution. The groups act with sign on each second coordinate since $\pi\eta V = V \oplus V^{\mathrm{ass}}$, not $V \oplus V$.

PROPOSITION 2.21. *The operation* $\boxtimes\!\boxtimes$ *is well defined.*

Proof. One need only mechanically verify that $(\Omega \mathbin{\hat{\times}} \Lambda)_k$ maps $(V \boxtimes\!\boxtimes W)_j$ into $(V \boxtimes\!\boxtimes W)_{k+j}$.

Note. It is easy to see that these operations are natural with respect to both restricting and inducing. Below V and W are sometimes graded and sometimes not. The operation makes this unambiguous.

PROPOSITION 2.22. *Recall* $\tau:\Omega \mathbin{\hat{\times}} \Lambda \to \Lambda \mathbin{\hat{\times}} \Omega$ *from* 1.5 i).

i) $V \boxtimes_1 W \cong \tau^*(W \boxtimes_1 V)$

ii) $(V \boxtimes_1 W)^{\mathrm{rev}} \cong V \boxtimes_1 (W^{\mathrm{rev}}) \cong V^{\mathrm{rev}} \boxtimes_1 W$

iii) $(V \boxtimes_2 W)^{\mathrm{ass}} \cong V \boxtimes_2 (W^{\mathrm{ass}}) \cong V^{\mathrm{rev}} \boxtimes_2 W$

iv) $(V \boxtimes\!\boxtimes W)^{\mathrm{rev}} \cong V^{\mathrm{ass}} \boxtimes\!\boxtimes W \cong V \boxtimes\!\boxtimes (W^{\mathrm{ass}})$
$\cong \tau^*(W \boxtimes\!\boxtimes V)$

v) $(V \boxtimes\!\boxtimes W)^{\mathrm{rev}} \oplus (V \boxtimes\!\boxtimes W) \cong (\eta V) \boxtimes_1 (\eta W)$

vi) $\pi(V \boxtimes_1 W) \cong V \boxtimes_2 \pi W$

vii) $\eta(V \boxtimes_2 W) \cong V \boxtimes_1 \eta W$

viii) $2\pi(V \boxtimes\!\boxtimes W) \cong \eta V \boxtimes_2 (W^{\mathrm{ass}} \oplus W) \cong \pi(\eta V \boxtimes_1 W)$
$\cong \tau^*\pi(\eta W \boxtimes_1 \eta V) \cong \tau^*[\eta W \boxtimes_2 (V^{\mathrm{ass}} \oplus V)]$

ix) $\pi(V \boxtimes\!\boxtimes W) \cong (\eta V) \boxtimes_2 W \cong \tau^*[(\eta W) \boxtimes_2 V]$

x) $(\pi V) \boxtimes\!\boxtimes W \cong \eta(V \boxtimes_2 W)$.

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Proof. i) Use the map

$$v \otimes w \mapsto (-1)^{t(v)t(w)} w \otimes v.$$

Checking that this commutes with the action uses

$$s(\lambda)t(v) + t(\omega v)t(\lambda w) \equiv t(v)t(w) + s(\omega)s(\lambda) + s(\omega)t(v) \pmod 2.$$

ii) The action and the graded parts are the same for all three of these.

iii) Use the maps

$$(V \boxtimes_2 W)^{\text{ass}} \to V \boxtimes_2 (W^{\text{ass}}) \to V^{\text{rev}} \boxtimes_2 W$$

$$v \otimes w \mapsto (-1)^{tv} v \otimes w; \; v \otimes w \mapsto v \otimes w$$

and check that the action is preserved.

iv) and v) A sub-graded representation $(V \boxtimes\!\boxtimes W)^{\text{comp}}$ of $V \boxtimes_1 W$ is given as follows, where the checking is exactly as in 2.20:

$$(V \boxtimes\!\boxtimes W)_0^{\text{comp}} = \text{Span}_{\mathbf{C}}\{ (v, v) \otimes (w, w) - i(v, -v) \otimes (w, -w) \}$$

$$(V \boxtimes\!\boxtimes W)_1^{\text{comp}} = \text{Span}_{\mathbf{C}}\{ (v, v) \otimes (w, -w) + i(v, -v) \otimes (w, w) \}.$$

Then

$$\begin{aligned}
&(V \boxtimes\!\boxtimes W)_0 + (V \boxtimes\!\boxtimes W)_0^{\text{comp}} \\
&= \text{Span}\{ (v, v) \otimes (w, w) \} + \text{Span}\{ (v, -v) \otimes (w, -w) \} \\
&= D_V \otimes D_W + A_V \otimes A_W = \eta V_0 \otimes \eta W_0 + \eta V_1 \otimes \eta W_1 \\
&= (\eta V \boxtimes_1 \eta W)_0.
\end{aligned}$$

Similarly

$$(V \boxtimes\!\boxtimes W)_1 + (V \boxtimes\!\boxtimes W)_1^{\text{comp}} = (\eta V \boxtimes_1 \eta W)_1.$$

It remains to show that the following are all isomorphic

$$\begin{array}{ccc}
(V \boxtimes\!\boxtimes W)^{\text{comp}} & (V \boxtimes\!\boxtimes W)^{\text{rev}} & V^{\text{ass}} \boxtimes\!\boxtimes W \\
\parallel & \parallel & \parallel \\
A & B & C \\
 & & \\
V \boxtimes\!\boxtimes W^{\text{ass}} & \tau^*(W \boxtimes\!\boxtimes V) & \\
\parallel & \parallel & \\
D & E &
\end{array}$$

For E, recall that if

$$\Omega \underset{\cong}{\overset{f}{\longrightarrow}} \Omega',$$

then $f^* U$ is the space U with the action

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$(\omega, u) \to f(\omega) \cdot u.$

Now determine maps by the following formulae:

$$A \to B \text{ by } (v_1, v_2) \otimes (w_1, w_2) \mapsto (v_1, v_2) \otimes (w_1, -w_2)$$

$$A \to C \text{ by } (v_1, v_2) \otimes (w_1, w_2) \mapsto (v_2, v_1) \otimes (w_1, w_2)$$

$$A \to D \text{ by } (v_1, v_2) \otimes (w_1, w_2) \mapsto (v_1, v_2) \otimes (w_2, w_1)$$

$A \to E$ by

$$(v, v) \otimes (w, w) - i(v, -v) \otimes (w, -w) \mapsto (w, w) \otimes (v, v) + i(w, -w) \otimes (v, -v)$$

$$(v, v) \otimes (w, -w) + i(v, -v) \otimes (w, w) \mapsto i[\,(w, w) \otimes (v, -v) - i(w, -w) \otimes (v, v)\,].$$

In each case, check that a linear map is determined by the formula; that grading is preserved; that the image contains a spanning set of the codomain; and that domain and codomain have the same dimension. We thus get isomorphisms which by lengthy calculations may be checked to preserve the group action.

vi) Both are $(V_0 \oplus V_1) \otimes (W_0 \oplus W_1)$ with the same action.

vii) An isomorphism

$$\{D_{(V_0 \oplus V_1) \otimes W}, A_{(V_0 \oplus V_1) \otimes W}\} \cong \{V_0 \otimes D_W + V_1 \otimes A_W, V_0 \otimes A_W + V_1 \otimes D_W\}$$

is given by

$$[\,(v_0, v_1) \otimes w, (v_0, v_1) \otimes w] \mapsto v_0 \otimes (w, w) + v_1 \otimes (w, -w)$$

$$[\,(v_0, v_1) \otimes w, -(v_0, v_1) \otimes w] \mapsto v_0 \otimes (w, -w) + v_1 \otimes (w, w).$$

viii)
$$\begin{aligned} 2\pi(V \boxtimes\!\boxtimes W) &\cong \pi(V \boxtimes\!\boxtimes W) + \pi(V \boxtimes\!\boxtimes W^{\mathrm{rev}}) \\ &\cong \pi[\,(V \boxtimes\!\boxtimes W)^{\mathrm{rev}} \oplus (V \boxtimes\!\boxtimes W)\,] \underset{v)}{\cong} \pi(\eta V \boxtimes_1 \eta W) \\ &\underset{\text{vi)}}{\cong} (\eta V) \boxtimes_2 (\pi\eta W) \underset{2.8}{\cong} (\eta V) \boxtimes_2 (W^{\mathrm{ass}} \oplus W). \end{aligned}$$

The last two follow from i).

ix) $(\eta V) \boxtimes_2 (W^{\mathrm{ass}}) \underset{\text{iii)}}{\cong} (\eta V)^{\mathrm{rev}} \boxtimes_2 W \underset{2.8}{\cong} \eta V \boxtimes_2 W.$

Thus

$$2\pi(V \boxtimes\!\boxtimes W) \underset{\text{viii)}}{\cong} \eta V \boxtimes_2 W^{\mathrm{ass}} + \eta V \boxtimes_2 W \cong 2\eta V \boxtimes_2 W.$$

Dividing by 2 in REP^- gives the result.

x) Similarly, this is deduced from earlier identities, dividing by 2:

$$2(\pi V) \boxtimes\!\boxtimes W \cong (\pi V) \boxtimes\!\boxtimes W + (\pi V)^{\mathrm{ass}} \boxtimes\!\boxtimes W$$

$$\underset{\mathrm{v)}}{\cong} (\eta\pi V) \boxtimes_1 \eta W \cong (V + V^{\mathrm{rev}}) \boxtimes_1 \eta W$$

$$\underset{\mathrm{ii)}}{\cong} V \boxtimes_1 \eta W + V \boxtimes_1 (\eta W)^{\mathrm{rev}} \cong 2V \boxtimes_1 \eta W.$$

Again in the proposition below, which of U, V, W are graded (and which are not) is clear from the context.

PROPOSITION 2.23. *With* Γ, Λ, Ω *resp. acting on* U, V, W *resp., we have* $\Gamma \hat{\times} \Lambda \hat{\times} \Omega$*-isomorphisms*

i) $U \boxtimes_1 (V \boxtimes_1 W) \cong (U \boxtimes_1 V) \boxtimes_1 W$

ii) $U \boxtimes_2 (V \boxtimes_2 W) \cong (U \boxtimes_1 V) \boxtimes_2 W$

iii) $U \boxtimes_1 (V \boxtimes\!\boxtimes W) \cong (U \boxtimes_2 V) \boxtimes\!\boxtimes W$

iv) $\tau^*[(V \boxtimes\!\boxtimes W) \boxtimes_2 U] \cong (U \boxtimes\!\boxtimes V) \boxtimes_2 W.$

Proof. In both i) and ii) all spaces are $U \otimes V \otimes W$ with the same action, and, in i), with the same grading. In iii), $(U \boxtimes_2 V) \boxtimes\!\boxtimes W$ is $\{Z_0, Z_1\}$ where Z_0 is the subspace of

$$D_{(U_0 \oplus U_1) \otimes V} \otimes D_W + A_{(U_0 \oplus U_1) \otimes V} \otimes A_W$$

generated by elements α and β below for all $u_i \in U_i$, $v \in V$ and $w \in W$:

$$\begin{aligned} \alpha(u_0, v, w) &= [(u_0, 0) \otimes v, (u_0, 0) \otimes v] \otimes [w, w] \\ &\quad + i[(u_0, 0) \otimes v, -(u_0, 0) \otimes v] \otimes [w, -w] \\ \beta(u_1, v, w) &= [(0, u_1) \otimes v, (0, u_1) \otimes v] \otimes [w, w] \\ &\quad + i[(0, u_1) \otimes v, -(0, u_1) \otimes v] \otimes [w, -w] \end{aligned}$$

and

$$Z_1 \subset D_{(U_0 \oplus U_1) \otimes V} \otimes A_W + A_{(U_0 \oplus U_1) \otimes V} \otimes D_W$$

has generators

$$\begin{aligned} \gamma(u_0, v, w) &= [(u_0, 0) \otimes v, (u_0, 0) \otimes v] \otimes [w, -w] \\ &\quad - i[(u_0, 0) \otimes v, -(u_0, 0) \otimes v] \otimes [w, w] \\ \delta(u_1, v, w) &= [(0, u_1) \otimes v, (0, u_1) \otimes v] \otimes [w, -w] \\ &\quad - i[(0, u_1) \otimes v, -(0, u_1) \otimes v] \otimes [w, w]. \end{aligned}$$

Similarly $U \boxtimes_1 (V \boxtimes\!\!\boxtimes W) = \{Y_0, Y_1\}$ with

$$Y_0 \subset U_0 \otimes [D_V \otimes D_W + A_V \otimes A_W] + U_1 \otimes [D_v \otimes A_W + A_V \otimes D_W]$$

being generated by

$$\alpha'(u_0, v, w) = u_0 \otimes [\,(v, v) \otimes (w, w) + i(v, -v) \otimes (w, -w)\,]$$
$$\beta'(u_1, v, w) = u_1 \otimes [\,(v, v) \otimes (w, -w) - i(v, -v) \otimes (w, w)\,]$$

and

$$Y_1 \subset U_0 \otimes [D_V \otimes A_W + A_V \otimes D_W] + U_1 \otimes [D_V \otimes D_W + A_V \otimes A_W]$$

has generators

$$\gamma'(u_0, v, w) = u_0 \otimes [\,(v, v) \otimes (w, -w) - i(v, -v) \otimes (w, w)\,]$$
$$\delta'(u_1, v, w) = u_1 \otimes [\,(v, v) \otimes (w, w) + i(v, -v) \otimes (w, -w)\,].$$

A linear map is uniquely determined by requiring $\alpha \mapsto \alpha'$, $\beta \mapsto i\beta'$, $\gamma \mapsto \gamma'$ and $\delta \mapsto -i\delta'$. Its image spans $U \boxtimes_1 (V \boxtimes\!\!\boxtimes W)$ which has the same dimensions as $(U \boxtimes_2 V) \boxtimes\!\!\boxtimes W$, so it is bijective. The proof of iii) is completed by a tedious calculation checking that the map commutes with the action of $\Gamma \hat{\times} \Lambda \hat{\times} \Omega$. To prove iv) we define an isomorphism

$$(U \boxtimes\!\!\boxtimes V) \boxtimes_2 W \to \tau^*[\,(V \boxtimes\!\!\boxtimes W) \boxtimes_2 U]$$

by $\phi \mapsto \phi' + \psi'$ and $\psi \mapsto -i\phi' + i\psi'$ where the domain has generators

$$\phi(u, v, w) = [\,(u, u) \otimes (v, v) + i(u, -u) \otimes (v, -v)\,] \otimes w$$
$$\psi(u, v, w) = [\,(u, u) \otimes (v, -v) - i(u, -u) \otimes (v, v)\,] \otimes w$$

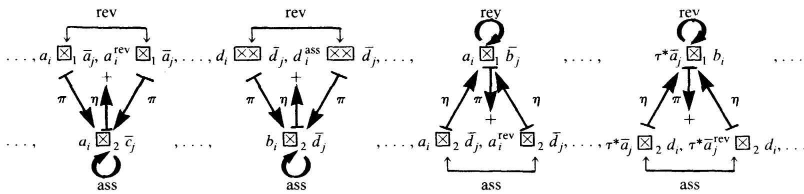
and the codomain has generators

$$\phi'(u, v, w) = [\,(v, v) \otimes (w, w) + i(v, -v) \otimes (w, -w)\,] \otimes u$$
$$\psi'(u, v, w) = [\,(v, v) \otimes (w, -w) - i(v, -v) \otimes (w, w)\,] \otimes u.$$

THEOREM 2.24. *Make lists as in* 2.10 *for $\mathcal{G}$-objects Ω and $\bar{\Omega}$:*

$$\mathrm{GIRREP}^-\Omega = \{a_1, a_1^{\mathrm{rev}}, \ldots, a_\mu^{\mathrm{rev}}, b_1, \ldots, b_\nu\};$$
$$\mathrm{GIRREP}^-\bar{\Omega} = \{\bar{a}_1, \ldots, \bar{b}_{\bar{\nu}}\};$$
$$\mathrm{IRREP}^-\Omega = \{c_1, \ldots, c_\mu, d_1, d_1^{\mathrm{ass}}, \ldots, d_\nu^{\mathrm{ass}}\};$$
$$\mathrm{IRREP}^-\bar{\Omega} = \{\bar{c}_1, \ldots, \bar{d}_{\bar{\nu}}^{\mathrm{ass}}\}.$$

Then the corresponding lists for $\Omega \hat{\times} \bar{\Omega}$ are given by the following diagram, which also gives the behaviour of rev, ass, π *and* η. (*The notation* $\underset{+}{\downarrow}$ *means "goes to the sum of the nearby elements"*):



In particular, the above lists are all distinct elements.

Before proving 2.24 we discuss some corollaries.

COROLLARY 2.25. *Counting in this diagram,*

$$\mu(\Omega \hat{\times} \bar{\Omega}) = \mu(\Omega)\mu(\bar{\Omega}) + \nu(\Omega)\nu(\bar{\Omega})$$
$$\nu(\Omega \hat{\times} \bar{\Omega}) = \mu(\Omega)\nu(\bar{\Omega}) + \nu(\Omega)\mu(\bar{\Omega}).$$

COROLLARY 2.26. *Define a functor W by*

$$W(\Omega) := \frac{\mathrm{GR}^{-}\Omega}{\{x:x^{\mathrm{rev}} = x\}} \oplus \frac{\mathrm{R}^{-}\Omega}{\{y:y^{\mathrm{ass}} = y\}} := W^{(0)} \oplus W^{(1)}.$$

Then there is an isomorphism

$$\boxtimes_4 : W\Omega \otimes W\bar{\Omega} \to W(\Omega \hat{\times} \bar{\Omega})$$

defined by using $\boxtimes_1$ *on* $W^{(0)} \otimes W^{(0)}$; $\boxtimes_2$ *on* $W^{(0)} \otimes W^{(1)}$; $\boxtimes\!\boxtimes$ *on* $W^{(1)} \otimes W^{(1)}$; *and* $\tau^* \circ \boxtimes_2 \circ \sigma$ *on* $W^{(1)} \otimes W^{(0)}$, *where* $\sigma(x \otimes y) = y \otimes x$.

Proof. This is immediate from 2.24 as long as $\boxtimes_4$ is well defined. A sample three of eight checks for this are as follows:

$$x = x^{\mathrm{rev}} \Rightarrow \begin{cases} (x \boxtimes_1 \bar{x})^{\mathrm{rev}} = x^{\mathrm{rev}} \boxtimes_1 \bar{x} = x \boxtimes_1 \bar{x}; \\ (x \boxtimes_2 \bar{y})^{\mathrm{ass}} = x^{\mathrm{rev}} \boxtimes_2 \bar{y} = x \boxtimes_2 \bar{y}. \end{cases}$$

$$y = y^{\mathrm{ass}} \Rightarrow (y \boxtimes\!\boxtimes \bar{y})^{\mathrm{rev}} = y^{\mathrm{ass}} \boxtimes\!\boxtimes \bar{y} = y \boxtimes\!\boxtimes \bar{y}.$$

COROLLARY 2.27. *Define*

$$\mathrm{R}^{-}_{sa}(\Omega) = \{y \in \mathrm{R}^{-}(\Omega): y = y^{\mathrm{ass}}\}.$$

Then we have a well defined homomorphism

$$\boxtimes_3 : \mathrm{R}^{-}_{sa}(\Omega) \otimes \mathrm{R}^{-}_{sa}(\bar{\Omega}) \to \mathrm{R}^{-}_{sa}(\Omega \hat{\times} \bar{\Omega})$$
$$y \otimes \bar{y} \mapsto \pi[\pi^{-1}y \boxtimes_1 \pi^{-1}\bar{y}].$$

This map is injective with image of index 2 *in its codomain. More precisely, in notation from* 2.24,

$$\mathrm{Coker}\, \boxtimes_3 \cong (\mathbf{Z}/2)^{\nu\bar{\nu}}$$

with generators being the images of the elements $b_i \boxtimes_2 \bar{d}_j$. *The isomorphism*

$$R_{sa}^{-}(\Omega) \otimes R_{sa}^{-}(\bar{\Omega}) \otimes \mathbf{Q} \cong R_{sa}^{-}(\Omega \mathbin{\hat{\times}} \bar{\Omega}) \otimes \mathbf{Q}$$

is also an isometry with respect to the canonical inner products in these **Q**-*spaces.*

Proof. This is again immediate from 2.24 modulo the well definition of $\boxtimes_3$. But $\pi x = \pi x'$ implies

$$\pi(x \boxtimes_1 \bar{y}) = (\pi x) \boxtimes_2 \bar{y} = (\pi x') \boxtimes_2 \bar{y} = \pi(x' \boxtimes_2 \bar{y}).$$

Similarly for the right hand factor. The last sentence is just the fact obtained from 2.24 that

$$\langle x \boxtimes_3 \bar{x}, y \boxtimes_3 \bar{y} \rangle = \langle x, y \rangle \langle \bar{x}, \bar{y} \rangle$$

for the canonical basis elements c_i, $d_j + d_j^{\text{ass}}$, $\bar{c}_i$, $\bar{d}_j + \bar{d}_j^{\text{ass}}$ for these spaces.

Proof of 2.24. The formulae for π, η, rev and ass are immediate from 2.22. Thus the list for $\text{GIRREP}^{-}(\Omega \mathbin{\hat{\times}} \bar{\Omega})$ follows from that for $\text{IRREP}^{-}(\Omega \mathbin{\hat{\times}} \bar{\Omega})$ by 2.11. We use 2.15 to check the latter. As for the $\dim^2$ condition, let $\gamma_i = \dim c_i$ and $\delta_k = \dim d_k$, so

$$\sum \gamma_i^2 + 2 \sum \delta_k^2 = \frac{1}{2}\#\Omega.$$

Let $\bar{\gamma}_j = \dim \bar{c}_j$ and $\bar{\delta}_l = \dim \bar{d}_l$, so

$$\sum \bar{\gamma}_j^2 + 2 \sum \bar{\delta}_l^2 = \frac{1}{2}\#\bar{\Omega}.$$

Then

$$\begin{aligned}
&\dim a_i \boxtimes_2 \bar{c}_j = \gamma_i \bar{\gamma}_j \\
&\dim a_i \boxtimes_2 \bar{d}_j = \gamma_i \bar{\delta}_l \\
&\dim \tau^*(\bar{a}_j \boxtimes_2 d_i) = \delta_k \bar{\gamma}_j \quad \text{and} \\
&\dim b_i \boxtimes_2 \bar{d}_j = \dim d_i \boxtimes\!\boxtimes\, \bar{d}_j = 2\delta_k \bar{\delta}_l.
\end{aligned}$$

Hence the sum of squares of dimensions of the listed elements for $\text{IRREP}^{-}(\Omega \mathbin{\hat{\times}} \bar{\Omega})$ is

$$\begin{aligned}
&\sum (\gamma_i \bar{\gamma}_j)^2 + \sum (2\delta_k \bar{\delta}_l)^2 + 2 \sum (\gamma_i \bar{\delta}_l)^2 + 2 \sum (\delta_k \bar{\gamma}_j)^2 \\
&= \left(\sum \gamma_i^2 + 2 \sum \delta_k^2\right)\left(\sum \bar{\gamma}_j^2 + 2 \sum \bar{\delta}_l^2\right) \\
&= \left(\frac{1}{2}\#\Omega\right)\left(\frac{1}{2}\#\bar{\Omega}\right) = \frac{1}{2}\#(\Omega \mathbin{\hat{\times}} \bar{\Omega}),
\end{aligned}$$

as required. The condition on inner products is immediately checked using 2.30 below. This is the necessary modification of the usual proof for ordinary exterior tensor products of irreducibles to give the (ungraded) irreducibles of the (untwisted) cartesian product of finite groups. Below we modify the usual adjointness to take account of mixtures of graded and ungraded representations.

