A CLASS OF HOMOMORPHISM THEORIES FOR GROUPOIDS

by GERALD LOSEY †

(Received 25th March 1963)

1. Introduction

In a well-behaved homomorphism theory for a class $\mathfrak{G}$ of algebraic systems certain " closed objects " relative to a given $G \in \mathfrak{G}$ are distinguished which act as kernels of homomorphisms. For example, if $\mathfrak{G}$ is the class of groups then the closed objects relative to a given group G are the normal subgroups of G; if $\mathfrak{G}$ is the class of semigroups with zero element then one can devise a homomorphism theory in which the closed objects relative to a given $S \in \mathfrak{G}$ are the ideals of S[cf. Rees (3)]; in the class of groupoids one may define the closed objects relative to a given groupoid G to be the congruence relations on G, that is, subsets $\pi \subseteq G \times G$ which are equivalence relations having the property that $(x_1y_1, x_2y_2) \in \pi$ whenever $(x_1, x_2), (y_1, y_2) \in \pi$. Given such a closed object N relative to G there exists a " factor " system G/N and a (canonical) homomorphism $\eta: G \to G/N$ characterised by the property: If $\sigma: G \to H$ is a homomorphism with kernel N then there is a unique homomorphism $\bar{\sigma}: G/N \to H$ such that $\bar{\sigma} . \eta = \sigma$ and the kernel of $\bar{\sigma}$ is trivial in the sense that the kernel of $\bar{\sigma}$ is the unique smallest closed object relative to G/N.

In order to construct a unified homomorphism theory for groupoids which will include the examples of the preceding paragraph it is desirable that the theory include a parameter whose specification reduces the general theory to a particular one. In the theory to be constructed in this paper the parameter will be a polynomial $P(X) \in Z[X_1, ..., X_n]$, Z being the ring of integers.

In Section **2** we shall describe a homomorphism theory for groupoids in which the closed objects relative to a given groupoid G are the ideals of the groupoid ring ZG. (This theory is essentially equivalent to that in which the closed objects are the congruence relations on G.) In Section **3** this theory will be used in defining the P-theory, $P = P(X)$ a polynomial. In Section **4** we shall show how the specification of P gives each of the theories mentioned above.

2. Groupoid rings and the r-theory

In this section we shall construct a homomorphism theory for groupoids which we shall call the r-theory (r for ring).

† This research was supported in part by the National Science Foundation under contract NSF GP-2273.

Let G be a groupoid and Z the ring of integers. Denote by ZG the set of all formal sums $\Sigma_{g \in G}\alpha(g)g$ where $\alpha(g) \in Z$ and $\alpha(g) = 0$ but for finitely many $g \in G$. Two sums are equal if and only if they are identical. Addition and multiplication in ZG are defined in the natural manner and the resulting (not necessarily associative) ring is called the *groupoid ring* of G. The mapping $g \to \Sigma\alpha(h)h$, $\alpha(g) = 1$, $\alpha(h) = 0$ if $h \neq g$, is an isomorphism of G into the multiplicative structure of ZG. We identify g with its image under this isomorphism.†

If ϕ: $G \to H$ is a (groupoid) homomorphism then ϕ induces by linearity a ring homomorphism ϕ^*: $ZG \to ZH$; that is,

$$\phi^*(\Sigma_{g \in G}\alpha(g)g) = \Sigma_{g \in G}\alpha(g)\phi(g) = \Sigma_{h \in H}(\Sigma_{\phi(g) = h}\alpha(g))h.$$

Denote the kernel of ϕ^* by Λ_ϕ; we shall call Λ_ϕ the *r-kernel* of ϕ. For each $h_i \in \phi(G)$ choose $g_i \in G$ such that $\phi(g_i) = h_i$ and set $X_\phi = \{g_i\}_{i \in I}$. Let

$$B(X_\phi) = \{g - g_i\colon g_i \in X_\phi,\ \phi(g) = \phi(g_i),\ g \neq g_i\}.$$

Then we have

Theorem 2.1. *The set $B(X_\phi)$ is a Z-basis for Λ_ϕ.*

Proof. The Z-independence of $B(X_\phi)$ is an immediate consequence of the Z-independence of the elements of G.