The Brothers HOM, GHOM, *and* TWOM. For $[U]$, $[V]$ in $\mathrm{REP}^{-}\Lambda$,

$$\mathrm{HOM}_{\Lambda}(U, V) = \{f \in \mathrm{HOM}_{\mathbf{C}}(U, V): f \text{ commutes with the action}\}.$$

For $[U]$, $[V]$ in $\mathrm{GREP}^{-}\Lambda$,

$$\mathrm{GHOM}_{\Lambda}(U, V) = \{ (f_0, f_1) \in \mathrm{HOM}(U_0, V_0) \times \mathrm{HOM}(U_1, V_1): \\ f = f_0 \oplus f_1 \text{ commutes with the action}\}.$$

For $[W]$, $[U'] \in \mathrm{REP}^{-}\Lambda$, define a $\mathbf{Z}/2$-graded vector space $\mathrm{TWOM}_{\Lambda}(W, U')$ by

$$\mathrm{TWOM}_{\Lambda}(W, U')_0 = \mathrm{HOM}_{\Lambda}(W, U')$$
$$\mathrm{TWOM}_{\Lambda}(W, U')_1 = \mathrm{HOM}_{\Lambda}(W, U'^{\mathrm{ass}}).$$

LEMMA 2.28. *If* $U \in \mathrm{REP}^{-}(\Gamma \hat{\times} \Lambda)$ *and* $U' = U|_{\Lambda}$ (*i.e.*, $U' = \nu^* U$ *where* $\nu(h) = (1, h)$), *then* $\mathrm{TWOM}_{\Lambda}(W, U')$ *becomes* $\mathbf{Z}/2$*-graded representation of* Γ *if we define an action by*

$$g \cdot f : w \mapsto (g, 1) \cdot f(w)$$

for all $g \in \Gamma$ *and* $f \in \mathrm{TWOM}_{\Lambda}(W, U')$.

Proof. It is obviously an action, and a direct calculation shows that gradings are mapped properly.

LEMMA 2.29. *If in* 2.28 *we have* $U = V' \boxtimes_2 W$ *where* V' *is irreducible, and* $W \not\cong W^{\mathrm{ass}}$ *is also irreducible, then the map*

$$v' \mapsto \left(w \overset{h_{v'}}{\mapsto} v' \otimes w \right)$$

is an isomorphism of $\mathbf{Z}/2$*-graded representations*

$$V' \cong \mathrm{TWOM}_{\Lambda}(W, V' \boxtimes_2 W|_{\Lambda}).$$

Proof. The dimensions of the graded parts agree and the map is clearly injective, so we need only calculate to check that $h_{g \cdot v'} = g \cdot h_{v'}$ and that $h_{v'} \in \mathrm{TWOM}_i$ if $v' \in V'_i$.

THEOREM 2.30. i) *Let* $[V]$, $[V']$ *be in* $\mathrm{GIRREP}^{-}\Gamma$ *and* $[W]$, $[W']$ *in* $\mathrm{IRREP}^{-}\Lambda$. *Assume either* $W \not\cong W^{\mathrm{ass}}$ *or* $V \not\cong V^{\mathrm{rev}}$. *Then*

$$\mathrm{HOM}_{\Gamma \hat{\times} \Lambda}[V \boxtimes_2 W, V' \boxtimes_2 W'] \cong \begin{cases} \mathbf{C} & \text{if } V \cong V' \text{ and } W \cong W' \\ \mathbf{C} & \text{if } V^{\mathrm{rev}} \cong V' \text{ and } W^{\mathrm{ass}} \cong W' \\ 0 & \text{otherwise.} \end{cases}$$

ii) *If also* $[Y] \in \mathrm{GIRREP}^{-}\Lambda$ *and* $[Z] \in \mathrm{IRREP}^{-}\Gamma$, *then*

$$\mathrm{HOM}_{\Gamma \hat{\times} \Lambda}[V \boxtimes_2 W, \tau^*(Y \boxtimes_2 Z)] \cong \begin{cases} \mathbf{C} \text{ if } V \cong \eta Z \text{ and } W \cong \pi Y \\ 0 \text{ otherwise.} \end{cases}$$

iii) *For* $[U]$, $[U']$ *in* $\mathrm{REP}^{-}(\Lambda \hat{\times} \Gamma)$, *there is an isomorphism of vector spaces*

$$\mathrm{HOM}_{\Gamma \hat{\times} \Lambda}(\tau^* U, \tau^* U') \cong \mathrm{HOM}_{\Lambda \hat{\times} \Gamma}(U, U').$$

Proof. The idea is first to prove that the map below is an isomorphism of vector spaces:

$$\mathrm{HOM}_{\Gamma \hat{\times} \Lambda}(V \boxtimes_2 W, U) \to \mathrm{GHOM}_{\Gamma}[V, \mathrm{TWOM}_{\Lambda}(W, U|_{\Lambda})]$$

$$f \mapsto (v \overset{e}{\mapsto} [w \overset{e_v}{\mapsto} f(v \otimes w)]).$$

One first checks that $v \in V_i$ implies $e_v \in \mathrm{TWOM}_i$; then that e commutes with the action of Γ. Bijectivity follows by writing down the obvious inverse. Now to prove i), note that $(V' \boxtimes_2 W')|_{\Lambda}$ is Λ-isomorphic to

$$(W')^{\dim V'_0} \oplus (W'^{\mathrm{ass}})^{\dim V'_1},$$

so

$$\mathrm{TWOM}_{\Lambda}(W, V' \boxtimes_2 W'|_{\Lambda}) = 0 \quad \text{if } W^{\mathrm{ass}} \ncong W' \ncong W.$$

So we are left with the cases $W' \cong W$ and $W' \cong W^{\mathrm{ass}}$. Below we assume $W \ncong W^{\mathrm{ass}}$; the alternative $V \ncong V^{\mathrm{rev}}$ is easier. In the first case

$$\begin{aligned} &\mathrm{HOM}_{\Gamma \hat{\times} \Lambda}(V \boxtimes_2 W, V' \boxtimes_2 W) \\ &\cong \mathrm{GHOM}_{\Gamma}[V, \mathrm{TWOM}_{\Lambda}(W, V' \boxtimes_2 W|_{\Lambda})] \\ &\underset{2.29}{\cong} \mathrm{GHOM}_{\Gamma}(V, V') \cong \begin{cases} \mathbf{C} \text{ if } V \cong V' \\ 0 \text{ if not.} \end{cases} \end{aligned}$$

In the second case, we get

$$\begin{aligned} &\mathrm{GHOM}_{\Gamma}[V, \mathrm{TWOM}_{\Lambda}(W, V' \boxtimes_2 W^{\mathrm{ass}}|_{\Lambda})] \\ &\cong \mathrm{GHOM}_{\Gamma}[V, \mathrm{TWOM}_{\Lambda}(W, V'^{\mathrm{rev}} \boxtimes_2 W|_{\Lambda})] \\ &\cong \mathrm{GHOM}_{\Gamma}(V, V'^{\mathrm{rev}}) \cong \begin{cases} \mathbf{C} \text{ if } V^{\mathrm{rev}} \cong V' \\ 0 \text{ if not.} \end{cases} \end{aligned}$$

To prove ii), again use the above adjointness isomorphism to yield

$$\mathrm{GHOM}_{\Gamma}[V, \mathrm{TWOM}_{\Lambda}(W, \tau^*(Y \boxtimes_2 Z)|_{\Lambda})].$$

Now

$$\tau^*(Y \boxtimes_2 Z)|_{\Lambda} \cong (\pi Y)^{\dim Z},$$

so we get zero if $W \ncong \pi Y$. On the other hand, the maps

$$D_Z \to \mathrm{HOM}_\Lambda(\pi Y,\ Y \boxtimes_2 Z|_\Lambda); \quad A_Z \to \mathrm{HOM}_\Lambda(\pi Y,\ Y \boxtimes_2 Z^{\mathrm{ass}}|_\Lambda)$$
$$(z, z) \mapsto (y \mapsto y \otimes z) \qquad ; \quad (z, -z) \mapsto (y \mapsto (-1)^{ty} y \otimes z)$$

give an isomorphism

$$\eta Z \to \mathrm{TWOM}_\Lambda[\pi Y,\ \tau^*(Y \boxtimes_2 Z)\,|_\Lambda].$$

Thus when $W \cong \pi Y$, we get

$$\mathrm{GHOM}_\Gamma(V, \eta Z) \quad \cong \quad \begin{cases} \mathbf{C} \text{ if } V \cong \eta Z \\ 0 \text{ if not.} \end{cases}$$

iii) More generally,

$$\mathrm{HOM}_\Omega(r^*U,\ r^*U') = \mathrm{HOM}_{\Omega'}(U,\ U')$$

for any isomorphism $r:\Omega \to \Omega'$ and any $[U]$, $[U']$ in REP Ω'.

3. Determination of $\mathrm{R}^-(\tilde{\Sigma}_n\langle\Gamma\rangle)$ **and** $\mathrm{GR}^-(\tilde{\Sigma}_n\langle\Gamma\rangle)$. To state certain assumptions and results it is convenient to introduce a category of doubly graded rings as follows.

Definition 3.1. Let $\mathscr{C}$ denote the category whose objects are pairs (H, p), where H is a $(\mathbf{Z}/2 \times \mathbf{N})$-graded ring with $1 \in H_{0,0}$ and $p \in H_{1,0}$, such that: $p^3 = 2p$; and, with $r := p^2 - 1 \in H_{0,0}$,

$$xy = r^{\epsilon\delta + ij} yx \quad \text{for all } x \in H_{\epsilon,i},\ y \in H_{\delta,j}.$$

Morphisms in $\mathscr{C}$ will be homomorphisms of graded rings mapping p to p. Note that $r^2 = 1$; $pr = rp = p$; and all elements commute with p and r.

Example. Let S be a finite set. Define a $\mathscr{C}$-object HS by generators and relations: The generators are $p \in HS_{1,0}$ and, for each $i > 0$ and $s \in S$, a generator $h_i^{(s)}$ where

$$h_{2k+1}^{(s)} \in HS_{0,2k+1} \quad \text{and} \quad h_{2k}^{(s)} \in HS_{1,2k}.$$

The relations are $p^3 = 2p$; $xy = r^{\epsilon\delta + ij} yx$ for all generators (hence all elements) and

$$(h_i^{(s)})^2 = (-1)^{i+1} p \left[h_{2i}^{(s)} + p \sum_{j=1}^{i-1} (-1)^j h_{2i-j}^{(s)} h_j^{(s)} \right].$$

The first theorem below gives conditions on a $\mathscr{C}$-object which suffice to verify that it is isomorphic to HS. The second one asserts that these conditions hold when $S = \mathrm{Con}(\Gamma)$ for the object $\bar{H}$ for which

$$\bar{H}_{0,n} = \mathrm{GR}^-(\tilde{\Sigma}_n\langle\Gamma\rangle) \quad \text{and} \quad \bar{H}_{1,n} = \mathrm{R}^-(\tilde{\Sigma}_n\langle\Gamma\rangle).$$

Together then they describe the structure of these latter groups and give considerably more information about additional structures (products, inner products, coproducts, etc.) on these groups.

Hypotheses. These will be the following conditions (I)-(VII) on a $\mathcal{C}$-object $\bar{H}$ relative to a given finite set S. Recall the sets of functions into partitions $\mathcal{D}'(n, S)$ etc. defined in 1.12.

(I) There exists a set $\{z_\phi:\phi \in \mathcal{D}(n, S)\}$ such that $\bar{H}_{0,n}$ and $\bar{H}_{1,n}$ are free abelian groups with bases

$$\{z_\phi, rz_\phi:\phi \in \mathcal{D}''(n, S)\} \cup \{pz_\phi:\phi \in \mathcal{D}'(n, S)\}$$

and

$$\{pz_\phi:\phi \in \mathcal{D}''(n, S)\} \cup \{z_\phi, rz_\phi:\phi \in \mathcal{D}'(n, S)\},$$

respectively.

Definition 3.2.

$$D^+ = \mathrm{Span}_{\mathbf{Z}}\{pz_\phi:\phi \in \mathcal{D}''(n, S)\}$$

$$D^- = \mathrm{Span}_{\mathbf{Z}}\{(1 + r)z_\phi:\phi \in \mathcal{D}'(n, S)\}$$

so that $D^+ \oplus D^- = D\bar{H}$, where for any $\mathcal{C}$-object H we define

$$DH := \bigoplus_{n \geqq 0} pH_{0,n}.$$

Define also

$$EH := H/IH,$$

where

$$IH := \{x \in H:rx = x\}.$$

We make D, E and I into functors in the obvious way. Clearly IH is a two sided ideal in H, so EH is a $(\mathbf{Z}/2 \times \mathbf{N})$-graded ring which is anti-commutative in the following graded sense:

$$xy = (-1)^{\epsilon\delta + ij}yx \quad \text{for } x \in EH_{\epsilon,i},\ y \in EH_{\delta,j}.$$

It is also clear that

$$\bigoplus_n \mathrm{Ker}\{H_{0,n} \xrightarrow{p} H_{1,n}\}$$

is a two-sided ideal in the $\mathbf{N}$-graded ring $H_{0,*}$. Thus the induced multiplication on DH, denoted $\circ$, makes DH into a commutative $\mathbf{N}$-graded ring. We have

$$(px) \circ (py) := pxy.$$

(II) There exist coproducts

$$\nabla:E\bar{H} \to E\bar{H} \otimes E\bar{H} \text{ and } \Delta:D\bar{H} \to D\bar{H} \otimes D\bar{H} \otimes \mathbf{Q}$$

making $E\bar{H}$ and $D\bar{H} \otimes \mathbf{Q}$ into graded Hopf algebras.

(III) $\triangle(D\bar{H}) \subset D^- \otimes D^- \otimes \frac{1}{2}\mathbf{Z} + D\bar{H} \otimes D\bar{H} \otimes \mathbf{Z}$.

(IV) There exist elements $\bar{h}_n^{(s)} \in \bar{H}$, $n > 0$, $s \in S$ with

$$\bar{h}_{2k+1}^{(s)} \in \bar{H}_{0,2k+1} \quad \text{and} \quad \bar{h}_{2k}^{(s)} \in \bar{H}_{1,2k}$$

such that ∇ and $\triangle$ act as follows: If

$$[\bar{h}_n^{(s)}] = \bar{h}_n^{(s)} + I\bar{H} \in E\bar{H}$$

then

$$\nabla[\bar{h}_n^{(s)}] = [\bar{h}_n^{(s)}] \otimes [1] + [1] \otimes [\bar{h}_n^{(s)}],$$

that is, $[\bar{h}_n^{(s)}]$ is primitive.

If

$$\bar{b}_{2k+1}^{(s)} := 2^{k+1}p\bar{h}_{2k+1}^{(s)} \in D\bar{H}_{2k+1}$$

and

$$\bar{b}_{2k}^{(s)} := 2^k(1 + r)\bar{h}_{2k}^{(s)} \in D\bar{H}_{2k},$$

then

$$\triangle\bar{b}_n^{(s)} = \sum_{i+j=n} \bar{b}_i^{(s)} \otimes \bar{b}_j^{(s)},$$

where $\bar{b}_0^{(s)}$ is the identity element of $D\bar{H}$, namely p.

(V) There exist positive definite symmetric bilinear forms

$$\langle\, ,\, \rangle : \bar{H}_{\epsilon,i} \times \bar{H}_{\epsilon,i} \to \mathbf{Z}$$

such that $\langle rx, ry\rangle = \langle x, y\rangle$ for all x, y.

(VI) If $\ll\, ,\, \gg$ is the induced inner product on $\bar{H} \otimes \bar{H}$ (that is,

$$\ll u \otimes v, x \otimes y \gg = \langle u, x\rangle\langle v, y\rangle\ )$$

then for x, y, z in $D\bar{H}$ we have

$$\langle x \circ y, z\rangle = \ll x \otimes y, \triangle z \gg.$$

(VII) For the elements in (IV), we have

$$\langle \bar{h}_n^{(s)}, \bar{h}_n^{(t)}\rangle = \delta_{s,t} \quad \text{and} \quad \langle \bar{h}_n^{(s)}, r\bar{h}_n^{(t)}\rangle = 0.$$

THEOREM 3.3. *The $\mathcal{C}$-object HS satisfies hypotheses* (I) *to* (VII), *and any $\mathcal{C}$-object $\bar{H}$ which satisfies* (I) *to* (VII) *is $\mathcal{C}$-isomorphic to HS. More precisely, there exists a unique $\mathcal{C}$-isomorphism $HS \to \bar{H}$ sending $h_n^{(s)}$ to $\bar{h}_n^{(s)}$. It is an isometry with respect to the inner products and induces isomorphisms*

$$EH(S) \cong E\bar{H} \quad \textit{and} \quad DH(S) \otimes \mathbf{Q} \cong D\bar{H} \otimes \mathbf{Q}$$

of Hopf algebras.

This theorem is formal algebra and is proved in the next section. We have separated it from the next theorem in order to emphasize the relative simplicity of the method of proof. The basic idea is similar to a Hopf algebra technique used by Liulevicius [5] to give a novel proof of the structure of $\oplus\, R(\Sigma_n)$. His method extends easily to $\oplus\, R(\Sigma_n\langle\Gamma\rangle)$[3]. Note that we did not assume that p and the $\bar{h}_n^{(s)}$ generate $\bar{H}$ as an algebra. This is the most important corollary to 3.3. Another corollary is that the $\bar{h}_n^{(s)}$ satisfy the relations given for $h_n^{(s)}$ in the definition of HS and "only" these relations.

THEOREM 3.4. *Let* Γ *be any finite group and let* $S = \mathrm{Con}\,\Gamma$, *the set of conjugacy classes in* Γ. *Then the* $\mathbf{Z}/2 \times \mathbf{N}$-*graded abelian group* $\bar{H}$ *where*

$$\bar{H}_{0,n} = \mathrm{GR}^-(\tilde{\Sigma}_n\langle\Gamma\rangle) \quad \textit{and} \quad \bar{H}_{1,n} = \mathrm{R}^-(\tilde{\Sigma}_n\langle\Gamma\rangle)$$

can be made into a $\mathscr{C}$-*object which satisifes* (I) *to* (VII). *More precisely, the multiplication* μ *on* $\bar{H}$ *is defined by commutative diagrams*:

$$\begin{tikzcd}
\bar{H}_{0,i} \otimes \bar{H}_{0,j} \arrow[rr, "\mu(0,0)"] \arrow[d, equal] & & \bar{H}_{0,i+j} \arrow[d, equal] \\
\mathrm{GR}^-(\tilde{\Sigma}_i\langle\Gamma\rangle) \otimes \mathrm{GR}^-(\tilde{\Sigma}_j\langle\Gamma\rangle) \arrow[r, "\boxtimes_1"] & \mathrm{GR}^-(\tilde{\Sigma}_i\langle\Gamma\rangle \hat{\times} \tilde{\Sigma}_j\langle\Gamma\rangle) \arrow[r, "\phi_*"] & \mathrm{GR}^-(\tilde{\Sigma}_{i+j}\langle\Gamma\rangle)
\end{tikzcd}$$

$$\begin{tikzcd}
\bar{H}_{0,i} \otimes \bar{H}_{1,j} \arrow[rr, "\mu(0,1)"] \arrow[d, equal] & & \bar{H}_{1,i+j}\langle\Gamma\rangle \arrow[d, equal] \\
\mathrm{GR}^-(\tilde{\Sigma}_i\langle\Gamma\rangle) \otimes \mathrm{R}^-(\tilde{\Sigma}_j\langle\Gamma\rangle) \arrow[r, "\boxtimes_2"] & \mathrm{R}^-(\tilde{\Sigma}_i\langle\Gamma\rangle \hat{\times} \tilde{\Sigma}_j\langle\Gamma\rangle) \arrow[r, "\phi_*"] & \mathrm{R}^-(\tilde{\Sigma}_{i+j}\langle\Gamma\rangle)
\end{tikzcd}$$

$$\begin{tikzcd}
\mu(1,0)\colon \bar{H}_{1,i} \otimes \bar{H}_{0,j} \arrow[r, "\sigma"] \arrow[dd, "="] & \bar{H}_{0,j} \otimes \bar{H}_{1,i} \arrow[r, "\mu(0,1)"] & \bar{H}_{1,i+j} \arrow[r, "\zeta_{j,i_*}"] & \bar{H}_{1,i+j} \arrow[d, equal] \\
& & \mathrm{R}^-(\tilde{\Sigma}_i\langle\Gamma\rangle \hat{\times} \tilde{\Sigma}_j\langle\Gamma\rangle) \arrow[r, "\phi_{i,j_*}"] & \mathrm{R}^-(\tilde{\Sigma}_{i+j}\langle\Gamma\rangle) \\
\mathrm{R}^-(\tilde{\Sigma}_i\langle\Gamma\rangle) \otimes \mathrm{GR}^-(\tilde{\Sigma}_j\langle\Gamma\rangle) \arrow[d, "\sigma"] & & \arrow[r, phantom, "1.8\mathrm{ii})"] & \\
\mathrm{GR}^-(\tilde{\Sigma}_j\langle\Gamma\rangle) \otimes \mathrm{R}^-(\tilde{\Sigma}_i\langle\Gamma\rangle) \arrow[rr, "\boxtimes_2"] & & \mathrm{R}^-(\tilde{\Sigma}_j\langle\Gamma\rangle \hat{\times} \tilde{\Sigma}_i\langle\Gamma\rangle) \arrow[r, "\phi_{j,i_*}"'] \arrow[uu, "\tau^*"] & \mathrm{R}^-(\tilde{\Sigma}_{i+j}\langle\Gamma\rangle) \arrow[uu, "\zeta_{j,i_*}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\bar{H}_{1,i} \otimes \bar{H}_{1,j} \arrow[rr, "\mu(1,1)"] \arrow[d, equal] & & \bar{H}_{0,i+j} \arrow[d, equal] \\
\mathrm{R}^-(\tilde{\Sigma}_i\langle\Gamma\rangle) \otimes \mathrm{R}^-(\tilde{\Sigma}_j\langle\Gamma\rangle) \arrow[r, "\boxtimes\boxtimes"] & \mathrm{GR}^-(\tilde{\Sigma}_i\langle\Gamma\rangle \hat{\times} \tilde{\Sigma}_j\langle\Gamma\rangle) \arrow[r, "\phi_{i,j_*}"] & \mathrm{GR}^-(\tilde{\Sigma}_{i+j}\langle\Gamma\rangle)
\end{tikzcd}$$

Furthermore the element p *is such that* π (*resp.* η) *is multiplication by* p *on* $\bar{H}_{0,*}$ (*resp.* $\bar{H}_{1,*}$); *whereas* rev (*resp.* ass) *is multiplication by* r *on* $\bar{H}_{0,*}$ (*resp.* $\bar{H}_{1,*}$).

Combining 3.3 and 3.4 we obtain much information about $GR^-(\widetilde{\Sigma}_n\langle\Gamma\rangle)$ and $R^-(\widetilde{\Sigma}_n\langle\Gamma\rangle)$. Together with the comments in the introduction and 2.12, this is equivalent when $n \geqq 4$ to information about projective representations of $\Sigma_n\langle\Gamma\rangle$ and $A_n\langle\Gamma\rangle$. In particular, as calculated in the next section, we obtain bases for the above groups. These consist of representations induced from certain products using $\boxtimes_1$, $\boxtimes\!\boxtimes$, $\boxtimes_2$, of representations coming from Clifford modules (see Section 5). When Γ is trivial, these bases coincide (modulo choices made of one from associated pairs of Clifford modules) with bases found by Schur [**9**]. Our proof is very different from his, which emphasizes characters and certain symmetric polynomials whose coefficients involve these characters. Elsewhere we shall explain more carefully this connection and generalize to the case of any Γ.

We should add here that Δ and ∇ on $\bar{H}$ are obtained essentially by "reversing" where possible the arrows in the definitions of the product. Thus its computation gives what are sometimes called branching rules. Branching rules for restriction from $\widetilde{\Sigma}_n\langle\Gamma\rangle$ to $\widetilde{A}_n\langle\Gamma\rangle$ are just multiplication by p. The inner product in $\bar{H}$ is the natural one for representations, so its computation has significance to the determination in Section 8 of the irreducibles, as well as to the proofs of 3.3 and 3.4.

Addendum to Section 3. There is a more elegant way to formulate the results of this section, sketched below. We have not presented this in detail for two reasons: it is further removed from the usual calculations made in this subject; and more importantly, it has not yet lead to any genuine simplification of the proofs of 3.3 and 3.4.

The idea is to regard HS and $\bar{H}$ as Hopf algebras over L, where

$$L := H_{*,0} \cong \mathbf{Z}[p]/(p^3 = 2p).$$

If the coproduct is $\square$ we have

$$\square(h_n^{(s)}) = h_n^{(s)} \otimes 1 + 1 \otimes h_n^{(s)} + p\left(\sum_{i=1}^{n-1} h_i^{(s)} \otimes h_{n-i}^{(s)}\right).$$

This is a definition in HS. The analogous formula for $\square\bar{h}_n^{(s)}$ in $\bar{H}$ follows from 7.1 below, where the existence of $\square$ on $\bar{H}$ is a consequence of the following corollary to 2.24:

If $V(\Omega) := GR^-(\Omega) \oplus R^-(\Omega)$ for $\Omega \in \mathcal{G}$, then

$$V(\Omega) \otimes_L V(\bar{\Omega}) \cong V(\Omega \hat{\times} \bar{\Omega})$$

(using $\boxtimes_1$, $\boxtimes_2$, $\boxtimes\!\boxtimes$ and $\tau^* \circ \boxtimes_2 \circ \sigma$).

Now (I) to (VII) can be simplified somewhat. For example: $\bar{H}$ is the free L-module on $\{z_\phi : \phi \in \mathcal{D}(n, S)\}$; and there are L-bilinear inner products

$$H_{*,n} \times H_{*,n} \to L$$

with good properties. The difficulty is that L is not even an integral domain. Furthermore Prim $\bar{H}$ is harder to work with. In even dimensions $2k$, it is

$$\cong \bigoplus_S (L/pL)$$

with generators $(r - 1)\bar{h}_{2k}^{(s)}$. In odd dimensions $(2k + 1)$, it is

$$\bigoplus_{s \in S} M_s,$$

where M_s has two generators

$$x^{(s)} = (r - 1)\bar{h}_{2k+1}^{(s)} \quad \text{and}$$

$$y^{(s)} = (2k + 1)\bar{h}_{2k+1}^{(s)} + p \sum_{i=1}^{k} (-1)^i (2k - 2i + 1)\bar{h}_{2k-i+1}\bar{h}_i$$

with relations

$$px^{(s)} = 0 \quad \text{and} \quad (r - 1)y^{(s)} = (2k + 1)x^{(s)}.$$

Using this to give direct arguments for 3.3 seems difficult. However the original (I) to (VII) can be recovered from their analogues here, so the proofs of 3.3 and 3.4 are also recovered, although more direct proofs should be possible.

Only Δ and ∇ are not obvious to recover, so we give this below. This should also help reconcile readers who have detected an odour of arbitrariness in our approach using D and E. To recover $E(\bar{H})$ and ∇ note that $p\bar{H} = \{x:rx = x\}$ since

$$\bar{H} \xrightarrow{p} \bar{H} \xrightarrow{r-1} \bar{H}$$

is exact, so that $E(\bar{H}) := \bar{H}/p\bar{H}$ is an algebra over $L/pL \cong \mathbf{Z}$. Furthermore,

$$E(\bar{H} \otimes_L \bar{H}) \cong (E\bar{H}) \otimes_{\mathbf{Z}} (E\bar{H}),$$

so we get ∇ from the diagram

$$\begin{tikzcd} H \arrow[r, "\square"] \arrow[d] & \bar{H} \otimes_L \bar{H} \arrow[d] \\ \nabla: E\bar{H} \arrow[r] & E(\bar{H} \otimes_L \bar{H}) \cong (E\bar{H}) \otimes_{\mathbf{Z}} (E\bar{H}). \end{tikzcd}$$

A similar argument gives the quotient $\bar{H}/(r - 1)\bar{H}$ the structure of a Hopf algebra over $L/(r-1)L$. The latter is $\mathbf{Z}[q]$, where q has $\mathbf{Z}/2$-grading 1 and minimal polynomial $x^2 - 2$. By the exactness of

$$\bar{H} \xrightarrow{r-1} \bar{H} \xrightarrow{p} \bar{H},$$

we have

$$D\bar{H} := p\bar{H}_{0,*} \cong \bar{H}_{0,*}/(r - 1)\bar{H}_{0,*}$$

(with $\circ$ chosen to make this an algebra isomorphism). Then $\triangle$ will be recovered by defining a coproduct $\triangle_1$ on $(\bar{H}_{0,*}/(r - 1)\bar{H}_{0,*})'$, where G' means G localized away from 2 (so

$$G' := G \otimes_R R\left[\frac{1}{2}\right]$$

where $R \subset \mathbf{Q}$ and G is an R-module). Now use the diagram

$$\begin{tikzcd}
(\bar{H}_{0,*}/(r - 1)\bar{H}_{0,*})' \arrow[r, dashed, "\triangle_1"] \arrow[dd, hook] & (\bar{H}_{0,*}/(r - 1)\bar{H}_{0,*})' \otimes_{\mathbf{Z}[1/2]} (\bar{H}_{0,*}/(r - 1)\bar{H}_{0,*})' \arrow[d, hook] \\
& (\bar{H}/(r - 1)\bar{H})' \otimes_{\mathbf{Z}[1/2]} (\bar{H}/(r - 1)\bar{H})' \arrow[d] \\
(\bar{H}/(r - 1)\bar{H})' \arrow[r, "\triangle_2"] & (H/(r - 1)\bar{H})' \otimes_{\mathbf{Z}[1/2,q]} (\bar{H}/(r - 1)\bar{H})'
\end{tikzcd}$$

Here $\triangle_2$ is the localization of the coproduct for $\bar{H}/(r - 1)\bar{H}$ over

$$\mathbf{Z}[q] = L/(r - 1)L.$$

The right hand vertical composite is clearly mono, and (because of the localization) maps onto the 0-component of its $\mathbf{Z}/2$-graded codomain. Thus we get the required $\triangle_1$.

Note that we have used exactly the two integral domain quotients of characteristic zero,

$$L/pL \cong \mathbf{Z} \quad \text{and} \quad L/(r - 1)L \cong \mathbf{Z}[\sqrt{2}],$$

of L.

4. Proof of 3.3. Recall the example HS after 3.1, and the squaring relation which expresses $(h_i^{(s)})^2$ in

$$\operatorname{Span}\{h_{2i-j}^{(s)}h_j^{(s)}: 0 \leqq j < i\}.$$

Definition 4.1. For $\phi \in \mathscr{D}(n, S)$, define

$$h_\phi = \prod_{s \in S} \prod_{i \in \phi(s)} h_i^{(s)} \in \begin{cases} HS_{1,n} & \text{if } \phi \in \mathscr{D}'(n, S) \\ HS_{0,n} & \text{if } \phi \in \mathscr{D}''(n, S). \end{cases}$$

In the above product, multiply with "i" decreasing from left to right and in some ordering for S fixed a priori. The dependence of h_ϕ on this choice is only up to a factor r.

PROPOSITION 4.2. *The groups $HS_{0,n}$ and $HS_{1,n}$ are free abelian with bases*

https://doi.org/10.4153/CJM-1986-070-1 Published online by Cambridge University Press

$$\{h_\phi, rh_\phi : \phi \in \mathscr{D}''(n, S)\} \cup \{ph_\phi : \phi \in \mathscr{D}'(n, S)\} \quad \textit{and}$$
$$\{ph_\phi : \phi \in \mathscr{D}''(n, s)\} \cup \{h_\phi, rh_\phi : \phi \in \mathscr{D}'(n, S)\}$$

respectively.

Proof. The list of generators plus the pseudo-commutativity relation shows that the elements $p^k \cdot$ (monomial in $h_i^{(s)}$) generate HS as a group. But using $p^3 = 2p$, replacing p^2 by $1 + r$, and eliminating higher powers of $h_i^{(s)}$ using the squaring relations, we see that the elements h_ϕ, rh_ϕ and ph_ϕ generate H as a group. Now take a free abelian group with these as a basis, and make it a ring using the relations for HS. Since HS maps to this, we have a basis, as required.

PROPOSITION 4.3. i) *The ring $E(HS)$ is "pseudo-exterior" on generators $[h_i^{(s)}]$: it has structure $\mathbf{Z}[[h_i^{(s)}]]/J$, where J is given by the relations $[h_i^{(s)}]^2 = 0$ and*

$$[h_i^{(s)}][h_j^{(t)}] = (-1)^{i+j+1}[h_j^{(t)}][h_i^{(s)}].$$

ii) *If*

$$d_\phi = \begin{cases} ph_\phi & \text{for } \phi \in \mathscr{D}''(n, S) \\ (1 + r)h_\phi & \text{for } \phi \in \mathscr{D}'(n, S), \end{cases}$$

then $D(HS)_n$ has basis $\{d_\phi : \phi \in \mathscr{D}(n, S)\}$. Its ring structure under $\circ$ is determined by a) *and* b):

a) *$D(HS)$ is torsion free.*

b) *If*

$$b_\phi := 2^{\left[\frac{|\phi| + l(\phi)}{2}\right]} d_\phi$$

where $\left[\ \right]$ denotes the integer part

$$BS := \mathrm{Span}_{\mathbf{Z}}\{b_\phi : \phi \in \mathscr{D}(n, S)\}$$

is a subring of $D(HS)$ whose structure is

$$\mathbf{Z}[b_i^{(s)}] \Big/ \left\langle b_i^{(s)} \circ b_i^{(s)} = 2 \sum_{j=0}^{i-1} (-1)^{i+j+1} b_{2i-j}^{(s)} \circ b_j^{(s)} \right\rangle.$$

Note. Here $b_i^{(s)}$ is b_ϕ for that ϕ mapping s to (i) and all other $t \in S$ to the empty partition. As seen below, b_ϕ is the $\circ$ product of $b_i^{(s)}$ for $s \in S$ and $i \in \phi(s)$. On the other hand d_ϕ is the analogous product of $d_i^{(s)}$ divided by a suitable power of 2. The relations for BS can be written more symmetrically

$$\sum_{j+k=2i} (-1)^j b_k^{(s)} \circ b_j^{(s)} = 0.$$

Proof of 4.3. i) Since $r \equiv -1 \bmod E(HS)$, the relations certainly hold. By 4.2, the set $\{ [h_\phi]:\phi \in \mathscr{D}(S) \}$ spans $E(HS)$. It remains only to prove linear independence. But if we assume

$$r \sum_{\phi \in \mathscr{D}''(n,S)} \mu_\phi h_\phi = \sum_{\phi \in \mathscr{D}''(n,S)} \mu_\phi h_\phi,$$

we immediately get $\mu_\phi = 0$ for all ϕ using 4.2. Similarly for $\mathscr{D}'(n, S)$.

ii) The set $\{d_\phi:\phi \in \mathscr{D}(n, S) \}$ is the image under multiplication by p of the basis in 4.2 for $H_{0,}*$, so it spans $D(HS)$. It is easy to check linear independence using 4.2. Since the $\{b_\phi\}$ basis for BS consists of scalar multiples of the $\{d_\phi\}$ basis for $D(HS)$, it is clear that the ring structure of $D(HS)$ is determined by that of BS. Now it is a matter of straightforward calculation that

$$b_\phi = \coprod_{\substack{s \in S \\ i \in \phi(s)}}^{\circ} b_i^{(s)} \quad \text{(product using } \circ)$$

and that the relations in BS hold (using the squaring relations defining HS). But these relations show that the ring generated by

$$\{b_i^{(s)}: \quad s \in S, i > 0\}$$

is spanned by $\{b_\phi:\phi \in \mathscr{D}(S) \}$. It therefore coincides with $B(S)$, and satisfies no further relations since $\{b_\phi\}$ is linearly independent.

PROPOSITION 4.4. *$E(HS)$ and $D(HS) \otimes \mathbf{Q}$ have Hopf algebra structures (graded over $\mathbf{Z}/2 \times \mathbf{N}$ and $\mathbf{N}$ respectively) given by maps*

$$\nabla:E \to E \otimes E \quad \textit{and} \quad \triangle:D \to D \otimes D \otimes \mathbf{Q}$$

for which

$$\nabla([h_n^{(s)}]) = [h_n^{(s)}] \otimes [1] + [1] \otimes [h_n^{(s)}]$$

and

$$\triangle(b_n^{(s)}) = \sum_{i+j=n} b_i^{(s)} \otimes b_j^{(s)},$$

with $b_0^{(s)} = p = 1_{DHS}$.

Proof. It is only necessary to check that the relations given in the structures for $E(HS)$ and $D(HS)$ in 4.3 are mapped to zero.

Definition 4.5. Define subgroups of $D(HS)$:

$$D' := \mathrm{Span}_{\mathbf{Z}}\{d_\phi:\phi \in \mathscr{D}'(n, S) \}$$

$$D'' := \mathrm{Span}_{\mathbf{Z}}\{d_\phi:\phi \in \mathscr{D}''(n, S) \}.$$

Define elements

$$p_{2n+1}^{(s)} := 2^{-n-1} \sum_{i=1}^{2n+1} (-1)^{i+1} i b_i^{(s)} \circ b_{2n+1-i}^{(s)}.$$

LEMMA 4.6.

i) $\triangle D(HS) \subset D' \otimes D' \otimes \frac{1}{2}\mathbf{Z} + D \otimes D \otimes \mathbf{Z}$;

thus, localized at odd primes q, $D(HS)_{(q)}$ is a Hopf algebra under $\triangle$.

ii) *For odd primes q, the space of primitives in the $\mathbf{Z}/q$-Hopf algebra $D(HS) \otimes \mathbf{Z}/q$ is spanned over $\mathbf{Z}/q$ by*

$$\{p_{2n+1}^{(s)} \otimes 1 : n \geqq 0, s \in S\}.$$

iii) *Define a subgroup $P' \subset D = D(HS)$ by*

$$P' = \{x \in D : \triangle x \in D \otimes D \quad \text{and}$$
$$2 \mid (\triangle x - x \otimes 1 - 1 \otimes x) \text{ in } D \otimes D/D' \otimes D'\}.$$

Then $2D \subset P'$ and $P'/2D$ is spanned over $\mathbf{Z}/2$ by the image of

$$\{d_{2k}^{(s)} : k > 0\} \cup \{p_{2l+1}^{(s)} : l \geqq 0\}.$$

Proof. See the appendix to this section.

Note. The first claim in 3.3 [that HS satisfies (I) to (VII)] follows from the rest of 3.3 plus the fact that an object $\bar{H}$ satisfying (I) to (VII) does exist (as follows from 3.4). In the preceding 4.2 to 4.6, we have given direct proofs of those properties from (I) to (VII) for HS which are needed in the proof of the main assertion of 3.3 and the proof of 3.4. These are (I) [taking $z_\phi = h_\phi$], (II), (III) [since D^- for HS with $z_\phi = h_\phi$ is exactly D'], and (IV) [taking $\bar{h}_n^{(s)}$ to be $h_n^{(s)}$ in HS].

THEOREM 4.7. *Let $H^{(1)}$ and $H^{(2)}$ be $\mathscr{C}$-objects. Assume that $H^{(1)}$ has no 2-torsion, and that, for both $H^{(1)}$ and $H^{(2)}$, the zero sequence*

$$H \xrightarrow{p} H \xrightarrow{1-r} H$$

is exact (that is, if $rh = h$ then $ph' = h$ for some h'). Let $\theta: H^{(1)}$ to $H^{(2)}$ be a $\mathscr{C}$-map. Then, for θ to be bijective, it suffices that $D(\theta)$ and $E(\theta)$ are both bijective.

Proof. For injectivity,

$$x \in \operatorname{Ker} \theta \Rightarrow [x] \in \operatorname{Ker} E(\theta) \Rightarrow rx = x.$$

If $x \in H_{0,*}^{(1)}$, $px \in \operatorname{Ker} D(\theta)$ so $px = 0$. But then

$$0 = p^2 x = (1 + r)x = 2x,$$

so $x = 0$ since $H^{(1)}$ has no 2-torsion. If $x \in H_{1,*}^{(1)}$, we can write

$$x = py \in \operatorname{Ker} D(\theta),$$

so again $x = 0$.

For surjectivity, first let $z \in H^{(2)}_{1,*}$. If $E(\theta)[x] = [z]$ in $EH^{(2)}$, we have

$$r(z - \theta x) = z - \theta x,$$

so $z - \theta x = pu$ for some $u \in H_{0,*}$. Since $D\theta$ is onto, write $pu = \theta(pv)$. Then $z = \theta(x + pv)$, as required. Finally, let $w \in H^{(2)}_{0,*}$ and find h with

$$r(w - \theta h) = w - \theta h$$

since $E\theta$ is epi. Then $w - \theta h = pg$ for some $g \in H^{(2)}_{1,*}$, and $g = \theta(f)$ for some f by the first part. Now $w = \theta(h + pf)$, as required.

A more learned (but no shorter) proof can be based on the 5-lemma. The proof of the substantial part of 3.3 now proceeds using 4.7 with $H^{(1)} = HS$ and $H^{(2)} = \bar{H}$. Note that the hypotheses of 4.7 hold by 4.2 and (I). First we must show

$$h_i^{(s)} \mapsto \bar{h}_i^{(s)}$$

determines a well defined $\mathscr{C}$-map by proving that the $\bar{h}_i^{(s)}$ satisfy the "same" squaring relations as do the $h_i^{(s)}$. These are deduced in 4.14 from knowing that the $\bar{b}_i^{(s)}$ satisfy the same relations as do the $b_i^{(s)}$. The latter follow in 4.13 by combining the Hopf algebra structure, which tends to force the rank to be large, with the rank restriction given by (I). At this point, one will know that the $\bar{h}_i^{(s)}$ generate $\bar{H} \otimes \mathbf{Q}$ as a $\mathbf{Q}$-algebra. To finish the proof we need to show that the $D(\theta)_n$ and $E(\theta)_n$, mappings between equal rank free abelian groups, are actually epi. This will follow by combining the Hopf structure with the inner products.

The completed ring $\prod_{n \geqq 0} D\bar{H}_n$ has a group of invertibles with subgroup $1 + \prod_{n>0} D\bar{H}_n$ of "special units". Also $\triangle$ extends to

$$\hat{\triangle}: \prod_{n \geqq 0} D\bar{H}_n \to \prod_{i,j \geqq 0} D\bar{H}_i \otimes D\bar{H}_j \otimes \mathbf{Q}.$$

Define elements

$$\bar{a}^{(s)} := \sum_{n>0} n\bar{b}_n^{(s)}$$

$$\bar{b}^{(s)} := 1 + \sum_{n>0} \bar{b}_n^{(s)} \quad \text{and}$$

$$\hat{p}^{(s)} := \bar{a}^{(s)} \circ (\bar{b}^{(s)^{-1}}) := \sum_{n>0} \hat{p}_n^{(s)}.$$

LEMMA 4.8.

i) $\hat{\triangle}\bar{b}^{(s)} = \bar{b}^{(s)} \otimes \bar{b}^{(s)}$.

ii) $\hat{\triangle}\bar{a}^{(s)} = \bar{a}^{(s)} \otimes \bar{b}^{(s)} + \bar{b}^{(s)} \otimes \bar{a}^{(s)}$.

iii) $\hat{\triangle}\hat{p}^{(s)} = 1 \otimes \hat{p}^{(s)} + \hat{p}^{(s)} \otimes 1$.

https://doi.org/10.4153/CJM-1986-070-1 Published online by Cambridge University Press

Proof.

i) $$\hat{\Delta}\bar{b}^{(s)} = \sum_{n \geqq 0} \Delta(\bar{b}_n^{(s)}) = \sum_{i,j \geqq 0} \bar{b}_i^{(s)} \otimes \bar{b}_j^{(s)}$$

$$= \sum_{i \geqq 0} \bar{b}_i^{(s)} \otimes \sum_{j \geqq 0} \bar{b}_j^{(s)} = \bar{b}^{(s)} \otimes \bar{b}^{(s)}.$$

ii) $$\hat{\Delta}\bar{a}^{(s)} = \sum_{n>0} n\Delta\bar{b}_n^{(s)} = \sum_{i+j>0} (i + j)\bar{b}_i^{(s)} \otimes \bar{b}_j^{(s)}$$

$$= \sum_{\substack{i>0 \\ j \geqq 0}} i\bar{b}_i^{(s)} \otimes \bar{b}_j^{(s)} + \sum_{\substack{i \geqq 0 \\ j>0}} \bar{b}_i^{(s)} \otimes j\bar{b}_j^{(s)}$$

$$= \bar{a}^{(s)} \otimes \bar{b}^{(s)} + \bar{b}^{(s)} \otimes \bar{a}^{(s)}.$$

iii) $$1 \otimes 1 = \hat{\Delta}1 = \hat{\Delta}(\bar{b}^{(s)} \circ \bar{b}^{(s)^{-1}}) = \hat{\Delta}(\bar{b}^{(s)}) \circ \hat{\Delta}(\bar{b}^{(s)^{-1}}),$$

so

$$\hat{\Delta}(\bar{b}^{(s)^{-1}}) = \hat{\Delta}(\bar{b}^{(s)})^{-1} = \bar{b}^{(s)^{-1}} \otimes \bar{b}^{(s)^{-1}}.$$

Hence

$$\begin{aligned}\hat{\Delta}\hat{p}^{(s)} &= \hat{\Delta}(\bar{a}^{(s)}) \circ \hat{\Delta}(\bar{b}^{(s)^{-1}}) \\ &= (\bar{a}^{(s)} \otimes \bar{b}^{(s)} + \bar{b}^{(s)} \otimes \bar{a}^{(s)}) \circ (\bar{b}^{(s)^{-1}} \otimes \bar{b}^{(s)^{-1}}) \\ &= \bar{a}^{(s)}\bar{b}^{(s)^{-1}} \otimes 1 + 1 \otimes \bar{a}^{(s)}\bar{b}^{(s)^{-1}} = \hat{p}^{(s)} \otimes 1 + 1 \otimes \hat{p}^{(s)}.\end{aligned}$$

Definition 4.9. Define $\bar{p}_n^{(s)}$ by

$$\hat{p}^{(s)} = \sum_{n>0} 2^{\left\lceil \frac{n+1}{2} \right\rceil} \bar{p}_n^{(s)}.$$

LEMMA 4.10. *We have*

$$\bar{p}_n^{(s)} \in D\bar{H}_n \quad \text{and} \quad \langle \bar{p}_{2j+1}^{(s)}, ph_{2j+1}^{(t)} \rangle = \delta_{st}.$$

Proof.

$$\hat{p}^{(s)} = \left(\sum_{i>0} i\bar{b}_i^{(s)}\right) \circ \left(\sum_{j \geqq 0} \tilde{b}_j^{(s)}\right)$$

where

$$\left(\sum_{j \geqq 0} \bar{b}_j^{(s)}\right) \circ \left(\sum_{j \geqq 0} \tilde{b}_j^{(s)}\right) = 1.$$

Since, by definition, $2^{\left\lceil \frac{j+1}{2} \right\rceil}$ divides $\bar{b}_j^{(s)}$ in $D\bar{H}$, it follows by induction on j that

$$2^{\left\lceil \frac{j+1}{2} \right\rceil} \text{ divides } \tilde{b}_j^{(s)} \text{ in } D\bar{H}.$$

Hence

$$2^{\left\lceil\frac{i+1}{2}\right\rceil + \left\lceil\frac{j+1}{2}\right\rceil} \text{ divides } \bar{b}_i^{(s)} \circ \tilde{b}_j^{(s)},$$

and since

$$\left\lceil\frac{i+1}{2}\right\rceil + \left\lceil\frac{j+1}{2}\right\rceil \geqq \left\lceil\frac{i+j+1}{2}\right\rceil,$$

we get that

$$2^{\left\lceil\frac{n+1}{2}\right\rceil} \text{ divides } \sum_{i+j=n} i\bar{b}_i^{(s)} \circ \tilde{b}_j^{(s)},$$

as required. For the inner product, it follows from the definitions that we must prove

$$\langle \hat{p}_{2j+1}^{(s)}, \bar{b}_{2j+1}^{(t)} \rangle = 4^{j+1}\delta_{st},$$

since $\hat{p}^{(s)} = \sum \hat{p}_n^{(s)}$; that is,

$$\hat{p}_{2j+1}^{(s)} = 2^{j+1}\bar{p}_{2j+1}^{(s)}.$$

The formula for $\triangle \bar{b}_n^{(t)}$ and the equation

$$\ll x \otimes y, \triangle z \gg = \langle x \circ y, z \rangle$$

imply that the map

$$F^{(t)}: \prod_{n \geqq 0} D\bar{H}_n \to \mathbf{Z}[[x]]$$

$$\sum_n e_n \mapsto \sum_n \langle e_n, \bar{b}_n^{(t)} \rangle x^n$$

is a ring homomorphism. Thus

$$\begin{aligned}
\sum \langle \hat{p}_n^{(s)}, \bar{b}_n^{(t)} \rangle x^n &= F^{(t)}(\hat{p}^{(s)}) \\
&= F^{(t)}(\bar{a}^{(s)} \circ \bar{b}^{(s)^{-1}}) \\
&= F^{(t)}(\bar{a}^{(s)})F^{(t)}(\bar{b}^{\,(s)})^{-1} \\
&= \left(\sum_n n2^{n+1}\delta_{st}x^n\right)\left(1 + \sum_{n>0} 2^{n+1}\delta_{st}x^n\right)^{-1},
\end{aligned}$$

since

$$\langle \bar{b}_n^{(s)}, \bar{b}_n^{(t)} \rangle = \delta_{st}$$

if $s \neq t$, giving the result in that case.