Let M be the Z-module generated by $B(X_\phi)$. Then, clearly, $M \subseteq \Lambda_\phi$. Let $\alpha = \Sigma\alpha(g)g \in \Lambda_\phi$. Then

$$\begin{aligned} 0 = \phi^*(\alpha) = \Sigma\alpha(g)\phi(g) &= \Sigma_{i \in I}\Sigma_{\phi(g) = \phi(g_i)}\alpha(g)\phi(g_i) \\ &= \Sigma_{i \in I}(\Sigma_{\phi(g) = \phi(g_i)}\alpha(g))h_i. \end{aligned}$$

Hence, $\Sigma_{\phi(g) = \phi(g_i)}\alpha(g) = 0$ for all $i \in I$ and so

$$\Sigma_{\phi(g) = \phi(g_i)}\alpha(g)g = \Sigma_{\phi(g) = \phi(g_i)}\alpha(g)(g - g_i).$$

Therefore, $\alpha = \Sigma_{i \in I}\Sigma_{\phi(g) = \phi(g_i)}\alpha(g)(g - g_i) \in M$. Consequently, $\Lambda_\phi \subseteq M$. This completes the proof of 2.1.

Let Λ be any ideal of ZG and let η: $ZG \to ZG/\Lambda$ be the canonical homomorphism. Denote by $\bar{\eta}$ the restriction of η to G. Then $\bar{\eta}$: $G \to ZG/\Lambda$ is a homomorphism of G into the multiplicative structure of ZG/Λ. Denote the image of G under $\bar{\eta}$ by $G/\Lambda = \{g + \Lambda\colon g \in G\}$. Thus given any ideal Λ of ZG we construct the "factor" groupoid G/Λ of G by Λ. The map $\bar{\eta}$: $G \to G/\Lambda$ defined by $g \to g + \Lambda$ is called the canonical homomorphism. We will generally write η for $\bar{\eta}$ if there is no danger of confusion.

Theorem 2.2. *Let ϕ: $G \to H$ be a homomorphism of groupoids and let Λ_ϕ be the r-kernel of ϕ. Let η: $G \to G/\Lambda_\phi$ be the canonical homomorphism. Then there is a unique isomorphism $\bar{\phi}$: $G/\Lambda_\phi \to H$ such that $\bar{\phi}\eta = \phi$.*

† If G has a zero element O' then, using the above definition of ZG, O' is not identified with the zero element $\Sigma_{g \in G}\, 0 \cdot g$ of ZG. If it is desired to make this identification then one must reduce ZG modulo the ideal $\theta = \{\Sigma\alpha(g)g\colon \alpha(g) = 0 \text{ for all } g \neq 0'\}$.

https://doi.org/10.1017/S0013091500025864 Published online by Cambridge University Press

Proof. Define $\bar{\phi}(g+\Lambda_\phi) = \phi(g)$. If $g, g' \in G$ then $\phi(g) = \phi(g')$ if and only if $g-g' \in \Lambda_\phi$, that is, if and only if $g+\Lambda_\phi = g'+\Lambda_\phi$. Thus $\bar{\phi}$ is both well-defined and one-one. It is easily verified to be a homomorphism. Moreover, $\bar{\phi}\eta(g) = \bar{\phi}(g+\Lambda_\phi) = \phi(g)$. The uniqueness of $\bar{\phi}$ follows from this last relation.

The following isomorphism theorems are also easily proved.