Continuing if $s = t$:

$$= (2x(d/dx)[(1-2x)^{-1}])(1+4x(1-2x)^{-1})^{-1}$$
$$= 4x(1-4x^2)^{-1} = \sum_{j\geqq 0} 4^{j+1}x^{2j+1},$$

as required.

Note. This also shows

$$\langle \bar{p}_{2j}^{(s)}, \bar{b}_{2j}^{(s)}\rangle = 0,$$

but in fact $\bar{p}_{2j}^{(s)} = 0$ as we see below in 4.12.

LEMMA 4.11. i) *The set of primitives* $\{\hat{p}_{2j+1}^{(s)}:j \geqq 0, s \in S\}$ *is algebraically independent.*

ii) $D\bar{H}_n$ *has rank* $\#\mathscr{D}(n, S)$, *as have the* n^{th} *groups in the subalgebras generated by*

$$\{\hat{p}_{2j+1}^{(s)}:j \geqq 0, s \in S\} \quad \text{and} \quad \{\bar{b}_i^{(s)}:i > 0, s \in S\}.$$

Proof. i) The given set is linearly independent by 4.10, and consists of primitives by 4.8 iii), so is algebraically independent. See [**2**] for this basic fact about graded Hopf algebras in characteristic zero.

ii) We have

$$\mathrm{Alg}\{\hat{p}_{2j+1}^{(s)}:j \geqq 0, s \in S\}_n \subset \mathrm{Alg}\{\bar{b}_i^{(s)}:i > 0, s \in S\}_n \subset D\bar{H}_n.$$

The first has rank $\#\mathscr{P}^{\mathrm{odd}}(n, S)$ by i), and the third has rank $\#\mathscr{D}(n, S)$ by (I). But these numbers are equal, as required.

LEMMA 4.12. $\mathrm{Prim}(D\bar{H} \otimes \mathbf{Q})_{2n} = 0$ *for all* n. (Prim *denotes the module of primitives.*)

Proof. If it were non-zero, we would have an element of $D\bar{H}$ which is transcendental over $\mathrm{Alg}\{\hat{p}_{2j+1}^{(s)}\}$, contradicting 4.11.

LEMMA 4.13.

$$\bar{b}_i^{(s)} \circ \bar{b}_i^{(s)} = 2\sum_{j=0}^{i-1} (-1)^{i+j+1}\bar{b}_{2i-j}^{(s)} \circ \bar{b}_j^{(s)}.$$

Proof. Let

$$w^{(s)} = \sum_{i\geqq 0} (-1)^i\bar{b}_i^{(s)}.$$

The relation to be proved is equivalent to $w^{(s)}\bar{b}^{(s)} = 1$, since

$$w^{(s)}\bar{b}^{(s)} = 1 + \sum_{i>0} v_{2i}$$

with v_{2i} equal to the difference of the two sides in the relation to be proved. Now

$$\hat{\triangle} w^{(s)} = w^{(s)} \otimes w^{(s)},$$

so

$$\hat{\triangle}(1 + \sum v_{2i}) = (1 + \sum v_{2i}) \otimes (1 + \sum v_{2i}).$$

Assuming inductively that $v_{2i} = 0$ for all $i < n$, we get

$$\triangle v_{2n} = v_{2n} \otimes 1 + 1 \otimes v_{2n},$$

so $v_{2n} = 0$ by 4.12.

Note. This shows that $\{\bar{b}_\phi : \phi \in \mathcal{D}(n, S)\}$ spans $\mathrm{Alg}\{\bar{b}_i^{(s)}\}$, and is therefore linearly independent by 4.11 ii). It is therefore a $\mathbf{Q}$-basis for $D\bar{H} \otimes \mathbf{Q}$. We need more work to prove the analogue over $\mathbf{Z}$, viz. that $\{\bar{d}_\phi\}$ spans $D\bar{H}$ over $\mathbf{Z}$.

LEMMA 4.14.

$$(\bar{h}_i^{(s)})^2 = (-1)^{i+1} p \left[\bar{h}_{2i}^{(s)} + p \sum_{j=1}^{i-1} (-1)^j \bar{h}_{2i-j}^{(s)} \bar{h}_j^{(s)} \right].$$

Proof. Drop the superscripts (s) for this proof. When $i = 2k$, the equation of 4.13 is

$$2^{2k}(p^2 \bar{h}_{2k}) \circ (p^2 \bar{h}_{2k}) = -2\bar{b}_{4k} - 2 \sum_{s=1}^{k-1} 2^{2k-s} 2^s p^2 \bar{h}_{4k-2s} \circ p^2 \bar{h}_{2s}$$

$$+ 2 \sum_{s=0}^{k-1} 2^{2k+1} p \bar{h}_{4k-2s-1} \circ p \bar{h}_{2s+1}.$$

Re-written in the multiplication of $\bar{H}$ (rather than the $\circ$ of $D\bar{H}$) we get

$$p \bar{h}_{2k}^2 = -p^2 \bar{h}_{4k} - 2p \sum_{r=1}^{2k-1} (-1)^r \bar{h}_{4k-r} \bar{h}_r,$$

after dividing by 2^{2k+1} in the torsion free $\bar{H}$. Since $\bar{h}_{2k}^2 = r\bar{h}_{2k}^2$ by "pseudo-commutativity", there exists y with $py = \bar{h}_{2k}^2$. Substitute this, multiply by p, replace p^3 by $2p$, divide by 2 and the required relation appears. The proof for odd i is exactly similar.

COROLLARY 4.15. *There is a unique $\mathcal{C}$-map $\theta : HS \to \bar{H}$ with*

$$\theta h_i^{(s)} = \bar{h}_i^{(s)} \quad \textit{for all } i, s.$$

Note. By (IV) and 4.4, $D\theta \otimes 1_{\mathbf{Q}}$ and $E\theta$ are homomorphisms with respect to $\triangle$ and ∇, respectively.

LEMMA 4.16. *For all* n, $D\theta_n$ *is a monomorphism between equal (finite) rank free abelian groups.*

Proof. This is immediate from 4.3 ii) and 4.13. (See the note after 4.13.)

LEMMA 4.17. *For all odd primes* q,

$$D\theta \otimes 1{:}D(HS) \otimes \mathbf{Z}/q \to D\bar{H} \otimes \mathbf{Z}/q$$

is injective.

Proof. Being a morphism of $\mathbf{Z}/q$-Hopf algebras, a non-zero element of least dimension in its kernel is primitive. If such an element exists, we have, by 4.6 ii), integers j and $\mu^{(s)}$ such that

$$\sum_s \mu^{(s)}\bar{p}^{(s)}_{2j+1} \text{ is divisible by } q \text{ in } D\bar{H},$$

but

$$\sum_s \mu^{(s)}p^{(s)}_{2j+1} \text{ is not divisible by } q \text{ in } D(HS).$$

But the first statement implies that for all $t \in S$, the integer

$$\mu^{(t)} = \left\langle \sum_s \mu^{(s)}\bar{p}^{(s)}_{2j+1}, p\bar{h}^{(t)}_{2j+1} \right\rangle$$

(by 4.10) is divisible by q, and this contradicts the second statement.

LEMMA 4.18. $D\theta \otimes 1{:}D(HS) \otimes \mathbf{Z}/2 \to D\bar{H} \otimes \mathbf{Z}/2$ *is mono.*

Proof. For a contradiction, let $x \otimes 1$ be a non-zero element of least dimension n in the kernel of $D\theta \otimes 1$. Then $2 \nmid x$ in $D(HS)_n$, but $2|\theta x$ in $D\bar{H}_n$. First we show $x \in P'$ (defined in 4.6 iii)). By 4.16 and minimality of n,

$$D\theta{:}D(HS)_i \to D\bar{H}_i$$

is bijective for all $i < n$. If $i + j = n$ with $i > 0$ and $j > 0$, the diagram

$$\begin{tikzcd}
D(HS)_n \arrow[rr, "D\theta"] \arrow[d, "\triangle"'] & & D\bar{H}_n \arrow[d, "\triangle"] \\
D(HS)_i \otimes D(HS)_j \otimes \mathbf{Q} \arrow[rr, "D\theta \otimes D\theta \otimes 1"] & & D\bar{H}_i \otimes D\bar{H}_j \otimes \mathbf{Q} \\
D(HS)_i \otimes D(HS)_j \arrow[rr] \arrow[u, phantom, "\bigcup" rotate=0] & & D\bar{H}_i \otimes D\bar{H}_j \arrow[u, phantom, "\bigcup"]
\end{tikzcd}$$

shows that

$$\triangle_{i,j}(x) \in D(HS)_i \otimes D(HS)_j,$$

since

$$\triangle_{i,j}(2D\bar{H}_n) \subset D\bar{H}_i \otimes D\bar{H}_j$$

by (III) for $\bar{H}$. Since

$$\triangle_{0,n}(x) = 1 \otimes x \quad \text{and} \quad \triangle_{n,0}(x) = x \otimes 1,$$

we have

$$\triangle\, x \in D(HS) \otimes D(HS).$$

The same diagram and the condition

$$\triangle(2D\bar{H}) \subset D^- \otimes D^- \otimes \mathbf{Z} + D\bar{H} \otimes D\bar{H} \otimes 2\mathbf{Z}$$

from (III) shows that

$$2 | \triangle_{i,j}(x) \text{ in } [D(HS)_i \otimes D(HS)_j]/[\,(D\theta \otimes D\theta]^{-1}(D_i^- \otimes D_j^-)\,].$$

But

$$D^-/2D\bar{H} = p^2\bar{H}_{1,*}/2D\bar{H}$$

by (I) for $\bar{H}$, and

$$D'/2D(HS) = p^2 HS_{1,*}/2D(HS)$$

by 4.3 ii). Thus

$$2 | \triangle_{i,j}(x) \text{ in } [D(HS)_i \otimes D(HS)_j]/[D_i' \otimes D_j'],$$

and we have proved $x \in P'$. By 4.6 iii), there exist integers $\nu^{(s)}$ such that:

if $n = 2l$,

$$2 \nmid \sum_s \nu^{(s)}(1 + r)h_{2l}^{(s)} \quad \text{in } D(HS),$$

but

$$2 | \sum_s \nu^{(s)}(1 + r)\bar{h}_{2l}^{(s)} \quad \text{in } D\bar{H};$$

if $n = 2l + 1$,

$$2 \nmid \sum_s \nu^{(s)} p_{2l+1}^{(s)} \quad \text{in } D(HS),$$

but

$$2 | \sum_s \nu^{(s)} \bar{p}_{2l+1}^{(s)} \quad \text{in } D\bar{H}.$$

The second case leads to a contradiction exactly as in 4.17. The first case is also an inner product argument, as follows. For all $t \in S$,

$$2|\nu^{(t)} = \left\langle \sum_s \nu^{(s)}(1 + r)\bar{h}_{2l}^{(s)}, \bar{h}_{2l}^{(t)} \right\rangle$$

by (VII) for $\bar{H}$ and the divisibility in $D\bar{H}$, but this contradicts the non-divisibility in $D(HS)$.

COROLLARY 4.19. *$D\theta$ is bijective.*

Proof. This is immediate from 4.16, 4.17 and 4.18.

Note. In 4.20 and 4.21 below, we use the elementary fact that for any commutative ring K, in a K-Hopf algebra which is "pseudo-exterior" on primitive generators, the K-module of primitives is spanned by these generators.

LEMMA 4.20. *For all* (ϵ, n),

$$(E\theta)_{\epsilon,n}:E(HS)_{\epsilon,n} \to E\bar{H}_{\epsilon,n}$$

is a monomorphism between equal (finite) rank free abelian groups.

Proof. By (I) and 4.3 i), we see that the ranks are equal (to $\#\mathcal{D}''(n, S)$ or $\#\mathcal{D}'(n, S)$ depending on whether ϵ is 0 or 1). To show $E\theta$ is injective, note that the subgroup of primitives

$$\mathrm{Span}_{\mathbf{Z}}\{ [h_i^{(s)}]:i > 0, s \in S\}$$

maps monomorphically to

$$\mathrm{Span}_{\mathbf{Z}}\{ [\bar{h}_i^{(s)}]:i > 0, s \in S\}.$$

LEMMA 4.21. *For all primes q,*

$$E\theta \otimes 1:E(HS) \otimes \mathbf{Z}/q \to E\bar{H} \otimes \mathbf{Z}/q$$

is a monomorphism.

Proof. Here the space of primitives of the domain is

$$\mathrm{Span}_{\mathbf{Z}/q}\{ [h_i^{(s)}] \otimes 1:i > 0, s \in S\},$$

so we must show that

$$q \nmid \sum_s \mu^{(s)}[\bar{h}_i^{(s)}] \quad \text{in } E\bar{H}_{*,i}$$

unless all the integers $\nu^{(t)}$ are divisible by q. This follows by noting that $\langle \ldots, (1 - r)\bar{h}_i^{(t)} \rangle$ defines a map

$$E\bar{H}_{*,i} \to \mathbf{Z},$$

since

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$$\langle x, (1 - r)\bar{h}_i^{(t)}\rangle = 0 \quad \text{if } rx = x,$$

and furthermore, that

$$\langle \bar{h}_i^{(s)}, (1 - r)\bar{h}_i^{(t)}\rangle = \delta_{st}.$$

COROLLARY 4.22. *$E\theta{:}E(HS) \to E\bar{H}$ is bijective.*

Proof. This is clear from 4.20 and 4.21.

Now 4.19 and 4.22 complete the proof of 3.3 using 4.7 as noted after 4.7.

Appendix – Proof of 4.6. We first eliminate dependence on S. Since HS is the tensor product over the $\mathbf{Z}/2$-graded $H_{*,0}$ of the algebras generated by $\{h_i^{(s)}{:}i > 0\}$, one for each $s \in S$, and for each of these the corresponding D is "coclosed" under $\triangle$, it is straightforward to see that we need only verify 4.6 for each of these separately. In effect, we have reduced to the case $\#S = 1$. We shall drop the superscripts (s).

For a sequence γ, let $\underline{\gamma}$ denote the underlying partition. We refer to elements d_γ and b_γ, even though they depend only on $\underline{\gamma}$. Sequences are convenient because the formula for $\triangle b_n$ leads immediately to

$$\triangle b_\gamma = \sum_{\sigma+\tau=\gamma} b_\sigma \otimes b_\tau,$$

where the summation is over all pairs (σ, τ) of sequences whose sum is γ under term-by-term addition. Thus, since

$$b_\gamma = 2^{[1/2(|\gamma|+l(\gamma))]} d_\gamma,$$

$$\triangle d_\gamma = \sum_{\sigma+\tau=\gamma} 2^{-\left[\frac{|\gamma|+l(\gamma)}{2}\right]} b_\sigma \otimes b_\tau.$$

SUBLEMMA (A). *Expand the above term*

$$2^{-\left[\frac{|\gamma|+l(\gamma)}{2}\right]} b_\sigma \otimes b_\tau = \sum_{\underline{\alpha},\underline{\beta}\in\mathscr{D}_n} z_{\alpha\beta} d_{\underline{\alpha}} \otimes d_{\underline{\beta}}$$

using the squaring relations in the definition of HS. Then:

i) $z_{\alpha\beta} \in \frac{1}{2}\mathbf{Z}$ *for all α and β;*

ii) $z_{\alpha\beta} \in \mathbf{Z}$ *if any of the following hold*: $\underline{\gamma} \in \mathscr{D}_n' \cup \mathscr{D}_n^{\text{odd}}$; *or* $\underline{\alpha} \in \mathscr{D}_n''$; *or* $\underline{\beta} \in \mathscr{D}_n''$; *or* $\underline{\sigma} \notin \mathscr{D}_n$; *or* $\underline{\tau} \notin \mathscr{D}_n$;

iii) $z_{\alpha\beta} \in 2\mathbf{Z}$ *if* $\underline{\gamma} \in \mathscr{D}_n'$ *and either* $\underline{\sigma} \notin \mathscr{D}_n$ *or* $\underline{\tau} \notin \mathscr{D}_n$.

Proof. Each time a squaring relation is used, a factor 2 appears. For some $s, t \geqq 0$, the term b_α arises from b_σ after "s" such uses, and the term

b_β from b_τ after "t" uses. Thus $2^{s+t}b_\alpha \otimes b_\beta$ will come from $b_\sigma \otimes b_\tau$, and

$$z_{\alpha\beta} \in 2^{s+t+\left\lceil\frac{|\alpha|+l(\alpha)}{2}\right\rceil+\left\lceil\frac{|\beta|+l(\beta)}{2}\right\rceil-\left\lceil\frac{|\gamma|+l(\gamma)}{2}\right\rceil}\mathbf{Z}.$$

The proof will analyze the above exponent of 2. Call it N.

Now $l(\sigma) \leqq l(\alpha) + s$ and $l(\tau) \leqq l(\beta) + t$, so

$$l(\gamma) \leqq \begin{cases} l(\sigma) + l(\tau) \leqq l(\alpha) + l(\beta) & \text{if } s = t = 0 \\ l(\sigma) + l(\tau) - 1 \leqq l(\alpha) + l(\beta) + s + t - 1 & \text{if } s + t > 0. \end{cases}$$

The term -1 occurs when $s + t > 0$ because $\underline{\gamma}$ has distinct parts and there must be at least one place where σ and τ both have positive parts, since either $\underline{\sigma} \notin \mathscr{D}_n$ or $\underline{\tau} \notin \mathscr{D}_n$ and $\sigma + \tau = \gamma$.

Proceed now case-by-case:

i) and ii) when $s + t > 0$: Here

$$|\gamma| + l(\gamma) \geqq (|\alpha| + l(\alpha)) + (|\beta| + l(\beta)) + (s + t - 1),$$

so

$$\begin{aligned} N \geqq s + t + \left\lceil\frac{|\alpha| + l(\alpha)}{2}\right\rceil + \left\lceil\frac{|\beta| + l(\beta)}{2}\right\rceil - \left\lceil\frac{|\alpha| + l(\alpha)}{2}\right\rceil \\ - \left\lceil\frac{|\beta| + l(\beta)}{2}\right\rceil - \left\lceil\frac{s + t - 1}{2}\right\rceil - 1 \\ = s + t - \left\lceil\frac{s + t - 1}{2}\right\rceil - 1 \geqq 0, \end{aligned}$$

using the fact that

$$\left\lceil\frac{x + y + z}{2}\right\rceil \leqq \left\lceil\frac{x}{2}\right\rceil + \left\lceil\frac{y}{2}\right\rceil + \left\lceil\frac{z}{2}\right\rceil + 1.$$

ii) when $\underline{\sigma} \notin \mathscr{D}_n$ or $\underline{\tau} \notin \mathscr{D}_n$ is the above.

i) when $s = t = 0$:

$$|\gamma| + l(\gamma) \leqq (|\alpha| + l(\alpha)) + (|\beta| + l(\beta)),$$

so

$$N \geqq \left\lceil\frac{|\gamma| + l(\gamma)}{2}\right\rceil - 1 - \left\lceil\frac{|\gamma| + l(\gamma)}{2}\right\rceil = -1,$$

using the fact that

$$\left\lceil\frac{x + y}{2}\right\rceil \leqq \left\lceil\frac{x}{2}\right\rceil + \left\lceil\frac{y}{2}\right\rceil + 1.$$

ii) when $s = t = 0$ and $\underline{\alpha} \in \mathscr{D}_n''$ or $\underline{\beta} \in \mathscr{D}_n''$ or $\underline{\gamma} \in \mathscr{D}_n'$: Proceed as immediately above, but use the fact that

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$$\left\lceil \frac{x+y}{2} \right\rceil \leqq \left\lceil \frac{x}{2} \right\rceil + \left\lceil \frac{y}{2} \right\rceil$$

if x is even or y is even or $x + y$ is odd. This yields $N \geqq 0$, as required.

ii) when $s = t = 0$ and $\underline{\gamma} \in \mathscr{D}_n^{\text{odd}}$: By the case above we can assume $\underline{\alpha} \in \mathscr{D}_n'$ and $\underline{\beta} \in \mathscr{D}_n'$. But here $\underline{\sigma} = \underline{\alpha}$, $\underline{\tau} = \underline{\beta}$ and $\sigma + \tau = \gamma$, so we have

$$l(\gamma) \leqq l(\alpha) + l(\beta) - 1$$

(otherwise $\underline{\gamma} \notin \mathscr{D}_n^{\text{odd}}$). Hence

$$\begin{aligned} N &= \left\lceil \frac{|\alpha| + l(\alpha)}{2} \right\rceil + \left\lceil \frac{|\beta| + l(\beta)}{2} \right\rceil - \left\lceil \frac{|\gamma| + l(\gamma)}{2} \right\rceil \\ &= \frac{|\alpha| + l(\alpha) - 1}{2} + \frac{|\beta| + l(\beta) - 1}{2} - \left(\frac{|\gamma| + l(\gamma)}{2} \right) \\ &\geqq \frac{|\alpha| + l(\alpha) - 1 + |\beta| + l(\beta) - 1 - (\,|\alpha| + |\beta|\,) - (l(\alpha) + l(\beta) - 1)}{2} = -\frac{1}{2}. \end{aligned}$$

Since $N \in \mathbf{Z}$, we have $N \geqq 0$, as required.

iii): Here $s + t > 0$ and $|\gamma| + l(\gamma)$ is odd, so

$$\begin{aligned} N &= s+t+\left\lceil \frac{|\alpha|+l(\alpha)}{2} \right\rceil + \left\lceil \frac{|\beta|+l(\beta)}{2} \right\rceil - \left(\frac{|\gamma|+l(\gamma)-1}{2} \right) \\ &\geq \frac{3}{2} + \left\lceil \frac{|\alpha|+l(\alpha)}{2} \right\rceil + \left\lceil \frac{|\beta|+l(\beta)}{2} \right\rceil - \left(\frac{|\alpha|+|\beta|+l(\alpha)+l(\beta)}{2} \right) \\ &= \frac{3}{2} + \left\lceil \frac{|\alpha|+l(\alpha)}{2} \right\rceil - \left(\frac{|\alpha|+l(\alpha)}{2} \right) + \left\lceil \frac{|\beta|+l(\beta)}{2} \right\rceil - \left(\frac{|\beta|+l(\beta)}{2} \right) \\ &\geq \frac{3}{2} + \left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right) = \frac{1}{2}. \end{aligned}$$

But $N \in \mathbf{Z}$, so $N \geqq 1$, as required.

This completes the proof of (A).

Proof of 4.6 i). This is immediate from (A) since terms $d_\alpha \otimes d_\beta$ when $\alpha \in \mathscr{D}_n''$ or $\beta \in \mathscr{D}_n''$ have integer coefficients by part ii) of (A).

SUBLEMMA (B). *If* $\theta_\lambda \in \mathbf{Z}$ *for all* $\lambda \in \mathscr{D}_n$ *and*

$$\triangle\left(\sum_\lambda \theta_\lambda d\lambda \right) \in D \otimes D \otimes \mathbf{Z},$$

then $\theta_\lambda \in 2\mathbf{Z}$ *for all* $\lambda \in \mathscr{D}_n'' - \mathscr{D}_n^{\text{odd}}$.

Proof. Fix $\lambda \in \mathscr{D}_n'' - \mathscr{D}_n^{\text{odd}}$, so λ has a positive even number of even parts. Let $2k$ be the largest even entry of λ. If $\lambda = (2k) \cup \mu$, then the

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coefficient of $d_{2k} \otimes d_\mu$ in $\triangle d_\lambda$ is $\frac{1}{2}$, since

$$-\left\lceil\frac{|\lambda| + l(\lambda)}{2}\right\rceil + \left\lceil\frac{2k + 1}{2}\right\rceil + \left\lceil\frac{|\mu| + l(\mu)}{2}\right\rceil = -1.$$

It suffices to prove that for all $\underline{\gamma} \in \mathcal{D}_n$ with $\underline{\gamma} \neq \lambda$, the coefficient of $d_{2k} \otimes d_\mu$ in $\triangle d_{\underline{\gamma}}$ is in $\mathbf{Z}$. Applying (A) we need only consider the case where $\underline{\sigma} = \underline{\alpha} = (2k)$, $\underline{\tau} = \underline{\beta} = \mu$ and $\underline{\gamma} \in \mathcal{D}_n''$. But since $\underline{\gamma} \neq \lambda$, $\underline{\gamma}$ must be μ with $2k$ added to one entry. Thus $\underline{\gamma} \in \mathcal{D}_n'$, and this case does not arise.

Proof of 4.6 iii).

$$\triangle D \subset D' \otimes D' \otimes \frac{1}{2}\mathbf{Z} + D \otimes D \otimes \mathbf{Z},$$

so

$$\triangle(2D) \subset D' \otimes D' \otimes \mathbf{Z} + D \otimes D \otimes 2\mathbf{Z} \subset D \otimes D$$

and the image of $\triangle(2D)$ in $D \otimes D/D' \otimes D'$ is contained in

$$2(D \otimes D/D' \otimes D').$$

Thus $2D \subset P'$, as required.

Now let $g \in P'$. By definition of P',

$$\triangle g \in D \otimes D \otimes \mathbf{Z},$$

so, using (B), for some $\theta_\lambda \in \mathbf{Z}$,

$$g \equiv \sum_{\lambda \in \mathcal{D}_n' \cup \mathcal{D}_n^{\text{odd}}} \theta_\lambda d_\lambda \pmod{2D}.$$

We proceed in three steps to show $g \equiv d_{2l}$ or p_{2l+1}, if g is homogeneous and non-zero.

Step I. If λ has more than one odd part, then $\theta_\lambda \in 2\mathbf{Z}$. To see this, proceed by contradiction. For those λ where it fails, choose one whose smallest odd part is largest, and call this smallest odd part $2l + 1$. Write

$$\lambda = (2l + 1) \cup \underline{\beta}.$$

The coefficient of $d_{2l+1} \otimes d_{\underline{\beta}}$ in $\triangle d_\lambda$ is 1, and

$$d_{2l+1} \otimes d_{\underline{\beta}} \in D'' \otimes D.$$

We therefore need only show that for all $\underline{\gamma} \in \mathcal{D}_n' \cup \mathcal{D}_n^{\text{odd}}$ with $\underline{\gamma} \neq \lambda$, either the coefficient of $d_{2l+1} \otimes d_\beta$ in $\triangle d_{\underline{\gamma}}$ is even, or else θ_γ is even. Using (A) iii), with $\underline{\sigma} = \underline{\alpha} = (2l + 1)$, we have γ equal to τ with $2l + 1$ added to one part. But $\underline{\tau} \neq \underline{\beta}$ gives

$$\underline{\gamma} = (2m + 2l + 1) \cup \delta, \underline{\tau} = (2m) \cup \delta$$

so $\tau \in \mathscr{D}$, a contradiction. So $\underline{\tau} = \underline{\beta}$ and $\underline{\gamma}$ is $\underline{\beta}$ with $2l + 1$ added to one of its parts. Since

$$\underline{\gamma} \in \mathscr{D}_n' \cup \mathscr{D}_n^{\text{odd}},$$

that part must be the unique even part in β. Thus

$$\gamma = (2m + 2l + 1) \cup \delta$$

where

$$\beta = (2m) \cup \delta \quad \text{and} \quad \alpha = (2l + 1, 2m) \cup \delta,$$

and δ has only odd parts larger than $2l + 1$. But now $\gamma \in \mathscr{D}_n^{\text{odd}}$ has smallest odd part larger than $2l + 1$ and more than 1 odd part and so θ_γ is even.

Step II. If λ has at most one odd part, but more than one even part, then $\theta_\lambda \in 2\mathbf{Z}$. To see this, let $2l$ be the smallest even part of λ, with $\lambda = (2l) \cup \beta$. The coefficient of $d_{2l} \otimes d_\beta$ in Δd_λ is 1, and

$$d_{2l} \otimes d_\beta \in D \otimes D'',$$

so it suffices to show that if $\gamma \in \mathscr{D}_n' \cup \mathscr{D}_n^{\text{odd}}$ has at most one odd part and $\gamma \neq \lambda$, then the coefficient of $d_{2l} \otimes d_\beta$ in Δd_γ is even. Taking $\alpha = (2l)$ in (A) iii), we are left with the case where γ is β with $2l$ added to one part. But then

$$\gamma \notin \mathscr{D}_n' \cup \mathscr{D}_n^{\text{odd}}.$$

Step III. We now know that $(P'/2D)_n$ is spanned by the image of

$$\{d_\lambda : \lambda \text{ has at most one odd part and one even part}\}.$$

If $n = 2l$, this leaves only d_{2l} as required. If $n = 2l + 1$, we must show that

$$d_{2l+1} + \sum_{i=1}^{l} \theta_i d_{2l+1-i} d_i \in P'$$

implies θ_j is odd for all j, since p_{2l+1} has this form mod 2. But this follows by applying Δ to the above element and considering the coefficient of $d_j \otimes d_{2l+1-j}$. This completes the proof of 4.6 iii).

Proof of 4.6 ii). A tedious combinatorial proof similar to the one above may be given. However we sketch a better approach, partly to indicate the kind of proof we would have preferred to have found for 4.6 iii).

Let $\Lambda = \mathbf{Z}[e_1(\underline{x}), e_2(\underline{x}), \dots]$ be the ring of stable symmetric polynomials in a "potential infinity" of variables $\underline{x}$, where $e_j(\underline{x})$ is the j^{th} elementary symmetric polynomial. Λ has basis $\{s(\pi) : \pi \in \mathscr{P}\}$ where $s(\pi)$ is the

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monomial symmetric function corresponding to π. Let I be the ideal generated by the subring of polynomials which are symmetric in the squares of the variables $\underline{x}$. Then I is generated as an ideal by $\{e_1(\underline{x}^2), e_2(\underline{x}^2), \dots\}$ and

$$e_k(\underline{x}^2) = e_k^2(\underline{x}) - 2e_{k+1}(\underline{x})e_{k-1}(\underline{x}) + 2e_{k+2}(\underline{x})e_{k-2}(\underline{x}) - \dots \pm 2e_{2k}(\underline{x}).$$

Thus Λ/I has the same structure as B, namely

$$\frac{\mathbf{Z}[e_1, e_2 \dots]}{\langle e_k^2 = 2e_{k+1}e_{k-1} - 2 \dots \pm 2e_{2k}\rangle}.$$

Under the isomorphism $B \to \Lambda/I$ sending b_k to e_k our coproduct on $B \otimes \mathbf{Z}_{(q)}$ corresponds to the canonical coproduct on Λ

$$\left(e_k \mapsto \sum_{i+j=k} e_i \otimes e_j\right)$$

for which it is well known that

$$s(\pi) \mapsto \sum_{\alpha\cup\beta=\pi} s(\alpha) \otimes s(\beta)$$

(see [**5**], on which this proof is modelled). Furthermore, since for all π,

$$s(\pi) \equiv s(\pi^{\text{ev}})s(\pi^{\text{od}}) \text{mod Span}\{s(\lambda): \#\lambda^{\text{ev}} < \#\pi^{\text{ev}}\},$$

where π^{ev} consists of the even parts of π, and π^{od} of its odd parts, it follows that

$$\{s(\pi) + I: \pi \in \mathscr{P}^{\text{odd}}\} \text{ is a basis for } \Lambda/I.$$

But in the coproduct for $(\Lambda/I) \otimes \mathbf{Z}/q$,

$$\{s(\pi) + I \mapsto \sum_{\alpha\cup\beta=\pi} (s(\alpha) + I) \otimes (s(\beta) + I).$$

But the term $(s(\alpha) + I) \otimes (s\beta + I)$ occurs only for $\pi = \alpha \cup \beta$, and the only π for which

$$[\alpha \cup \beta = \pi \Rightarrow \alpha = \phi \text{ or } \beta = \phi]$$

is $\pi = (2l + 1)$ for some l. Thus the primitives in $(\Lambda/I) \otimes \mathbf{Z}_{(q)}$ are spanned by

$$\{s(2l + 1): l \geqq 0\}.$$

The result for $B \otimes \mathbf{Z}/q$ now follows, since $s(2l + 1)$ corresponds to $\hat{p}_{2l+1}$ under the isomorphism. (This last statement needn't be checked, since all we needed to prove was that

$$\text{Prim}_n(B \otimes \mathbf{Z}/q)$$

was a cyclic module and zero for n even, knowing already that $\hat{p}_{2l+1}$ is primitive and indivisible in $B \otimes \mathbf{Z}_{(q)}$.)

5. Clifford modules. We have yet to exhibit a single "negative" representation (although they obviously exist for any $\mathcal{G}$-object). This is remedied here. Sufficiently many are produced to give the required generators $\bar{h}_i^{(s)}$ for $\bar{H}$ in 3.4.

Let $\mathbf{R}^k$ have a positive definite form g with orthonormal basis $e_1, \ldots, e_k$ (the standard ones, if desired). Denote by $CL(k)$ the real Clifford algebra of the negative definite form $-g[1]$. Then $CL(k)$ has $\mathbf{R}$-basis

$$\{e_{i_1}e_{i_2} \ldots e_{i_r} : r \geqq 0, 1 \leqq i_1 < i_2 \ldots \leqq k\},$$

with multiplication determined by

$$e_i^2 = -1 \quad \text{and} \quad e_ie_j = -e_je_i \quad \text{for } j \neq i.$$

We have $CL(k) = CL_0(k) \oplus CL_1(k)$ is a $\mathbf{Z}/2$-graded algebra, where the above basis element is in $CL_j(k)$ if and only if $r \equiv j (\text{mod } 2)$.

We shall find a subgroup of invertibles in $CL(n - 1)$ which is isomorphic to $\tilde{\Sigma}_n$, and use this to convert each $CL(n - 1)$-module into $\tilde{\Sigma}_n$-representation, which will be "negative" since z corresponds to -1 in $CL(n - 1)$. The idea is fairly obvious: Σ_n has an irreducible faithful representation on $\mathbf{R}^{n-1}$; namely, the non-trivial summand in the standard permutation representation. This gives a map

$$\Sigma_n \hookrightarrow O(n - 1),$$

which lifts to $\tilde{\Sigma}_n \to \text{Pin}(n - 1)$.

But $\text{Pin}(n - 1) \subset CL(n - 1)^*$, as required. Making suitable choices, a formula is

$$t_i \quad \mapsto \quad -\sqrt{\frac{i-1}{2i}}\, e_{i-1} + \sqrt{\frac{i+1}{2i}}\, e_i = t_i'.$$

Since $t_i' \in \mathbf{R}^{n-1} \subset CL_1(n - 1)$ and the generators $t_i \in (\tilde{\Sigma}_n)_1$, we see that

$$\tilde{A}_n = (\tilde{\Sigma}_n)_0 \text{ embeds in } CL_0(n - 1), \text{ and}$$
$$(\tilde{\Sigma}_n)_1 \text{ embeds in } CL_1(n - 1).$$

Note that $\{t_1', \ldots, t_{n-1}'\}$ is a basis for $\mathbf{R}^{n-1}$, and so it generates $CL(n - 1)$ as an algebra. Thus we have proved all but the last part of the following. (See also [**1**] and [**7**].)

PROPOSITION 5.1. *There is an embedding*

$$\xi_n : \tilde{\Sigma}_n \to CL(n - 1)^*$$

of groups which preserves $\mathbf{Z}/2$*-grading and whose image generates* $CL(n - 1)$ *as an algebra. Furthermore, there is an isometric embedding*

$$\kappa_{a,b}:\mathbf{R}^{a+b-2} \rightarrow \mathbf{R}^{a+b-1}$$

whose induced Clifford algebra morphism fits into the following commutative diagram:

$$\begin{tikzcd}
(x, y) \in \tilde{\Sigma}_a \hat{\times} \tilde{\Sigma}_b \arrow[rr, "\hat{\phi}_{a,b}"] \arrow[d] & & \tilde{\Sigma}_{a+b} \arrow[d, "\xi_{a+b}"] \\
\xi_a x \otimes \xi_b y \in CL(a - 1) \hat{\otimes} CL(b - 1) \arrow[r, "\gamma", "\simeq"'] & CL(a + b - 2) \arrow[r, "CL(\kappa_{a,b})"] & CL(a + b - 1)
\end{tikzcd}$$

Here $\hat{\otimes}$ *is the graded tensor product of algebras, and* γ *is the standard isomorphism*

$$e_i \otimes 1 \mapsto e_i \quad \textit{and} \quad 1 \otimes e_j \mapsto e_{a-1+j}[1].$$

Proof. To construct κ, note that the diagram need only be checked on generators $(t_i, 1)$ and $(1, t_j)$. The map γ is essentially $CL(\delta)$ for the orthogonal decomposition

$$\mathbf{R}^{a-1} \oplus \mathbf{R}^{b-1} \xrightarrow{\delta} \mathbf{R}^{a+b-2}; \quad (e_i, 0) \mapsto e_i; \quad (0, e_j) \mapsto e_{a-1+j}.$$

We find that $\kappa \circ \delta$ must satisfy

$$(e_i, 0) \mapsto e_i \quad \text{and} \quad (0, t_j') \mapsto t'_{a+j}.$$

These formulae do in fact give an isometric embedding, so we compose with δ^{-1} to obtain the required κ.

Now let M_i denote the Grothendieck group of finitely generated complex modules over $CL(i)$, and GM_i the one generated by $\mathbf{Z}/2$-graded complex modules. There are operations rev, ass, π and η relating these just as in the group representation case, and in fact

$$\bigoplus_{n=0}^{\infty} (GM_n \oplus M_n)$$

has the structure of a $\mathbf{Z}/2 \times \mathbf{N}$-algebra, in fact a $\mathscr{C}$-object. We don't really use these last facts, so they will be treated elsewhere in detail. We'll use only information essentially contained in **[1]**, but express it using a convenient notation dependent on the structure of the above ring. That structure is in fact

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$$\mathbf{Z}[p, m]/\langle p^3 = 2p\rangle$$

where $p \in M_0$ and $m \in M_1$. Thus

$$GM_{2j} \cong \mathbf{Z} \oplus \mathbf{Z} \text{ with generators } m^{2j}, rm^{2j}$$

$$M_{2j+1} \cong \mathbf{Z} \oplus \mathbf{Z} \text{ with generators } m^{2j+1}, rm^{2j+1}$$

$$GM_{2j+1} \cong \mathbf{Z} \text{ with generator } pm^{2j+1}$$

$$M_{2j} \cong \mathbf{Z} \text{ with generator } pm^{2j}.$$

Here $r = p^2 - 1$. These generators actually represent modules, where m^{2j+1}, rm^{2j+1}, and pm^{2j} have dimension 2^j, whereas the graded modules m^{2j} and rm^{2j} have graded parts each of dimension 2^{j-1}, but the graded parts of pm^{2j+1} have dimension 2^j.

We need to calculate the effect of restriction along $CL(\kappa_{a,b})$, giving maps

$$M_{a+b-1} \rightarrow M_{a+b-2} \quad \text{and}$$

$$GM_{a+b-1} \rightarrow GM_{a+b-2}.$$

Suppose given an isometric embedding $(V, q) \hookrightarrow (V', q')$ of inner product spaces. When $V = V'$, this is given by conjugation with an element of $CL(V', q')$, and so induces the identity on $M[CL(V', q')]$. On $GM[CL(V', q')]$ it induces either the identity or multiplication by r (i.e., reversing), depending on whether the element is in CL_0 or CL_1 i.e., whether the isometry of (V', q') preserves orientation or not.

Now consider the case where $\dim V < \dim V'$. Any two isometric embeddings differ by an orientation preserving isometry of (V', q'). By the previous paragraph they have the same effect on both M and GM. Thus we can replace κ by the standard inclusion

$$\mathbf{R}^{a+b-2} \hookrightarrow \mathbf{R}^{a+b-1},$$

and its effect has already been calculated in [**1**] (or follows very easily by their methods in cases where it is not made explicit). We state the result.

PROPOSITION 5.2. *Any isometric embedding* $\mathbf{R}^{i-1} \hookrightarrow \mathbf{R}^i$ *has the following effect on modules*:

$\underline{i = 2j}$: $m^{2j} \mapsto pm^{2j-1}$, $rm^{2j} \mapsto pm^{2j-1}$; $pm^{2j} \mapsto (1 + r)m^{2j-1}$

$\underline{i = 2j + 1}$: $pm^{2j+1} \mapsto (1 + r)m^{2j}$; $m^{2j+1} \mapsto pm^{2j}$, $rm^{2j+1} \mapsto pm^{2j}$.

In effect, "replace one copy of m by p".

In fact this proposition can also be proved directly by counting dimensions and taking note of the effects of rev and ass.

6. How to get a Hopf algebra. Let $\mathscr{G}'$ be the subcategory of $\mathscr{G}$ consisting of all $\mathscr{G}$-objects and those $\mathscr{G}$-maps which are injective as functions. Here we give a slightly abstract version of how functors on $\mathscr{G}'$ can give graded algebras and Hopf algebras when applied to the objects $\tilde{\Sigma}_n\langle\Gamma\rangle$. This is analogous to the classical case of ordinary groups and Σ_n, but slightly more complicated. In the next section we apply the considerations below to several different functors, so this abstractness is merely for efficiency. First we state some standing assumptions, then make a list of labelled hypotheses which will occur in different combinations in the following propositions.

Suppose given two functors (both called W) from $\mathscr{G}'$ to $\mathbf{Z}/2$-graded abelian groups. These functors are to agree on objects (denoted

$$\Omega \mapsto W\Omega = W^{(0)}\Omega \oplus W^{(1)}\Omega),$$

but one of them is contravariant (denoted $\beta \mapsto \beta^*$), while the other is covariant (denoted $\beta \mapsto \beta_*$).

Suppose also given natural transformations

$$\omega: W\Lambda \otimes W\Omega \to W(\Lambda \hat{\times} \Omega)$$

$$\rho: W\Lambda \to W\Lambda$$

(natural for both the contra- and covariant versions). Here the tensor product has the usual grading:

$$(W \otimes W)^{(0)} = W^{(0)} \otimes W^{(0)} + W^{(1)} \otimes W^{(1)}$$

$$(W \otimes W)^{(1)} = W^{(0)} \otimes W^{(1)} + W^{(1)} \otimes W^{(0)}.$$

Assume that ρ is an involution: $\rho \circ \rho = 1$. Assume finally that the following diagram is commutative:

$$\begin{tikzcd}
W\Lambda \otimes W\Omega \arrow[r, "\omega"] \arrow[d, "\rho \otimes 1"'] & W(\Lambda \hat{\times} \Omega) \arrow[d, "\rho"] & W\Lambda \otimes W\Omega \arrow[l, "\omega"'] \arrow[d, "1 \otimes \rho"] \\
W\Lambda \otimes W\Omega \arrow[r, "\omega"] & W(\Lambda \hat{\times} \Omega) & W\Lambda \otimes W\Omega \arrow[l, "\omega"']
\end{tikzcd}$$

Now fix a finite group Γ, and for the rest of this section denote $\tilde{\Sigma}_n\langle\Gamma\rangle$ by Γ_n. The first two "hypotheses" are as follows:

(UN) There exists $1 \in W^{(0)}\Gamma_0$ such that

$$\phi_*\omega(1 \otimes x) = x = \phi_*\omega(x \otimes 1) \quad \text{for all } x \in W(\Gamma_j), \text{ all } j.$$

(Here ϕ is $\phi_{0,j}$ and $\phi_{j,0}$ from 1.7.)