Theorem 2.3. *Let ϕ: $G \to G'$ be a homomorphism of G onto G'. If Λ' is an ideal of ZG' then $\Lambda = \phi^{*-1}(\Lambda')$ is an ideal of ZG containing Λ_ϕ and there is a unique isomorphism $\bar{\phi}$: $G/\Lambda \to G'/\Lambda'$ such that $\eta'\phi = \bar{\phi}\eta$, where η: $G \to G/\Lambda$ and η': $G' \to G'/\Lambda'$ are the canonical homomorphisms.*

Theorem 2.4. *Let ϕ: $G \to H$ be a homomorphism with r-kernel Λ_ϕ. Let K be a subgroupoid of G and τ: $K \to H$ the restriction of ϕ to K. Then $\Lambda_\tau = \Lambda_\phi \cap ZK$ and, thus, there is a unique isomorphism*

$$\bar{\tau}: K/\Lambda_\phi \cap ZK \to H$$

such that $\bar{\tau}\eta = \tau$, where η: $K \to K/\Lambda_\phi \cap ZK$ is the canonical homomorphism.

3. The P-theory.

A collection $\mathfrak{G}$ of groupoids will be said to be *complete* if for every $G \in \mathfrak{G}$, $\mathfrak{G}$ also contains all homomorphic images of G. For example, the collection of all semigroups with zero is complete but the collection of all loops is not [cf. (**1**)].

Let $X_1, \ldots, X_n$ be a finite set of non-associative, non-commuting indeterminates and let $P(X) \in Z[X_1, \ldots, X_n]$. The polynomial P is compatible with the complete collection of groupoids $\mathfrak{G}$ if for every element $(k) = (k_1, \ldots, k_n)$ in $G^n = G \times \ldots \times G$, $G \in \mathfrak{G}$, $P(k_1, \ldots, k_n) \in ZG$. If P has zero constant term then P is compatible with all collections $\mathfrak{G}$; if P has a non-zero constant term then P is compatible with $\mathfrak{G}$ if and only if every groupoid $G \in \mathfrak{G}$ has an identity element 1_G in which case $P(k) = a_0 1_G + a_1 k_1 + \ldots$.

Throughout the remainder of this section $\mathfrak{G}$ will be a fixed complete collection of groupoids and P a fixed polynomial compatible with $\mathfrak{G}$.

Let $G \in \mathfrak{G}$ and let K be a non-empty subset of G^n. We define $\Delta_P(K)$ to be the ideal of ZG generated by the elements $P(k)$, $(k) \in K$. We define

$$Cl_P(K) = \{(g) \in G^n : P(g) \in \Delta_P(K)\}.$$

If K is the empty set $\varnothing$ then we set

$$Cl_P(\varnothing) = I_P(G) = \{(g) \in G^n : P(g) = 0\}.$$

A subset K of G^n is said to be P-closed if $Cl_P(K) = K$. Note that $I_P(G)$ is the unique minimal P-closed subset of G^n; $I_P(G)$ may be empty. We call $I_P(G)$ the *trivial* closed subset of G^n.

Theorem 3.1. *The operator Cl_P satisfies the following:*

(a) *$K \subseteq Cl_P(K)$;*

(b) *For any $K \subseteq G^n$, $Cl_P(K)$ is P-closed;*

https://doi.org/10.1017/S0013091500025864 Published online by Cambridge University Press

(c) *If* $\{K_i\}_{i\in I}$ *is a collection of P-closed subsets of* G^n *then* $K = \bigcap_{i\in I}K_i$ *is P-closed;*

(d) $I_P(G)$ *and* G^n *are P-closed.*

Proof. (a) and (d) follow trivially from the definition. Also from the definition we have $P(g)\in\Delta_P(K)$ for all $(g)\in Cl_P(K)$. Hence, $\Delta_P(Cl_P(K)) = \Delta_P(K)$ and so $Cl_P(Cl_P(K)) = Cl_P(K)$.

Let $\{K_i\}_{i\in I}$ be a collection of P-closed sets. Then $(g)\in Cl_P(\bigcap K_i)$ only if $P(g)\in\Delta_P(\bigcap K_i)\subseteq\bigcap\Delta_P(K_i)$. Thus $P(g)\in\Delta_P(K_i)$ for all i and hence, $(g)\in Cl_P(K_i) = K_i$ for all i. Thus, $(g)\in\bigcap K_i$. Therefore, $Cl_P(\bigcap K_i)\subseteq\cap K_i$. Now (c) follows from (a).

Let G, $H\in\mathfrak{G}$ and let $\sigma: G\to H$ be a homomorphism. We extend σ to a mapping $\tilde{\sigma}: G^n\to H^n$ componentwise, that is, $\tilde{\sigma}(g_1, \ldots, g_n) = (\sigma(g_1), \ldots, \sigma(g_n))$.

Lemma 3.2. *If* $\sigma: G\to H$ *is a homomorphism then*

$$\sigma^*(\Delta_P(K)) = \Delta_P(\tilde{\sigma}(K))$$

for all subsets $K\subseteq G^n$.

Proof. $\Delta_P(K)$ is generated by the elements $P(k)$, $(k)\in K$, and so $\sigma^*(\Delta_P(K))$ is generated by the elements $\sigma^*P(k) = P(\tilde{\sigma}(k))\in\Delta_P(\sigma(K))$. Hence

$$\sigma^*(\Delta_P(K))\subseteq\Delta_P(\tilde{\sigma}(K)).$$

On the other hand, $\Delta_P(\tilde{\sigma}(K))$ is generated by the elements $P(\tilde{\sigma}(k)) = \sigma^*P(k)$. Therefore $\Delta_P(\tilde{\sigma}(K))\subseteq\sigma^*(\Delta_P(K))$.

Using this lemma we can show that all homomorphisms are " continuous " in the following sense:

Theorem 3.3. *Let* $\sigma: G\to H$ *be a homomorphism and let* K' *be a P-closed subset of H. Then* $K = \tilde{\sigma}^{-1}(K')$ *is a P-closed subset of G.*

Proof. If $(g)\in Cl_P(K)$ then $P(g)\in\Delta_P(K)\subseteq\sigma^{*-1}(\Delta_P(K'))$, by Lemma 4.2. Therefore, $\sigma^*P(g) = P(\tilde{\sigma}(g))\in\Delta_P(K')$ and so $\sigma(g)\in Cl_P(K') = K'$ or

$$(g)\in\tilde{\sigma}^{-1}(K') = K.$$

Let $\sigma: G\to H$ be a homomorphism. Define the *P-kernel* of σ to be the set $N = \{(x)\in G^n: \tilde{\sigma}(x)\in I_P(G')\}$. Since $N = \tilde{\sigma}^{-1}(I_P(G'))$ is the inverse image of a P-closed set, N is a P-closed set of G^n. The homomorphism σ is said to be a *P-isomorphism* if the P-kernel of σ is $I_P(G)$.

If N is any P-closed subset of G^n then we define the *P-factor-groupoid* $(G/N)_P = G/\Delta_P(N)$. The canonical homomorphism $\eta: G\to(G/N)_P$ is the homomorphism $g\to g+\Delta_P(N)$ as defined in Section **2**.

Theorem 3.4. *The P-kernel of the natural homomorphism* $\eta: G\to(G/N)_P$ *is* N. *Moreover,*

$$I_P((G/N)_P) = \{(\bar{g}_1, \ldots, \bar{g}_n): \bar{g}_i = g_i+\Delta_P(N), (g_i)\in N\}.$$

Proof. $I((_PG/N)_P)$ is the set of all $(\bar{g}_1, \ldots, \bar{g}_n)$, $\bar{g}_i = g_i+\Delta_P(N)$, such that $P(\bar{g}_1, \ldots, \bar{g}_n) = P(g_1, \ldots, g_n)+\Delta_P(N) = \Delta_P(N)$. Thus, $(g)\in G^n$ is in the P-kernel of η if and only if $P(g)\in\Delta_P(N)$, that is, if and only if $(g)\in N$.