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(AS) For all Λ, Λ', Λ'' in $\mathcal{G}$, the following commutes

$$\begin{tikzcd}
W\Lambda \otimes W\Lambda' \otimes W\Lambda'' \arrow[r, "\omega \otimes 1"] \arrow[dd, "1 \otimes \omega"'] & W(\Lambda \hat{\times} \Lambda') \otimes W\Lambda'' \arrow[d, "\omega"] \\
 & W[(\Lambda \hat{\times} \Lambda') \hat{\times} \Lambda''] \arrow[d, "\alpha_*"] \\
W\Lambda \otimes W(\Lambda' \hat{\times} \Lambda'') \arrow[r, "\omega"] & W[\Lambda \hat{\times} (\Lambda' \hat{\times} \Lambda'')]
\end{tikzcd}$$

PROPOSITION 6.1. *In order that* $\bigoplus_{i \geqq 0} W\Gamma_i$ *be a ring (graded over* $\mathbf{Z}/2 \times \mathbf{N}$ *and associative with* 1) *under the multiplication*

$$W\Gamma_i \otimes W\Gamma_j \xrightarrow{\omega} W(\Gamma_i \hat{\times} \Gamma_j) \xrightarrow{\phi_*} W\Gamma_{i+j},$$

it suffices that (UN) *and* (AS) *hold.*

Proof. Obviously we get a 1 by (UN), and associativity is proved by the following diagram:

$$\begin{tikzcd}
W\Gamma_i \otimes W\Gamma_j \otimes W\Gamma_k \arrow[r, "\omega \otimes 1"] \arrow[dd, "1 \otimes \omega"'] & W(\Gamma_i \hat{\times} \Gamma_j) \otimes W\Gamma_k \arrow[r, "\phi_* \otimes 1"] \arrow[d, "\omega"] \arrow[dr, phantom, "\text{(naturality)}"] & W\Gamma_{i+j} \otimes W\Gamma_k \arrow[d, "\omega"] \\
\arrow[r, phantom, "\text{(AS)}"] & W[(\Gamma_i \hat{\times} \Gamma_j) \hat{\times} \Gamma_k] \arrow[r, "(\phi \hat{\times} 1)_*"] \arrow[d, "\alpha_*"] & W(\Gamma_{i+j} \hat{\times} \Gamma_k) \arrow[dd, "\phi_*"] \\
W\Gamma_i \otimes W(\Gamma_j \hat{\times} \Gamma_k) \arrow[r, "\omega"] \arrow[d, "1 \otimes \phi_*"'] \arrow[dr, phantom, "\text{(naturality)}"] & W[\Gamma_i \hat{\times} (\Gamma_j \hat{\times} \Gamma_k)] \arrow[d, "(1 \hat{\times} \phi)_*"] \arrow[r, phantom, "\text{[1.8 iii)]}"] & \\
W\Gamma_i \otimes W\Gamma_{j+k} \arrow[r, "\omega"] & W(\Gamma_i \hat{\times} \Gamma_{j+k}) \arrow[r, "\phi_*"] & W\Gamma_{i+j+k}
\end{tikzcd}$$

The next hypotheses are:

(IS) ω is an isomorphism;

(IN) If β is a bijective $\mathcal{G}$-map, then $\beta^* = \beta_*^{-1}$.

PROPOSITION 6.2. *With assumptions* (UN), (AS), (IS) *and* (IN), *it follows that* $\bigoplus_{i \geqq 0} W\Gamma_i$ *is a graded coalgebra (coassociative with counit) under the comultiplication*

$$\triangle_{i,j}: W\Gamma_{i+j} \xrightarrow{\phi^*} W(\Gamma_i \hat{\times} \Gamma_j) \xrightarrow{\omega^{-1}} W\Gamma_i \otimes W\Gamma_j.$$

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Proof. Just reverse all arrows in 6.1, replacing ϕ_* by ϕ^* and ω by ω^{-1}.

The last five hypotheses are

(C)* The following diagram commutes

$$\begin{tikzcd}
x \otimes y \arrow[d, leftrightarrow] & W^{(\epsilon)}\Lambda \otimes W^{(\delta)}\Omega \arrow[r, "\omega"] \arrow[d, leftrightarrow, "\sigma"'] & W^{(\epsilon+\delta)}(\Lambda \mathbin{\hat{\times}} \Omega) \arrow[d, "\tau^* \circ \rho^{\epsilon\delta}"] \\
y \otimes x & W^{(\delta)}\Omega \otimes W^{(\epsilon)}\Lambda \arrow[r, "\omega"] & W^{(\epsilon+\delta)}(\Omega \mathbin{\hat{\times}} \Lambda)
\end{tikzcd}$$

$(C)_*$ Replace τ^* by τ_* in (C)*.

(S) The map $\Lambda \to \Lambda$ given by $x \to z^{s(x)}x$ induces

$$\rho^\epsilon : W^{(\epsilon)}\Lambda \to W^{(\epsilon)}\Lambda$$

on both functors, covariant and contravariant.

(IA) For all $w \in \Lambda$, the inner automorphism $\iota(w)$ induces

$$\rho^{s(w)(\epsilon+1)} : W^{(\epsilon)}\Lambda \to W^{(\epsilon)}\Lambda$$

on both functors.

(M) Mackey's theorem holds for W in the following sense: In the statement of 2.16, replace GR^- by $W^{(\epsilon)}$, replace rev by $\rho^{1+\epsilon}$, and reinterpret α_*, α^*, etc. (We are assuming this for both $\epsilon = 0$ and $\epsilon = 1$.)

PROPOSITION 6.3. (IN) $\Rightarrow$ [(C)* $\Leftrightarrow$ $(C)_*$].

Proof. Since $\tau \circ \tau = 1$, and $\tau^* \circ \tau_* = 1$ by (IN), we have $\tau^* = \tau_*$.

Note. We could have similarly divided both (S) and (IA) into covariant and contravariant parts, but enough is enough.

PROPOSITION 6.4. (M) $\Rightarrow$ (IN).

Proof. In (M) take all groups equal, $g_\triangle = 1$, $\alpha = \beta$ and $\alpha_\triangle = \beta_\triangle = 1$. We get $\beta^* \circ \beta_* = 1$. But β^* and β_* are bijective.

PROPOSITION 6.5. (M) $\Rightarrow$ (IA).

Proof. In (M), take all groups equal, $g_\triangle = w$, $\alpha_\triangle = \beta_\triangle = \beta = 1$ and $\alpha = \iota(w)$. We get

$$\iota(w)^* 1_*(x) = \rho^{s(w)(\epsilon+1)} 1_* 1^*(x)$$

or

$$\iota(w)^* = \rho^{s(w)(\epsilon+1)}.$$

The argument for $\iota(w)_*$ is the same, except $\alpha = 1$, $\beta = \iota(w)$. (Alternatively, use 6.4.)

COROLLARY 6.6 *If* (IA) *and* (S) *both hold, then for*

$$\zeta_{a,b}:\Gamma_{a+b} \to \Gamma_{a+b}$$

as in 1.8 ii), *we have*

$$\zeta^*_{a,b} = \zeta_{a,b_*} = \rho^{ab}:W^{(\epsilon)} \to W^{(\epsilon)} \text{ for } \epsilon = 0 \text{ and } 1.$$

Proof. The argument is exactly the same as given in the proof of 2.14.

PROPOSITION 6.7. *Assuming* (UN), (AS), (IA), (S), (C)$_*$, *the algebra of* 6.1 *is pseudo-commutative in that, for all* $x \in W^{(\epsilon)}\Gamma_i$ *and* $y \in W^{\delta}\Gamma_j$, *we have*

$$xy = \rho^{\epsilon\delta + ij}(yx).$$

Proof. Use the diagram

$$\begin{tikzcd}
W^{(\epsilon)}\Gamma_i \otimes W^{(\delta)}\Gamma_j \arrow[r, "\omega"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "(C)_*"] & W^{(\epsilon+\delta)}(\Gamma_i \hat{\times} \Gamma_j) \arrow[r, "\phi_*"] \arrow[d, "\tau_* \circ \rho^{\epsilon\delta}"] \arrow[dr, phantom, "{[1.8\text{ ii) and naturality of }\rho]}"] & W^{(\epsilon+\delta)}\Gamma_{i+j} \arrow[d, "\rho^{\epsilon\delta} \circ \zeta_{i,j_*}"] \\
W^{(\delta)}\Gamma_j \otimes W^{(\epsilon)}\Gamma_i \arrow[r, "\omega"] & W^{(\epsilon+\delta)}(\Gamma_j \hat{\times} \Gamma_i) \arrow[r, "\phi_*"] & W^{(\epsilon+\delta)}\Gamma_{i+j}
\end{tikzcd}$$

and note that $\rho^{\epsilon\delta} \circ \zeta_{i,j_*} = \rho^{\epsilon\delta + ij}$ *by* 6.6.

PROPOSITION 6.8. *Assuming* (UN), (AS), (IS), (IA), (IN), (S), (C)*, *the coalgebra of* 6.2 *satisfies the pseudo-co-commutative condition*:

$$\triangle_{(i,\epsilon),(j,\delta)}(z) = \sigma\triangle_{(j,\delta),(i,\epsilon)}\rho^{ij+\epsilon\delta}(z)$$

for all $z \in W^{(\epsilon+\delta)}\Gamma_{i+j}$, *where* $\triangle_{(i,\epsilon),(j,\delta)}$ *is* $\triangle$ *followed by projection to* $W^{(\epsilon)}\Gamma_i \otimes W^{(\delta)}\Gamma_j$.

Proof. Reverse the arrows in 6.7.

THEOREM 6.9. *Assume* (UN), (AS), (IS), (S), (C)$_*$ *and* (M). *Then the algebra-co-algebra* $\bigoplus_{i \geqq 0} W\Gamma_i$ *becomes a Hopf algebra which satisfies the commutativity conditions of* 6.7 *and* 6.8.

Proof. We get (C)*, (IA), and (IN) for free by 6.3, 6.4 and 6.5. All that remains is to prove that $\triangle$ is an algebra homomorphism. Note that in defining the multiplication in the tensor product algebra of two algebras in this category of $\mathbf{Z}/2 \times \mathbf{N}$-graded algebras which are pseudo-commutative, we introduce the appropriate "sign", namely $\rho^{ij+\epsilon\delta}$ when interchanging the two middle factors; that is

$$(x_1 \otimes x_2)(y_1 \otimes y_2) := \rho^{ij+\epsilon\delta}(x_1y_1 \otimes x_2y_2)$$

where

$$x_2 \in W^{(\epsilon)}\Gamma_i \quad \text{and} \quad y_1 \in W^{(\delta)}\Gamma_j.$$

The homomorphism property is equality of the outside arrows in the commutative diagram below, where we are given

$$a + b = k = c + d$$

$$\epsilon + \delta = \gamma = \epsilon' + \delta'$$

and the direct sums $\coprod$ are over all matrices

$$\mathscr{J} = \begin{bmatrix} i_1 & i_2 \\ j_1 & j_2 \end{bmatrix}$$

with non-negative integer entries, row sums $\begin{pmatrix} a \\ b \end{pmatrix}$ and column sums (c, d), and in two cases also over all matrices

$$\mathscr{K} = \begin{bmatrix} \epsilon_1 & \epsilon_2 \\ \delta_1 & \delta_2 \end{bmatrix}$$

with $\mathbf{Z}/2$-entries, row sums $\begin{pmatrix} \epsilon \\ \delta \end{pmatrix}$ and column sums (ϵ', δ'). (Here and below we write $W^{(\epsilon)}$ as $\overset{\epsilon}{W}$).

$$\begin{tikzcd}
\overset{\epsilon}{W}\Gamma_a \otimes \overset{\delta}{W}\Gamma_b \arrow[r, "\phi^* \otimes \phi^*"] \arrow[d, "\omega"'] \arrow[dr, phantom, "\text{(naturality)}"] & \coprod_{\mathscr{J}} \overset{\epsilon}{W}(\Gamma_{i_1} \hat{\times} \Gamma_{i_2}) \otimes \overset{\delta}{W}(\Gamma_{j_1} \hat{\times} \Gamma_{j_2}) \arrow[r, "\coprod \omega^{-1} \otimes \omega^{-1}"] \arrow[d, "\coprod \omega"] \arrow[dr, phantom, "(*)"] & \coprod_{\mathscr{J},\mathscr{K}} \overset{\epsilon_1}{W}\Gamma_{i_1} \otimes \overset{\epsilon_2}{W}\Gamma_{i_2} \otimes \overset{\delta_1}{W}\Gamma_{j_1} \otimes \overset{\delta_2}{W}\Gamma_{j_2} \arrow[d, "\coprod \rho^{i_2 j_1 + \epsilon_2 \sigma_1}\, 1 \otimes \sigma \otimes 1"] \\
\overset{\gamma}{W}(\Gamma_a \hat{\times} \Gamma_b) \arrow[r, "(\phi \hat{\times} \phi)^*"] \arrow[dd, "\phi_*"'] \arrow[ddr, phantom, "\text{(M) and (1.9)}"] & \coprod_{\mathscr{J}} \overset{\gamma}{W}(\Gamma_{i_1} \hat{\times} \Gamma_{i_2} \hat{\times} \Gamma_{j_1} \hat{\times} \Gamma_{j_2}) \arrow[d, "\coprod \rho^{j1j2} \circ (1 \hat{\times} \tau \hat{\times} 1)_*"] & \coprod_{\mathscr{J},\mathscr{K}} \overset{\epsilon_1}{W}_{i_1} \otimes \overset{\delta_1}{W}\Gamma_{j_1} \otimes \overset{\epsilon_2}{W}\Gamma_{i_2} \otimes \overset{\delta_2}{W}\Gamma_{j_2} \arrow[d, "\coprod \omega \otimes \omega"] \\
 & \coprod_{\mathscr{J}} \overset{\gamma}{W}(\Gamma_{i_1} \hat{\times} \Gamma_{j_1} \hat{\times} \Gamma_{i_2} \hat{\times} \Gamma_{j_2}) \arrow[r, "\coprod \omega^{-1}"] \arrow[d, "(\phi \hat{\times} \phi)_*"] \arrow[dr, phantom, "\text{(naturality)}"] & \coprod_{\mathscr{J}} \overset{\epsilon'}{W}(\Gamma_{i_1} \hat{\times} \Gamma_{j_1}) \otimes \overset{\delta'}{W}(\Gamma_{i_2} \hat{\times} \Gamma_{j_2}) \arrow[d, "\phi_* \otimes \phi_*"] \\
\overset{\gamma}{W}(\Gamma_k) \arrow[r, "\phi^*"'] & \overset{\gamma}{W}(\Gamma_c \hat{\times} \Gamma_d) \arrow[r, "\omega^{-1}"'] & \overset{\epsilon'}{W}\Gamma_c \otimes \overset{\delta'}{W}\Gamma_d
\end{tikzcd}$$

To apply (M) and (1.9) for the bottom left square, we must alter the diagram in 1.9, replacing the bottom arrow by just $\iota(w)$, and composing $1 \hat{\times} \tau \hat{\times} 1$ on the top right with

$$(x \mapsto z^{j_1 i_2 s(x)} \cdot x).$$

The new diagram still commutes, and (M) is now applicable. We get a factor

$$\rho^{i_2 j_1(\epsilon_1+\epsilon_2+\delta_1+\delta_2+1)}$$

from the Mackey formula assumed in (M). But we have

$$[x \mapsto z^{s(x)}x]_* = \rho^{\epsilon} \text{ on } W^{(\epsilon)} \text{ by (S),}$$

so

$$[x \mapsto z^{i_2 j_1 s(x)}x]_* = \rho^{i_2 j_1(\epsilon_1+\epsilon_2+\delta_1+\delta_2)}$$

here. Thus the "sign" here is $\rho^{i_2 j_1}$, as required. The top right square commutes as follows, (where here $\Gamma_1, \ldots, \Gamma_4$ may be any $\mathscr{G}$-objects):

$$\begin{tikzcd}
\overset{\epsilon_1}{W}\Gamma_1 \otimes \overset{\epsilon_2}{W}\Gamma_2 \otimes \overset{\delta_1}{W}\Gamma_3 \otimes \overset{\delta_2}{W}\Gamma_4 \arrow[rrr, "\omega \otimes \omega"] \arrow[rrd, "\omega \otimes 1"] \arrow[rdd, "1 \otimes \omega \otimes 1"'] \arrow[dddd, "1 \otimes \sigma \otimes 1"'] & & & \overset{\epsilon_1+\epsilon_2}{W}(\Gamma_1 \overset{\wedge}{\times} \Gamma_2) \otimes \overset{\delta_1+\delta_2}{W}(\Gamma_3 \overset{\wedge}{\times} \Gamma_4) \arrow[dd, "\omega"] \\
& \text{(AS)} & \overset{\epsilon_1+\epsilon_2}{W}(\Gamma_1 \overset{\wedge}{\times} \Gamma_2) \otimes \overset{\delta_1}{W}\Gamma_3 \otimes \overset{\delta_2}{W}\Gamma_4 \arrow[ru, "1 \otimes \omega"] \arrow[d, "\omega \otimes 1"] \arrow[r, phantom, "\text{(AS)}"] & \\
& \overset{\epsilon_1}{W}\Gamma_1 \otimes \overset{\epsilon_1+\delta_2}{W}\Gamma(\Gamma_2 \overset{\wedge}{\times} \Gamma_3) \otimes \overset{\delta_2}{W}\Gamma_4 \arrow[r, "\omega \otimes 1"] \arrow[d, "1 \otimes (\tau_* \circ \rho^{\epsilon_2 \delta_1}) \otimes 1"] & W(\Gamma_1 \overset{\wedge}{\times} \Gamma_2 \overset{\wedge}{\times} \Gamma_3) \otimes W\Gamma_4 \arrow[r, "\omega"] \arrow[d, "(1 \overset{\wedge}{\times} \tau)_* \circ \rho^{\epsilon_2 \delta_1} \otimes 1"] & W(\Gamma_1 \overset{\wedge}{\times} \Gamma_2 \overset{\wedge}{\times} \Gamma_3 \overset{\wedge}{\times} \Gamma_4) \arrow[d, "(1 \overset{\wedge}{\times} \tau \overset{\wedge}{\times} 1)_* \circ \rho^{\epsilon_2 \delta_1}"] \\
& W\Gamma_1 \otimes W(\Gamma_3 \overset{\wedge}{\times} \Gamma_2) \otimes W\Gamma_4 \arrow[r, "\omega \otimes 1"] \arrow[l, phantom, "(C)_*"] & W(\Gamma_1 \overset{\wedge}{\times} \Gamma_3 \overset{\wedge}{\times} \Gamma_2) \otimes W\Gamma_4 \arrow[r, "\omega"] \arrow[d, phantom, "\text{(same as top third with } \Gamma_2, \Gamma_3 \text{ interchanged)}"] & W(\Gamma_1 \overset{\wedge}{\times} \Gamma_3 \overset{\wedge}{\times} \Gamma_2 \overset{\wedge}{\times} \Gamma_4) \\
\overset{\epsilon_1}{W}\Gamma_1 \otimes \overset{\delta_1}{W}\Gamma_3 \otimes \overset{\epsilon_2}{W}\Gamma_2 \otimes \overset{\delta_2}{W}\Gamma_4 \arrow[ru, "1 \otimes \omega \otimes 1"] \arrow[rrr, "\omega \otimes \omega"] & & & W(\Gamma_1 \overset{\wedge}{\times} \Gamma_3) \otimes W(\Gamma_2 \overset{\wedge}{\times} \Gamma_4) \arrow[u, "\omega"']
\end{tikzcd}$$

The squares in the middle commute by naturality and by the diagram relating ω and ρ given at the beginning. We have included only sufficient ϵ's and δ's to make the diagram unambiguous.

7. Proof of 3.4. We must verify that $\bar{H}$ is a $\mathscr{C}$-object and that (I) to (VII) hold, with definitions as given in the statement of 3.4.

To prove $\bar{H}$ is a $\mathscr{C}$-object, take the functor W in Section 6 to be $\mathrm{GR}^- \oplus R^-$, that is $W^{(0)} = \mathrm{GR}^-$, $W^{(1)} = \mathrm{R}^-$. Take ρ to be rev on $W^{(0)}$, and ass on $W^{(1)}$. To make the product defined in 3.4 compatible with that in 6.1, we take ω to be the following:

$\boxtimes_1$: $W^{(0)} \otimes W^{(0)} \to W^{(0)}$;

$\boxtimes\!\boxtimes$ $: W^{(1)} \otimes W^{(1)} \to W^{(0)}$;

$\boxtimes_2$ $: W^{(0)} \otimes W^{(1)} \to W^{(1)}$;

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and the fourth component of ω will be the composite

$$W^{(1)} \otimes W^{(0)} \xrightarrow{\sigma} W^{(0)} \otimes W^{(1)} \xrightarrow{\boxtimes_2} W^{(1)} \xrightarrow{\tau_*} W^{(1)}.$$

Finally take

$$1 := [\{\mathbf{C}, 0\}] \in W^{(0)}(\Gamma_0) = \mathrm{GR}^-(\widetilde{\Sigma}_0\langle\Gamma\rangle) = \mathrm{GR}^-(\widetilde{\Sigma}_0).$$

Here $\widetilde{\Sigma}_0 = \{1, z\}$ and z acts as -1 on $\mathbf{C}$. Let $r = \mathrm{rev}(1)$ and $p = \pi(1)$.

Note first that $(\mathrm{C})_*$ holds for $\epsilon = \delta = 0$ by 2.22 i), for $\epsilon = \delta = 1$ by 2.22 iv) and for $\epsilon = 0, \delta = 1$ or $\epsilon = 1, \delta = 0$ tautologically. Verification of (UN) is trivial from the definition of 1. The proof of (AS) divides into eight cases by restricting to

$$W^{(\epsilon_1)}\Lambda \otimes W^{(\epsilon_2)}\Lambda' \otimes W^{(\epsilon_3)}\Lambda''.$$

The four cases $(\epsilon_1, \epsilon_2, \epsilon_3) = (0, 0, 0), (1, 0, 0,), (1, 1, 0)$ and $(1, 1, 1)$ are exactly the four parts of 2.23. But the other four cases follow from these four, as discussed below.

First of all we have the formulae

$$\mathrm{rev}(xy) = (\mathrm{rev}\, x)y = x\, \mathrm{rev}\, y \quad (x, y \in W^{(0)})$$

$$\mathrm{rev}(xy) = (\mathrm{ass}\, x)y = x\, \mathrm{ass}\, y \quad (x, y \in W^{(1)})$$

$$\mathrm{ass}(xy) = (\mathrm{rev}\, x)y = x(\mathrm{ass}\, y) \quad (x \in W^{(0)}, y \in W^{(1)})$$

$$\pi(xy) = x(\pi y) = (\pi x)y \quad (x \in W^{(0)}, y \in W^{(0)})$$

$$\eta(xy) = x\eta(y) = (\pi x)y \quad (x \in W^{(0)}, y \in W^{(1)}).$$

These are immediate using the definition of the multiplication, naturality and 2.22 ii), iv), iii).

It now follows that

$$\begin{aligned} rx &= xr = \mathrm{rev}(x) && (x \in W^{(0)}) \\ rx &= xr = \mathrm{ass}(x) && (x \in W^{(1)}) \\ px &= xp = \pi(x) && (x \in W^{(0)}) \\ px &= xp = \eta(x) && (x \in W^{(1)}) \end{aligned}$$

verifying that last statement of 3.4. To prove these, for example

$$\mathrm{rev}(x) = \mathrm{rev}(1 \cdot x) = (\mathrm{rev}\, 1)x = rx \quad \text{for } x \in W^{(0)},$$

and similarly for the others.