Theorem 3.5. *Let $\sigma: G \to H$ be a homomorphism with P-kernel N. Then there exists a unique P-isomorphism $\bar{\sigma}: (G/N)_P \to H$ such that $\bar{\sigma}\eta = \sigma$, where $\eta: G \to (G/N)_P$ is the canonical homomorphism.*

Proof. Define $\bar{\sigma}: (G/N)_P \to H$ by $g + \Delta_P(N) \to \sigma(g)$. If $g + \Delta_P(N) = g' + \Delta_P(N)$ then $g - g' \in \Delta_P(N)$ and so

$$\sigma(g) - \sigma(g') = \sigma(g - g') \in \sigma^*(\Delta_P(N)) = \Delta_P(\tilde{\sigma}(N)) = \Delta_P(I_P(H)) = \{0\}$$

since $P(h) = 0$ for all $(h) \in I_P(H)$. Therefore, $\sigma(g) = \sigma(g')$ and so $\bar{\sigma}$ is well defined. It is clearly a homomorphism. An element $(\bar{g}) \in (G/N)_P^n$ belongs to the P-kernel of $\bar{\sigma}$ if and only if

$$(\bar{\sigma}(\bar{g}_1), \dots, \bar{\sigma}(\bar{g}_n)) = (\sigma(g_1), \dots, \sigma(g_n)) = \sigma(g_1, \dots, g_n) \in I_P(H),$$

that is, if and only if $(g) \in N$. Thus, by 4.4, $(\bar{g})$ is in the kernel of $\bar{\sigma}$ if and only if $(\bar{g}) \in I_P((G/N)_P)$. Hence $\bar{\sigma}$ is a P-isomorphism. Moreover

$$\bar{\sigma}\eta(g) = \bar{\sigma}(g + \Delta_P(N)) = \sigma(g).$$

The uniqueness of $\bar{\sigma}$ follows from this relation.

The property of theorem 3.5 characterises the P-factor-group up to isomorphism.

Theorem 3.6. *Let G be a groupoid and N a P-closed subset of G^n. Let K be a groupoid and $\mu: G \to K$ a homomorphism with P-kernel N. Suppose that for every homomorphism $\phi: G \to H$ with P-kernel N there exists a P-isomorphism ϕ': $K \to H$ such that $\phi'\mu = \phi$. Then K and $(G/N)_P$ are isomorphic.*

Proof. By 3.5 there exists a homomorphism $\bar{\mu}: (G/N)_P \to K$ such that $\bar{\mu}\eta = \mu$. By the hypothesis, there exists a homomorphism $\eta': K \to (G/N)_P$ such that $\eta'\mu = \eta$. Hence $(\bar{\mu}\eta')\mu = \mu$ and $(\eta'\bar{\mu})\eta = \eta$. Thus $\bar{\mu}\eta'$ is the identity mapping on K and $\eta'\bar{\mu}$ is the identity mapping on $(G/N)_P$. Therefore $\bar{\mu}$ and η' are inverse isomorphisms.

The following " isomorphism " theorems can be proved in a manner similar to that of 3.5.

Theorem 3.7. *Let $\sigma: G \to G'$ be a homomorphism of G onto G' with P-kernel N. Then there is a one-one correspondence between the P-closed subsets H' of G'^n and the P-closed subsets H of G^n containing N given by $H = \tilde{\sigma}^{-1}(H')$. If H and H' are corresponding P-closed subsets then there exists a unique P-isomorphism $\bar{\sigma}: (G/H)_P \to (G'/H')_P$ such that $\eta'\sigma = \bar{\sigma}\eta$, where $\eta: G \to (G/H)_P$ and $\eta': G' \to (G'/H')_P$ are the canonical homomorphisms.*

Theorem 3.8. *Let $\sigma: G \to H$ be a homomorphism with P-kernel N. Let K be a subgroupoid of G and let $\tau: K \to H$ be the restriction of σ to K. Then the P-kernel of τ is $N \cap K^n$ and, thus, there exists a unique P-isomorphism*

$$\bar{\tau}: (K/N \cap K^n)_P \to H$$

such that $\bar{\tau}\eta = \tau$, where $\eta: K \to (K/N \cap K^n)_P$ is the canonical homomorphism.

4. Examples

I. Let $\mathfrak{G}$ be the collection of all groups and let $P(X) = X-1$. Then the P-closed subsets of a group G will be subsets of $G^1 = G$. We claim that the P-closed subsets of G are precisely the normal subgroups of G.

Suppose $H \subseteq G$ and H is P-closed. Then $x, y \in H$ implies that $x-1$, $y-1 \in \Delta_P(H)$ and thus that $x-1$, $-y^{-1}(y-1) = y^{-1}-1 \in \Delta_P(H)$. Therefore, $xy^{-1}-1 = (x-1)(y^{-1}-1)+(x-1)+(y^{-1}-1) \in \Delta_P(H)$ and so $xy^{-1} \in H$. Thus H is a subgroup of G. Moreover, for any $x \in H$, $z \in G$ we have

$$z^{-1}xz-1 = z^{-1}(x-1)z \in \Delta_P(H).$$

Therefore, H is a normal subgroup of G. Conversely, suppose H is a normal subgroup of G and $G = G/H$. Then the kernel of the homomorphism $ZG \to Z\bar{G}$ defined by $\Sigma\alpha(g)g \to \Sigma\alpha(g)\bar{g}$, $\bar{g} = gH$, is precisely $\Delta_P(H)$. [This follows from 2.2; see also Jennings, (**2**).] Therefore, if $g \in Cl_P(H)$ then $g-1 \in \Delta_P(H)$ and so $\bar{g} = \bar{1}$, that is $g \in H$. Hence $H = Cl_P(H)$.

The trivial P-closed subset of G is $\{1\}$ and it is easy to see that every P-isomorphism is an isomorphism. By 3.6, $(G/H)_P \cong G/H$ and this isomorphism is given by $g+\Delta_P(H) \leftrightarrow gH$.

II. Let $\mathscr{S}$ be the collection of all semigroups with 0 and let $Q(X) = X$. The Q-closed sets are subsets of $S^1 = S$. We claim that the Q-closed subsets of S are precisely the ideals of S.

Suppose J is a Q-closed subset of S. Then $x \in J$ if and only if $x \in \Delta_Q(J)$. Thus if $x \in J$ and $y \in S$ then both yx and xy belong to $\Delta_Q(J)$ and, hence, to J. Therefore, J is an ideal of S. Conversely, suppose J is an ideal of S. Then it is clear that $\Delta_Q(J)$ is the set of all finite sums $\Sigma_{x \in J} n(x)x$ where $n(x) \in Z$. If $y \in Cl_Q(J)$ then $y \in S$ and $y = \Sigma_{x \in J} n(x)x$. By the linear independence of the elements of S in ZS, this implies $y = x$ for some $x \in J$. Consequently, $Cl_Q(J) = J$ and J is Q-closed.

Let S be a semigroup with $\bar{0}$ and J an ideal of S. Then

$$(S/J)_Q = \{x+\Delta_Q(J) : x \in S\}.$$

Let $\bar{x} = x+\Delta_Q(J)$. Then $\bar{x} = \bar{0}$ if and only if $x \in \Delta_Q(J)$, that is, if and only if $x \in J$. If $x, y \in S$ then $\bar{x} = \bar{y}$ if and only if $x-y \in \Delta_Q(J)$. But this can happen if and only if $x = y$ or x and $y \in J$, that is, $\bar{x} = \bar{y}$ implies $x = y$ or $\bar{x} = \bar{y} = \bar{0}$. Thus, under the canonical homomorphism, the elements of J are mapped onto $\bar{0}$ and the elements $x \in S-J$ are mapped onto distinct elements $\bar{x} = \bar{0}$. This is precisely the definition of the Rees quotient of S by J [cf. Rees (**3**)].