Next we get $p^2 = 1 + r$, $pr = rp = p$ and $p^3 = 2p$. The first since

$$p^2 = \eta(p) = \eta\pi(1) = (1) + \mathrm{rev}(1) = 1 + r.$$

Then

$$pr = \pi(r) = \pi(1^{\text{rev}}) = \pi(1) = p = (\pi 1)^{\text{ass}} = rp,$$

and

$$p^3 = p(1 + r) = p + pr = p + p = 2p.$$

The verification of the four remaining cases of (AS) can now be done, but it is easier to see how to deduce directly the four cases of associativity for the multiplication that these would be used to deduce. For example, in the case $(\epsilon_1, \epsilon_2, \epsilon_3) = (0, 1, 1)$, with

$$x \in W^{(0)}\Gamma_i, \quad y \in W^{(1)}\Gamma_j, \quad z \in W^{(1)}\Gamma_k,$$

we have

$$\begin{aligned}(xy)z &= (r^{ij}yx)z = r^{ij}[(yx)z] \\ &= r^{ij+(i+j)k}[(z)(yx)] = r^{ij+ik+jk}[(zy)x] \\ &= r^{ij+ik+jk}r^{jk}[(yz)x] = r^{i(j+k)}[(yz)x] \\ &= x(yz),\end{aligned}$$

as required. We have used only cases of (AS) already proved, the pseudo-commutativity which follows from $(\mathrm{C})_*$, and the fact that r can be moved around at will because of the previous identities. The other three cases are similar manipulations.

It remains to verify (I) to (VII).

(I) We can take the required bases to be

$$\mathrm{GIRREP}^-(\tilde{\Sigma}_n\langle\Gamma\rangle) \quad \text{and} \quad \mathrm{IRREP}^-(\tilde{\Sigma}_n\langle\Gamma\rangle),$$

by 2.10 with $\Omega = \tilde{\Sigma}_n\langle\Gamma\rangle$, as long as we verify that the integers μ and ν in 2.10 are $\#\mathscr{D}''(n, S)$ and $\#\mathscr{D}'(n, S)$ respectively, with $S = \mathrm{Con}(\Gamma)$. Using the numerical equality of #IRREP and #Con, this goes as follows:

Since

$$\mathrm{R}(\tilde{\Sigma}_n\langle\Gamma\rangle) = \mathrm{R}^+(\tilde{\Sigma}_n\langle\Gamma\rangle) \oplus \mathrm{R}^-(\tilde{\Sigma}_n\langle\Gamma\rangle), \quad \text{and}$$

$$\mathrm{R}^+(\tilde{\Sigma}_n\langle\Gamma\rangle) \cong \mathrm{R}(\tilde{\Sigma}_n\langle\Gamma\rangle),$$

the conjugacy class count for $\tilde{\Sigma}_n\langle\Gamma\rangle$ in 1.13 yields

$$2\nu + \mu = 2\#\mathscr{D}'(n, S) + \#\mathscr{D}''(n, S).$$

Using ad hoc arguments for $n = 0$ and 1 (where $A_n = \Sigma_n$), and the isomorphism

$$\mathrm{GR}(\tilde{\Sigma}_n\langle\Gamma\rangle) \cong \mathrm{R}(\tilde{A}_n\langle\Gamma\rangle) \quad \text{for } n \geqq 2 \text{ of } 2.12,$$

together with the conjugacy class count of $\tilde{A}_n\langle\Gamma\rangle$ in 1.13, we obtain

$$\nu + 2\mu = \#\mathscr{D}'(n, S) + 2\#\mathscr{D}''(n, S),$$

as required.

Next we must verify (II). To construct ∇, we again use Section 6, but this time let

$$W^{(0)} = \mathrm{GR}^-/\{x:x^{\mathrm{rev}} = x\}, \quad W^{(1)} = \mathrm{R}^-/\{y:y^{\mathrm{ass}} = y\}.$$

This is exactly the functor W of 2.26, and for ω we use $\boxtimes_4$ as defined there. Since $\boxtimes_4$ is just passing to the quotient with the previous ω used to define multiplication in $\bar{H}$, verifications of (UN) and (AS) are immediate. Define ρ here to be multiplication by -1. Since $[r] = [-1]$ in $E\bar{H}$, the diagram connecting ω and ρ and hypothesis $(\mathrm{C})_*$ follow from the fact that they held in the previous application of Section 6. To get a Hopf algebra, apply 6.9. We must still verify (IS), (S) and (M). But (IS) is given by 2.26. (S) is immediate, since $[x \mapsto z^{s(x)}x]_*$ is ass on R^-, and is the identity on GR^- because $\mathbf{Z}/2$-gradable representations are self-associate. Finally, using the classical Mackey theorem for R^- (with no "sign" in the formula) and using 2.16 for GR^- (where the "sign" rev occurs), then passing to the quotient, we get (M) for this choice of W.

To complete the verification of (II) we make yet another application of Section 6. Here we take $W^{(0)} = \mathrm{R}^-_{sa} \otimes \mathbf{Q}$ as defined in 2.27, and take $W^{(1)} = 0$, so the $\mathbf{Z}/2$-grading here is more apparent than real. Take ω to be $\boxtimes_3$ from 2.27, and $\rho = 1$. The hypotheses of 6.9 are now easy to verify. (IS) is given by 2.27. For (UN), we take "1" to be $p = \pi(1)$, since

$$\phi_*[\pi(1) \boxtimes_3 \pi(x)] = \phi_*\pi(1 \boxtimes_1 x) = \pi x.$$

For $\boxtimes_3$, (AS) follows immediately from (AS) for $\boxtimes_1$ given by 2.23 i). (S) is clear, since, by definition, ass is the identity on R^-_{sa}. $(\mathrm{C})_*$ holds for $\omega = \boxtimes_3$, $\rho = 1$ since we have only $\epsilon = \delta = 0$ to consider, and since $(\mathrm{C})_*$ holds in that case for $\omega = \boxtimes_1$. Finally (M) holds by the ordinary Mackey theorem for (ungraded) representations.

Next we must check hypothesis (III). But this is given by the third sentence of Corollary 2.27, in view of the definition of $\triangle$ and the choice of $\{z_\phi\}$ in (I).

For hypothesis (IV), we use the modules in Section 5 to define the elements $\bar{h}_n^{(s)}$. Pick some bijective function (recalling $S = \mathrm{Con}\,\Gamma$)

$$S \to \mathrm{IRREP}\,\Gamma; \quad s \mapsto [V_s].$$

Let N_{2r} be a module representing $m^{2r-1} \in M_{2r-1}$, and let $\{N_{2r+1}^{(0)}, N_{2r+1}^{(1)}\}$ be a graded module representing $m^{2r} \in GM_{2r}$. Using the embeddings

$$\xi_i:\tilde{\Sigma}_i \hookrightarrow CL(i - 1),$$

we get an action of $\tilde{\Sigma}_{2r}$ on N_{2r} and of $\tilde{\Sigma}_{2r+1}$ on $\{N_{2r+1}^{(0)}, N_{2r+1}^{(1)}\}$. Finally, let

$$\bar{h}_{2r}^{(s)} = [V_s^{\otimes 2r} \otimes N_{2r}] \in \mathrm{R}^-(\tilde{\Sigma}_{2r}\langle\Gamma\rangle) = \bar{H}_{1,2r}$$

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$$\bar{h}^{(s)}_{2r+1} = [\,\{V_s^{\otimes 2r+1} \otimes N^{(0)}_{2r+1},\ V_s^{\otimes 2r+1} \otimes N^{(1)}_{2r+1}\}\,]$$
$$\in \mathrm{GR}^-(\widetilde{\Sigma}_{2r+1}\langle\Gamma\rangle) = \bar{H}_{0,2r+1}.$$

Here the action of $\widetilde{\Sigma}_{2r}\langle\Gamma\rangle$ on $V_s^{\otimes 2r} \otimes N_{2r}$ is

$$(g_1, \dots, g_{2r};\ s)(v_1 \otimes \dots \otimes v_{2r} \otimes n)$$
$$= (g_1 v_{\theta(s)^{-1}1}) \otimes \dots \otimes (g_{2r} v_{\theta(s)^{-1}2r}) \otimes (s \cdot n).$$

The formula for the action of $\widetilde{\Sigma}_{2r+1}\langle\Gamma\rangle$ on

$$V_s^{\otimes(2r+1)} \otimes [N^{(0)}_{2r+1} \oplus N^{(1)}_{2r+1}]$$

is the same, with $2r$ replaced by $2r + 1$. In this case, we see that the action respects the $\mathbf{Z}/2$-grading. In both cases, it is easily checked that we have a well defined linear action. To check the required formulae for ∇ and $\triangle$ given in (IV) one simply runs through the definitions of these coproducts, and uses the following result.

PROPOSITION 7.1. *If* $a > 0$ *and* $b > 0$, *we have*

$$\phi^*_{a,b}\bar{h}^{(s)}_{a+b} = \begin{cases} \eta\bar{h}^{(s)}_a \boxtimes_1 \bar{h}^{(s)}_b & (a \text{ even}, b \text{ odd}) \\ \eta\bar{h}^{(s)}_a \boxtimes_2 \bar{h}^{(s)}_b & (a \text{ even}, b \text{ even}) \\ \pi(\bar{h}^{(s)}_a \boxtimes_1 \bar{h}^{(s)}_b) & (a \text{ odd}, b \text{ odd}). \end{cases}$$

Proof. In the case $(a, b) = (2i, 2j)$, we must show that, when restricted to

$$\widetilde{\Sigma}_{2i}\langle\Gamma\rangle \mathbin{\hat{\times}} \widetilde{\Sigma}_{2j}\langle\Gamma\rangle,$$

$V_s^{\otimes(2i+2j)} \otimes N_{2i+2j}$ is isomorphic to

$$\eta(V_s^{\otimes 2i} \otimes N_{2i}) \boxtimes_2 (V_s^{\otimes 2j} \otimes N_{2j}).$$

The latter is

$$[V_s^{\otimes 2i} \otimes (N_{2i} \oplus N^{\mathrm{ass}}_{2i})\,] \boxtimes_2 (V_s^{\otimes 2j} \otimes N_{2j}).$$

The question thus becomes whether N_{2i+2j} restricted to $\widetilde{\Sigma}_{2i} \mathbin{\hat{\times}} \widetilde{\Sigma}_{2j}$ is isomorphic to $\eta N_{2i} \boxtimes_2 N_{2j}$. This follows immediately by using 5.2 to follow the motion around the diagram below of the elements $m^{2i+2j-1}$ and $pm^{2i-1} \otimes m^{2j-1}$ from the bottom corners:

$$\begin{tikzcd}
\mathrm{R}^-(\widetilde{\Sigma}_{2i+2j}) \arrow[rr] & & \mathrm{R}^-(\widetilde{\Sigma}_{2i} \mathbin{\hat{\times}} \widetilde{\Sigma}_{2j}) & \mathrm{GR}^-(\widetilde{\Sigma}_{2i}) \otimes \mathrm{R}^-(\widetilde{\Sigma}_{2j}) \arrow[l, "\boxtimes_2"'] \\
M[CL(2i+2j-1)\,] \arrow[u] \arrow[r] & M[CL(2i+2j-2)\,] \arrow[r] & M[CL(2i-1) \hat{\otimes} CL(2j-1)\,] \arrow[u] & \mathrm{GM}[CL(2i-1) \otimes M[CL(2j-1)\,] \arrow[l] \arrow[u]
\end{tikzcd}$$

The left half of the diagram is induced by the diagram in 5.1. The bottom right horizontal map is the tensor product of Clifford modules with the same action formula as for $\boxtimes_2$.

The other two formulae are proved in an exactly similar way. The case $(a, b) = (2i, 2j + 1)$ reduces to showing by 5.1 and 5.2 that

$$\{N^{(0)}_{2i+2j+1}, N^{(1)}_{2i+2j+1}\}$$

when restricted to $\tilde{\Sigma}_{2i} \hat{\times} \tilde{\Sigma}_{2j+1}$, becomes isomorphic to

$$\eta(N_{2i}) \boxtimes_1 \{N^{(0)}_{2j+1}, N^{(1)}_{2j+1}\}.$$

The remaining case $(a, b) = (2i + 1, 2j - 1)$ reduces to using 5.1 and 5.2 to show that N_{2i+2j}, when restricted to $\tilde{\Sigma}_{2i+1} \hat{\times} \tilde{\Sigma}_{2j-1}$, becomes isomorphic to

$$\pi[\{N^{(0)}_{2i+1}, N^{(1)}_{2i+1}\} \boxtimes_1 \{N^{(0)}_{2j-1}, N^{(1)}_{2j-1}\}].$$

The verification of (V) for $\bar{H}$ is immediate. One uses the usual inner product of representations on $\bar{H}_{1,i}$, and the inner product of graded representations constructed in 2.5 and 2.6 on $\bar{H}_{0,i}$. The formula

$$\langle rx, ry \rangle = \langle x, y \rangle$$

comes from the obvious

$$\mathrm{HOM}_{\Omega}(V, W) = \mathrm{HOM}_{\Omega}(V^{\mathrm{ass}}, W^{\mathrm{ass}})$$

for representations, and

$$\mathrm{GHOM}_{\Omega}(V, W) = \mathrm{GHOM}_{\Omega}(V^{\mathrm{rev}}, W^{\mathrm{rev}})$$

for graded representations.

The verification of (VI) is an application of Frobenius reciprocity. If

$$x \in \mathrm{R}^{-}_{sa}(\tilde{\Sigma}_i\langle\Gamma\rangle), \ y \in \mathrm{R}^{-}_{sa}(\tilde{\Sigma}_j\langle\Gamma\rangle)$$

and

$$z \in \mathrm{R}^{-}_{sa}(\tilde{\Sigma}_{i+j}\langle\Gamma\rangle)$$

we have

$$\begin{aligned}\langle x \circ y, z\rangle &= \langle \phi_{i,j_*}(x \boxtimes_3 y), z\rangle \\ &= \langle x \boxtimes_3 y, \phi^*_{i,j}z\rangle = \ll x \otimes y, \boxtimes_3^{-1} \phi^*_{i,j}z \gg \\ \langle x \circ \gamma, z\rangle &= \ll x \otimes y, \triangle_{i,j}z \gg\end{aligned}$$

as required. The second last equality is the last sentence of 2.27.

Finally, to verify (VII), we must show that the $\bar{h}^{(s)}_n$ are all irreducible, with

$$r\bar{h}^{(s)}_n \neq \bar{h}^{(s)}_n \neq \bar{h}^{(t)}_n \quad \text{if } s \neq t.$$

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In the case where Γ is the trivial group, this is immediate from the fact in 5.1 that the image of $\widetilde{\Sigma}_n$ in $CL(n-1)$ generates $CL(n-1)$ as an algebra, so that irreducible modules remain irreducible on restriction to $\widetilde{\Sigma}_n$, and non-isomorphic pairs of modules remain non-isomorphic. Thus we have

$$\mathrm{HOM}_{\widetilde{\Sigma}_{2i}}(N_{2i}, N_{2i}) = \mathbf{C}$$

$$\mathrm{HOM}_{\widetilde{\Sigma}_{2i}}(N_{2i}, N_{2i}^{\mathrm{ass}}) = 0$$

$$\mathrm{GHOM}_{\widetilde{\Sigma}_{2i+1}}(\{N_{2i+1}^{(0)}, N_{2i+1}^{(1)}\}, \{N_{2i+1}^{(0)}, N_{2i+1}^{(1)}\}) = \mathbf{C}$$

and

$$\mathrm{GHOM}_{\widetilde{\Sigma}_{2i+1}}(N_{2i+1}^{(0)\oplus(1)}, (N_{2i+1}^{(0)\oplus(1)})^{\mathrm{rev}}) = 0.$$

The case of general Γ now goes as follows: First take $n = 2i$. To show

$$\langle \bar{h}_{2i}^{(s)}, r\bar{h}_{2i}^{(t)} \rangle = 0:$$

On restriction to $\widetilde{\Sigma}_{2i} \subset \widetilde{\Sigma}_{2i}\langle\Gamma\rangle$,

$$\bar{h}_{2i}^{(s)} = [V_s^{\otimes 2i} \otimes N_{2i}]$$

goes to a direct sum of $(\dim V_s)^{2i}$ copies of N_{2i}, whereas $r\bar{h}_{2i}^{(t)}$ goes to copies of N_{2i}^{ass}, and $N_{2i} \not\cong N_{2i}^{\mathrm{ass}}$ from above (and both are irreducible). Thus

$$\dim \mathrm{HOM}_{\widetilde{\Sigma}_{2i}\langle\Gamma\rangle}(V_s^{\otimes 2i} \otimes N_{2i}, V_t^{\otimes 2i} \otimes N_{2i}^{\mathrm{ass}})$$
$$\leqq \dim \mathrm{HOM}_{\widetilde{\Sigma}_{2i}}(V_s^{\otimes 2i} \otimes N_{2i}|_{\widetilde{\Sigma}_{2i}}, V_t^{\otimes 2i} \otimes N_{2i}^{\mathrm{ass}}|_{\widetilde{\Sigma}_{2i}}) = 0.$$

To show

$$\langle \bar{h}_{2i}^{(s)}, \bar{h}_{2i}^{(t)} \rangle = \delta_{st},$$

we restrict instead to

$$\Gamma^{2i} \subset \widetilde{\Sigma}_{2i}\langle\Gamma\rangle.$$

Note that $V_s^{\otimes 2i} \otimes N_{2i}$, on restriction to Γ^{2i}, goes to $\dim N_{2i}$ copies of the irreducible representation $V_s^{\otimes 2i}$ of Γ^{2i}. The case $s \neq t$ now gives zero as required, since $V_s^{\otimes 2i} \not\cong V_t^{\otimes 2i}$ as representations of Γ^{2i}. For the case $s = t$, we get

$$\mathrm{HOM}_{\widetilde{\Sigma}_{2i}\langle\Gamma\rangle}[V_s^{\otimes 2i} \otimes N_{2i}, V_s^{\otimes 2i} \otimes N_{2i}\}$$
$$\cong \mathrm{HOM}_{\Gamma^{2i}}[V_s^{\otimes 2i}, \mathrm{HOM}_{\widetilde{\Sigma}_{2i}}(N_{2i}, V_s^{\otimes 2i} \otimes N_{2i}|_{\widetilde{\Sigma}_{2i}})] = Z \quad \text{(say)},$$

by adjointness. But $V_s^{\otimes 2i}$ is irreducible, and

$$\mathrm{HOM}_{\widetilde{\Sigma}_{2i}}(N_{2i}, V_s^{\otimes 2i} \otimes N_{2i}|_{\widetilde{\Sigma}_{2i}})$$

has dimension at most (in fact, equal to) $(\dim V_s)^{2i}$ since N_{2i} is irreducible. Thus Z has dimension at most one, as required.

The case general Γ with $n = 2i + 1$ proceeds in exactly analogous fashion, with HOM replaced by GHOM and ass by rev.

This completes the proof of 3.4.

8. Inner products and irreducibles. We now have the isomorphism $HS \cong \bar{H}$. We shall identify these groups with each other, and refer to the ring generators as $h_n^{(s)}$. Also abbreviate $D(HS) \cong D\bar{H}$ to D. It has basis $\{d_\phi\}$ and elements

$$b_\phi = 2^s d_\phi, \quad s = \left[\frac{|\phi| + l(\phi)}{2}\right].$$

In most of this section we work in D, and shall therefore from here until the proof of 8.5 write its multiplication (called $\circ$ previously) as juxtaposition.

The identity

$$\langle xy, z\rangle = \langle\langle x \otimes y, \Delta z\rangle\rangle$$

plus the formula

$$\langle b_n^{(s)}, b_n^{(t)}\rangle = 2^{n+1}\delta_{st}$$

allow one to calculate completely the inner product in D. In fact, when Γ is trivial, a calculation analogous to [**3**, 5.7] yields

$$\langle b_\alpha, b_\beta\rangle = 2^n \sum 2^{\#(\text{non-zero entries of } M)},$$

where the summation is over matrices M with non-negative integer entries, row sums α, and column sums β, and where $|\alpha| = |\beta| = n$. A calculation analogous to [**3**, 7.6] will yield a specific and even less enlightening formula for $\langle b_\phi, b_\theta\rangle$ in the case of general S. One then gets $\langle d_\phi, d_\theta\rangle$ by dividing by a power of 2. When combined with 8.6 below, one can compute inner products for the basis of $\bar{H}$ in 4.2. In principle, one then has

$$\mathrm{IRREP}^-(\tilde{\Sigma}_n\langle\Gamma\rangle) \text{ and } \mathrm{GIRREP}^-(\tilde{\Sigma}_n\langle\Gamma\rangle),$$

since they are characterized up to sign by being orthonormal bases. In practice, it turns out to be better to first determine the subset

$$\mathrm{IRREP}^-_{sa}(\tilde{\Sigma}_n\langle\Gamma\rangle)$$

of D consisting of those self-associates which are irreducible with respect to the monoid of self-associate representations. Its elements are those $c \in \mathrm{IRREP}^-$ for which $c^{\mathrm{ass}} = c$, plus the elements $d + d^{\mathrm{ass}}$ where

$$d^{\mathrm{ass}} \neq d \in \mathrm{IRREP}^-.$$

These elements are characterized up to sign as forming an orthogonal basis all of whose elements e satisfy

$$\langle e, e\rangle = \text{either 1 or 2.}$$

This is clear from the fact that for integers α_i and β_j we have:

$$\sum \alpha_i^2 + 2 \sum \beta_j^2 = 1 \text{ implies that all } \beta_j = 0$$

whereas

$$\alpha_i = 0 \text{ for all } i \text{ except one } i_0 \text{ for which } \alpha_{i_0} = 1;$$

and

$$\sum \alpha_i^2 + 2 \sum \beta_j^2 = 2 \text{ gives one } \beta = 1,$$

all other α and $\beta = 0$.

Now let us define certain elements $\hat{p}$, g and f of D (the last being in $D \otimes \mathbf{Q}$ until we have proved "integrality").

Definition 8.1. For $\psi \in \mathscr{P}^{\text{odd}}(n, S)$, let

$$\hat{p}_\psi = \prod_{s \in S} \prod_{2i+1 \in \psi s} \hat{p}_{2i+1}^{(s)} \quad \text{(defined before 4.8).}$$

For $\alpha \in \mathscr{D}$, let $b_\alpha^{(s)}$ denote that b_ϕ where ϕ maps s to α, all other t to the empty partition.

Define $g_\alpha^{(s)}$ inductively on $l(\alpha)$:

$$g_n^{(s)} = b_n^{(s)}; \text{ if } i > j,$$

$$g_{ij}^{(s)} = b_{ij}^{(s)} - 2b_{i+1,j-1}^{(s)} + 2b_{i+2,j-2}^{(s)} - \ldots + (-1)^j 2b_{i+j}^{(s)}.$$

If $m > 1$ is odd, define

$$g_{i_1,\ldots,i_m}^{(s)} = b_{i_1}^{(s)} g_{i_2,\ldots,i_m}^{(s)} - b_{i_2}^{(s)} g_{i_1,i_3,\ldots,i_m}^{(s)} + \ldots + b_{i_m}^{(s)} g_{i_1,\ldots,i_{m-1}}^{(s)}.$$

If $m > 2$ is even, define

$$g_{i_1,\ldots,i_m}^{(s)} = g_{i_1 i_2}^{(s)} g_{i_3,\ldots,i_m}^{(s)} - g_{i_1,i_3}^{(s)} g_{i_2,i_4,\ldots,i_m}^{(s)} + \ldots + g_{i_1 i_m}^{(s)} g_{i_2,\ldots,i_{m-1}}^{(s)}.$$

For $\phi \in \mathscr{D}(S)$, let

$$g_\phi = \prod_s g_{\phi(s)}^{(s)}.$$

Let

$$f_\phi = 2^{-\left\lceil \frac{|\phi| + l(\phi)}{2} \right\rceil} g_\phi,$$

specializing to $f_\alpha^{(s)}$ for that ϕ mapping s to α, all other t to the empty partition.

THEOREM 8.2. *With* $S = \operatorname{Con}\Gamma$, *we have*

$$\mathrm{IRREP}^{-}_{sa}(\tilde{\Sigma}_n\langle\Gamma\rangle) = \{f_\phi : \phi \in \mathscr{D}(n, S)\}.$$

The proof proceeds using three lemmas:

LEMMA 8.3. *An orthonormal basis for* $D_n \otimes \mathbf{R}$ *is given by*

$$\{(2^{n+l(\psi)}\zeta(\psi))^{-\frac{1}{2}}\hat{p}_\psi : \psi \in \mathscr{P}^{\mathrm{odd}}(n, S)\},$$

where $\zeta : \mathscr{P}^{\mathrm{odd}}(S) \to \mathbf{N}$ *is defined by*

$$\zeta(\psi) = \prod_s \zeta'(\psi(s))$$

with

$$\zeta'(1^{i_1}3^{i_3}\dots) = \prod_j j^{i_j}\cdot i_j!.$$

LEMMA 8.4. *In* $D \otimes D \otimes \mathbf{Q}$ *we have*

$$\sum_\psi \hat{p}_\psi \otimes \hat{p}_\psi/2^{l(\psi)}\zeta(\psi) = \sum_\phi g_\phi \otimes g_\phi/2^{l(\phi)},$$

summations over $\psi \in \mathscr{P}^{\mathrm{odd}}(S)$ *and* $\phi \in \mathscr{D}(S)$.

LEMMA 8.5. *For all* $\phi \in \mathscr{D}(S)$, *we have* $f_\phi \in D$. *Furthermore*

$$f_\phi \equiv d_\phi \bmod \operatorname{Span}_{\mathbf{Q}}\{d_\theta : \phi \prec \theta\},$$

where $\prec$ *is any linear order on* $\mathscr{D}(n, S)$ *which satisfies the partial order condition*

$$[(\forall s, \phi(s) \leqq \theta(s)) \text{ and } (\exists t, \phi(t) < \theta(t))] \Rightarrow \phi \prec \theta,$$

with $<$ *denoting lexicographic order.*

Note. It follows from the proof of 8.5 that we could replace d_θ by $2d_\theta$ except in the cases when $\phi \in \mathscr{D}''(S)$ and $\theta \in \mathscr{D}'(S)$, and replace $\operatorname{Span}_{\mathbf{Q}}$ by $\operatorname{Span}_{\mathbf{Z}}$.

The proof of 8.2 from these lemmas is straightforward. By 8.3, 8.4, and elementary multilinear algebra (namely:

$$\sum_1^m a_i \otimes a_i = \sum_1^m b_i \otimes b_i$$

for some orthonormal basis $\{a_i\}$ of an inner product space implies $\{b_i\}$ is also orthonormal), we see that

$$\{2^{-\frac{1}{2}(n+l(\phi))}g_\phi : \phi \in \mathscr{D}(n, S)\}$$

is also an orthonormal basis for $D \otimes \mathbf{R}$. A quick calculation then shows that $\{f_\phi : \phi \in \mathcal{D}(n, S)\}$, a subset of D by 8.5, is an orthogonal basis satisfying

$$\langle f_\phi, f_\phi \rangle = \begin{cases} 1 \text{ if } \phi\ \mathcal{D}''(n, S) \\ 2 \text{ if } \phi \in \mathcal{D}'(n, S), \end{cases}$$

as required. That f_ϕ, and not $(-f_\phi)$, is a representation is clear from the second part of 8.5.

Proof of 8.3. If neither x nor y is in D_0, then

$$\langle xy, \hat{p}^{(s)}_{2j+1} \rangle = 0$$

by the primitivity of $\hat{p}^{(s)}_{2j+1}$ and the formula

$$\langle xy, z \rangle = \ll x \otimes y, \triangle z \gg.$$

In particular, if $l(\mu) > 1$,

$$\langle \hat{p}^{(s)}_\mu, \hat{p}^{(s)}_{2j+1} \rangle = 0.$$

An easy induction on $l(\mu)$ yields

$$\triangle \hat{p}^{(s)}_\mu = \sum_{\alpha \cup \beta = \mu} \hat{p}^{(s)}_\alpha \otimes \hat{p}^{(s)}_\beta,$$

summation over ordered pairs (α, β) of partitions whose union is μ. Thus, if $\lambda = \alpha_0 \cup \beta_0$ for non-empty α_0 and β_0, we have

$$\langle \hat{p}^{(s)}_\lambda, \hat{p}^{(s)}_\mu \rangle = \ll \hat{p}^{(s)}_{\alpha_0} \otimes \hat{p}^{(s)}_{\beta_0}, \sum_{\alpha \cup \beta = \mu} \hat{p}^{(s)}_\alpha \otimes \hat{p}^{(s)}_\beta \gg$$

$$= \sum_{\alpha \cup \beta = \mu} \langle \hat{p}^{(s)}_{\alpha_0}, \hat{p}^{(s)}_\alpha \rangle \langle \hat{p}^{(s)}_{\beta_0}, \hat{p}^{(s)}_\beta \rangle.$$

This yields by induction on $|\lambda|$, that

(*) $\lambda \neq \mu$ implies $\langle \hat{p}^{(s)}_\lambda, \hat{p}^{(s)}_\mu \rangle = 0.$

If $\lambda = \mu$, we get

$$\langle \hat{p}^{(s)}_\lambda, \hat{p}^{(s)}_\lambda \rangle = \#\{\beta : \beta_0 = \beta \subset \lambda\} \cdot \langle \hat{p}^{(s)}_{\alpha_0}, \hat{p}^{(s)}_\alpha \rangle \langle \hat{p}^{(s)}_{\beta_0}, \hat{p}^{(s)}_\beta \rangle.$$

Since

$$\zeta'(\alpha_0 \cup \beta_0) = \#\{\beta : \beta_0 = \beta \subset \alpha_0 \cup \beta_0\} \cdot \zeta'(\alpha_0) \cdot \zeta'(\beta_0),$$

we find by induction $l(\lambda)$ that

$$\langle \hat{p}^{(s)}_\lambda, \hat{p}^{(s)}_\lambda \rangle = 2^{|\lambda| + l(\lambda)} \zeta'(\lambda).$$

The initial case of this induction is

$$\langle \hat{p}^{(s)}_{2j+1}, \hat{p}^{(s)}_{2j+1} \rangle = \langle \hat{p}^{(s)}_{2j+1}, (2j + 1) b^{(s)}_{2j+1} + \text{products} \rangle$$

$$= (2j + 1)2^{2j+2},$$

using a formula from the proof of 4.10. Finally, to do the general case,

$$\langle \hat{p}_\psi, \hat{p}_\theta \rangle = \left\langle \prod_s \hat{p}^{(s)}_{\psi(s)}, \prod_s \hat{p}^{(s)}_{\theta(s)} \right\rangle$$

$$= \left\langle \triangle^{\#S}\left(\prod_s \hat{p}^{(s)}_{\psi(s)}\right), \bigotimes_s \hat{p}^{(s)}_{\theta(s)} \right\rangle_{\#S},$$

where

$$\triangle^{\#S}:D \rightarrow D \otimes D \otimes \ldots \otimes D \quad (\#S \text{ copies})$$

is the appropriate iterate of $\triangle$, and $\langle\ ,\ \rangle_{\#S}$ is the induced inner product on $D \otimes \ldots \otimes D$; continuing,

$$= \left\langle \prod_s \triangle^{\#S}\hat{p}^{(s)}_{\psi(s)}, \bigotimes_s \hat{p}^{(s)}_{\theta(s)} \right\rangle_{\#S}$$

$$= \sum_{\alpha_{i1} \cup \ldots \cup \alpha_{ik} = \psi s_i} \left\langle \prod_{i=1}^{k} \hat{p}^{(s_i)}_{\alpha_{i1}} \otimes \ldots \otimes \hat{p}^{(s_i)}_{\alpha_{ik}}, \hat{p}^{(s_1)}_{\theta(s_1)} \otimes \ldots \otimes \hat{p}^{(s_k)}_{\theta(s_k)} \right\rangle_k$$

(where $\#S = k$, $S = \{s_1, \ldots, s_k\}$)

$$= \sum_{\alpha_{i1} \cup \ldots = \psi s_i} \langle \hat{p}^{(s_1)}_{\alpha_{11}} \ldots \hat{p}^{(s_k)}_{\alpha_{k1}} \otimes \hat{p}^{(s_1)}_{\alpha_{12}} \ldots \hat{p}^{(s_k)}_{\alpha_{k2}} \otimes$$

$$\ldots, \hat{p}^{(s_1)}_{\theta(s_1)} \otimes \ldots \otimes \hat{p}^{(s_k)}_{\theta(s_k)} \rangle_k.$$

The equations

$$\langle h_i^{(s)}, h_i^{(t)} \rangle = 0 = \langle h_i^{(s)}, rh_i^{(t)} \rangle$$

for $s \neq t$ imply easily that $D^{(s)}$ is orthogonal to the ideal generated by $\cup_{t \neq s} D^{(t)}$, where

$$D^{(s)} = \mathbf{Q}[b_i^{(s)}:i > 0] \subset D \otimes \mathbf{Q}.$$

Thus we get zero above as required, unless, for all i, $\alpha_{ii} = \psi s_i$ and α_{ij} is empty for $i \neq j$. In the latter case, we get

$$\prod_i \langle \hat{p}^{(s_i)}_{\alpha_{ii}}, \hat{p}^{(s_i)}_{\theta(s_i)} \rangle = \prod_i \langle \hat{p}^{(s_i)}_{\psi s_i}, \hat{p}^{(s_i)}_{\theta s_i} \rangle.$$

By (*), we get zero again, as required, unless $\psi s_i = \theta s_i$ for all i, that is, $\psi = \theta$. In the latter case, we get

$$\prod_s \langle \hat{p}^{(s)}_{\psi s}, \hat{p}^{(s)}_{\psi s} \rangle = \prod_s 2^{|\psi s| + l(\psi s)} \zeta'(\psi s)$$

$$= 2^{|\psi| + l(\psi)} \zeta(\psi),$$

as required.

Proof of 8.4. We shall violate the spirit of the rest of this paper and essentially quote from Schur [**9**] the special case of this identity when Γ is trivial, namely

$$\sum_{\lambda \in \mathscr{P}_n^{\mathrm{odd}}} 2^{-l(\lambda)} \zeta'(\lambda)^{-1} \hat{p}_\lambda^{(s)} \otimes \hat{p}_\lambda^{(s)} = \sum_{\alpha \in \mathscr{D}_n} 2^{-l(\alpha)} g_\alpha^{(s)} \otimes g_\alpha^{(s)}.$$

This maps under $F \otimes F$ to precisely the identity (92) from [**9**], where F is the isomorphism

$$D^{(s)} = \mathbf{Q}[b_1^{(s)}, b_2^{(s)}, \dots] \mapsto \Lambda \otimes \mathbf{Q}$$

given by

$$b_i^{(s)} \mapsto q_i(\underline{x}).$$

Here Λ is the ring of stable symmetric polynomials, and

$$q_i(\underline{x}) = \sum h_j(\underline{x}) e_{i-j}(\underline{x}),$$

where h_j (resp. e_j) is the j^{th} complete (resp. elementary) symmetric function. The reason that $F \otimes F$ maps our asserted identity to Schur's proven identity is that Schur's $\mathbf{Q}_\alpha(\underline{x})$ are defined in terms of $q_i(\underline{x})$ exactly as $g_\alpha^{(s)}$ were defined here in terms of $b_i^{(s)}$, and further, that the odd power sums are the same polynomials in $q_i(\underline{x})$ that $\hat{p}_{\mathrm{odd}}^{(s)}$ are in $b_i^{(s)}$. (Another proof of Schur's identity may be obtained by substituting $t = -1$ in a general identity for Hall-Littlewood polynomials given by MacDonald [**6**, III (4.1) and III (4.4)]).

To complete the proof for the case of general S:

$$\begin{aligned}
&\sum_{\psi \in \mathscr{P}^{\mathrm{odd}}(S)} 2^{-l(\psi)} \zeta(\psi)^{-1} \hat{p}_\psi \otimes \hat{p}_\psi \\
&= \sum_{\psi \in \mathscr{P}^{\mathrm{odd}}(S)} \prod_s \left[2^{-l(\psi s)} \zeta'(\psi s)^{-1} \left(\prod_{i \in \psi s} \hat{p}_i^{(s)} \otimes \hat{p}_i^{(s)} \right) \right] \\
&= \prod_s \sum_{\lambda \in \mathscr{P}^{\mathrm{odd}}} 2^{-l(\lambda)} \zeta'(\lambda)^{-1} \left(\prod_{i \in \lambda} \hat{p}_i^{(s)} \otimes \prod_{i \in \lambda} \hat{p}_i^{(s)} \right) \\
&= \prod_s \sum_{\alpha \in \mathscr{D}} 2^{-l(\alpha)} g_\alpha^{(s)} \otimes g_\alpha^{(s)}
\end{aligned}$$

(by the special case above)

$$= \sum_{\phi \in \mathscr{D}(S)} \prod_s 2^{-l(\phi s)} g_{\phi(s)}^{(s)} \otimes g_{\phi(s)}^{(s)} = \sum_{\phi \in \mathscr{D}(S)} 2^{-l(\phi)} g_\phi \otimes g_\phi.$$

Proof of 8.5. First we prove, by induction downwards on the lexicographic order of μ, that $\mu \in \mathscr{P}_n$ implies

https://doi.org/10.4153/CJM-1986-070-1 Published online by Cambridge University Press

$$b_\mu^{(s)} \in \mathrm{Span}_{\mathbf{Z}}\{2^{l(\mu)-l(\nu)}b_\nu^{(s)}:\nu \in \mathscr{D}_n, \nu \geqq \mu, l(\nu) \leqq l(\mu)\}.$$

This is vacuous for $\mu \in \mathscr{D}_n$, taking care of the initial case $\mu = (n)$ and of the inductive step unless μ has a repeated entry "i". In that case, writing $\mu = (i, i) \cup \alpha$ with $l(\alpha) = l(\mu) - 2$,

$$b_\mu^{(s)} = b_\alpha^{(s)}(b_i^{(s)})^2 = 2b_\alpha^{(s)}(b_{i+1}^{(s)}b_{i-1}^{(s)} - b_{i+2}^{(s)}b_{i-2}^{(s)} + \ldots \pm b_{2i}^{(s)}).$$

But now apply the inductive hypothesis to the partitions $(i + j, i - j) \cup \alpha$ and $(2i) \cup \alpha$ to get the required result.

Next proceed by induction on $l(\gamma)$ to prove

$$g_\gamma^{(s)} \equiv b_\gamma^{(s)} \bmod \mathrm{Span}_{\mathbf{Z}}\{2^{1+\left[\frac{l(\gamma)-l(\beta)}{2}\right]}b_\beta^{(s)}:$$
$$\beta \in \mathscr{D}_n, \beta > \gamma, l(\beta) \leqq l(\gamma)\}.$$

This is done by direct calculation with the definition of $g_\gamma^{(s)}$ and simple arguments about the integer part function $[\ \ ]$.

Now we obtain

$$(*) \qquad f_\gamma^{(s)} = d_\gamma^{(s)} + \sum_{\substack{\beta>\gamma \\ l(\beta)\leqq l(\gamma)}} \xi_{\gamma,\beta}d_\beta^{(s)}$$

where

$$\xi_{\gamma,\beta} \in \begin{cases} 2\mathbf{Z} & \text{if } \beta \in \mathscr{D}_n' \text{ or } \gamma \in \mathscr{D}_n'' \\ \mathbf{Z} & \text{otherwise.} \end{cases}$$

This is obtained by dividing by $2^{\left[\frac{n+l(\gamma)}{2}\right]}$ the relation expressing $g_\gamma^{(s)}$ in terms of $b_\gamma^{(s)}$. The condition $\xi_{\gamma,\beta} \in 2\mathbf{Z}$ follows in those cases because

$$1 + \left[\frac{l(\gamma) - l(\beta)}{2}\right] + \left[\frac{n + l(\beta)}{2}\right] - \left[\frac{n + l(\gamma)}{2}\right]$$

is positive there. Furthermore, noting that

$$d_\beta^{(s)} \in p^2\bar{H}_{1,n} \quad \text{for } \beta \in \mathscr{D}_n',$$

we obtain

$$(**) \qquad f_\gamma^{(s)} \in p^2\bar{H}_{1,n} + 2D \quad \text{for } \gamma \in \mathscr{D}_n'.$$

Now

$$(***) \qquad f_\phi = 2^{\sum_s \left[\frac{|\phi(s)|+l(\phi(s))}{2}\right] - \left[\frac{|\phi|+l(\phi)}{2}\right]} \prod_s f_{\phi(s)}^{(s)}.$$

If we multiply relations (*) together as γ ranges over $\phi(s)$ for $s \in S$ we then obtain the second part of 8.5. The first part follows since $f_{\phi(s)}^{(s)} \in D$ and since $\prod_s f_{\phi(s)}^{(s)}$ is divisible in D by the reciprocal of the power of 2 in (***). To prove the last point, let $S = \{s_1, s_2, \dots\}$ with $\phi(s_i) = \gamma_i$, $|\gamma_i| = n_i$, $l(\gamma_i) = l_i$. We must show $\prod_i f_{\gamma_i}^{(s_i)}$ is divisible by

$$2^{\left[\frac{\Sigma n_i + l_i}{2}\right] - \Sigma\left[\frac{n_i + l_i}{2}\right]}$$

in D. If all $n_i + l_i$ are even there is nothing to prove. Let N be the number of $n_i + l_i$ which are odd and suppose $N > 0$. The previous power of 2 is $2^{[N/2]}$. But $n_i + l_i$ odd is equivalent to $\gamma_i \in \mathscr{D}'$ so apply (**). It remains to show that for $I + J = N$, $I \geqq 0$, $J \geqq 0$ and elements $x_i \in \bar{H}_{1,*}$ and $y_i \in D$, we have

$$(p^2 x_1) \circ (p^2 x_2) \circ \dots \circ (p^2 x_I) \circ (2y_1) \circ (2y_2) \circ \dots \circ (2y_J)$$

is divisible by $2^{[N/2]}$ in D. Here the product $\circ$ is that in D, reintroducing the old notation, whereas $p^2 x$ is product in $\bar{H}$. But

$$(p^2 x') \circ (p^2 x'') = p^3 x' x'' = 2px'x''.$$

Iterating we get the required power of 2.

This completes the proof of 8.5.

In order to pass from $\mathrm{IRREP}_{sa}^-(\tilde{\Sigma}_n\langle\Gamma\rangle)$ to IRREP^- and GIRREP^-, we need one more lemma.

LEMMA 8.6. *If ϕ and θ are distinct elements of $\mathscr{D}(S)$, then*

$$\langle h_\phi, rh_\theta\rangle = \langle h_\phi, h_\theta\rangle.$$

Proof. By definition of products in $\bar{H}$ and reciprocity,

$$\left\langle \prod_n h_{i_n}^{(s_n)}, \prod_m h_{j_m}^{(t_m)} \right\rangle$$

has the form

$$\langle x, \phi_{i_1,i_2,\dots}^* \phi_{j_1,j_2,\dots *} y\rangle = \langle x, z\rangle$$

(say), where y is a product using $\boxtimes_1$, $\boxtimes_2$, $\boxed{\times\times}$ of

$$h_{j_1}^{(t_1)}, h_{j_2}^{(t_2)}, \dots$$

(and similarly x). To show

$$\langle x, z^\rho\rangle = \langle x, z\rangle,$$

where ρ is ass or rev as appropriate, apply Mackey's theorem to z. Unless $i_1 = j_1$, $i_2 = j_2, \dots$, we'll have a factor

$$\phi_{k_1,k_2}^* \dots (h_k^{(t)})$$

with more than one k_i. By 7.1 this is invariant under ρ. In the case $i_1 = j_1$, etc., the same argument applies to all terms except one, namely $\langle x, y\rangle$. But this term is zero since $t_i \neq s_i$ for at least one i, using iterations of 2.30.

THEOREM 8.7. *If we list the basis from* 4.2 *for* $\mathrm{R}^-(\tilde{\Sigma}_n\langle\Gamma\rangle)$ *with subscripts* ϕ *in non-increasing order with respect to* $\prec$, *and with* h_ϕ *and* rh_ϕ *adjacent in either order for* $\phi \in \mathscr{D}'(n, S)$, *then the Gram-Schmidt process will produce* $\mathrm{IRREP}^-(\tilde{\Sigma}_n\langle\Gamma\rangle)$. *The same is true for* GR^-, *except* h_ϕ *and* rh_ϕ *are adjacent for* $\phi \in \mathscr{D}''(n, S)$, *and Gram-Schmidt produces* $\mathrm{GIRREP}^-(\tilde{\Sigma}_n\langle\Gamma\rangle)$. *Specific formulae are as follows*:

$$\mathrm{GIRREP}^-(\tilde{\Sigma}_n\langle\Gamma\rangle) = \{c_\phi, rc_\phi : \phi \in \mathscr{D}''(n, S)\} \cup \{pc_\phi : \phi \in \mathscr{D}'(n, S)\}$$

$$\mathrm{IRREP}^-(\tilde{\Sigma}_n\langle\Gamma\rangle) = \{pc_\phi : \phi \in \mathscr{D}''(n, S)\} \cup \{c_\phi, rc_\phi : \phi \in \mathscr{D}'(n, S)\}$$

where, if $\phi \in \mathscr{D}''(n, S)$ *and*

$$f_\phi = d_\phi + \sum_{\phi \prec \theta \in \mathscr{D}''(n,S)} 2\beta_{\phi\theta}d_\theta + \sum_{\phi \prec \theta \in \mathscr{D}'(n,S)} \beta_{\phi\theta}d_\theta,$$

then

$$c_\phi = h_\phi + \sum_{\phi \prec \theta \in \mathscr{D}''(n,S)} \beta_{\phi\theta}(h_\theta + rh_\theta) + \sum_{\phi \prec \theta \in \mathscr{D}'(n,S)} \beta_{\phi\theta}ph_\theta;$$

whereas, if $\phi \in \mathscr{D}'(n, S)$ *and*

$$f_\phi = d_\phi + \sum_{\phi \prec \theta \in \mathscr{D}'(n,S)} 2\beta_{\phi\theta}d_\theta + \sum_{\phi \prec \theta \in \mathscr{D}''(n,S)} 2\beta_{\phi\theta}d_\theta,$$

then

$$c_\phi = h_\phi + \sum_{\phi \prec \theta \in \mathscr{D}'(n,S)} \beta_{\phi\theta}(h_\theta + rh_\theta) + \sum_{\phi \prec \theta \in \mathscr{D}''(n,S)} \beta_{\phi\theta}ph_\theta.$$

Note that the $\beta_{\phi\theta}$ are easily computed from the inductive definition for g_ϕ.

Proof. Let c_ϕ be the irreducible for which

$$f_\phi = \begin{cases} pc_\phi & \phi \in \mathscr{D}''(n, S) \\ p^2c_\phi & \phi \in \mathscr{D}'(n, S). \end{cases}$$

By 8.5 and 8.6, $c_\phi - h_\phi$ is a linear combination of those ph_θ and p^2h_θ for which $\phi \prec \theta$. But the given linear combinations are the only ones compatible with the above relations between f_ϕ and c_ϕ.

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