III. Let $\mathscr{M}$ be the class of all groupoids and $R(X, Y) = X-Y$. The R-closed sets will be subsets of $G \times G$, $G \in \mathscr{M}$. The trivial R-closed subset will be the diagonal: $I_R(G) = \{(g, g) : g \in G\}$. We claim that the R-closed subsets of $G \times G$ are precisely the congruence relations in $G \times G$.

Suppose $\pi \subseteq G \times G$ is R-closed. Clearly $(x, x) \in \pi$ for all $x \in G$ since $I_R(G)$ is contained in all R-closed sets. If $(x, y) \in \pi$ then $R(y, x) = -R(x, y) \in \Delta_R(\pi)$ and so $(y, x) \in \pi$. If (x, y), $(y, z) \in \pi$ then

$$(x-y)+(y-z) = (x-z) = R(x, z) \in \Delta_R(\pi)$$

and so $(x, z) \in \pi$. Thus π is an equivalence relation. Let $(x_1, y_1), (x_2, y_2) \in \pi$. Then $(x_1 - y_1)x_2 + y_1(x_2 - y_2) = x_1x_2 - y_1y_2 \in \Delta_R(\pi)$ and so $(x_1x_2, y_1y_2) \in \pi$. Therefore π is a congruence relation in $G \times G$.

Conversely, let π be a congruence relation in $G \times G$. Let G′ be the groupoid of congruence classes, $G' = G/\pi$ and let $\tilde{\pi}: G \to G'$ be the natural mapping, that is, $\tilde{\pi}(g)$ is the congruence class containing g. Then $\tilde{\pi}^*: ZG \to ZG'$ has kernel $\Lambda_{\tilde{\pi}}$ which is spanned over Z by the elements $x_i - y_i$, $(x_i, y_i) \in \pi$. Since $\Delta_R(\pi)$ is spanned over ZG by the elements $x_i - y_i$, $(x_i, y_i) \in \pi$, it follows that

$$\Delta_R(\pi) = \Lambda_{\tilde{\pi}}.$$

Hence every element of $\Delta_R(\pi)$ can be expressed in the form $\Sigma n_i(x_i - y_i)$ where $(x_i, y_i) \in \pi$, $n_i \in Z$. Since $(x_i, y_i) \in \pi$ implies $(y_i, x_i) \in \pi$ we can further assume that every element of $\Delta_R(\pi)$ can be written $\Sigma(x_i - y_i)$ with $(x_i, y_i) \in \pi$ (with repetitions allowed). Suppose that in such an expression $y_i = x_j$ for some i and j; then we can replace $(x_i - y_i) + (x_j - y_j)$ by $x_i - y_j$ since $(x_i, y_i), (x_j, y_j) \in \pi$ implies $(x_i, y_j) \in \pi$. Thus every element of $\Delta_R(\pi)$ can be written $\Sigma(x_i - y_i)$ where $(x_i, y_i) \in \pi$ and no $y_i = x_j$. Now suppose $(x, y) \in Cl_R(\pi)$; then

$$x - y = \Sigma(x_i - y_i) \in \Delta_R(\pi).$$

If the sum on the right included more than one term then, in order that the appropriate cancellations occur, we would have $y_1 = x_j$ for some $j > 1$. But this cannot occur. Therefore $x - y = x_1 - y_1$ and so $x = x_1$ and $y = y_1$. Thus $(x, y) = (x_1, y_1) \in \pi$. Hence π is R-closed.

The isomorphism $G/\pi \cong (G/\pi)_R$ is given by $\tilde{\pi}(g) \leftrightarrow g + \Delta_R(\pi)$.

REFERENCES

(1) R. H. Bruck, *A Survey of Binary Systems* (Berlin, 1958).

(2) S. A. Jennings, The group ring of a class of infinite nilpotent groups, *Canadian J. Math.*, **7** (1955), 169-187.

(3) D. Rees, On semigroups, *Proc. Cambridge Phil. Soc.*, **36** (1940), 387-400.

The University of Wisconsin