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Twisted jets, motivic measures and orbifold cohomology

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Abstract

We introduce the notion of twisted jets. For a Deligne–Mumford stack $\mathcal{X}$ of finite type over $\mathbb{C}$, a twisted ∞-jet on $\mathcal{X}$ is a representable morphism $\mathcal{D} \to \mathcal{X}$ such that $\mathcal{D}$ is a smooth Deligne–Mumford stack with the coarse moduli space $\operatorname{Spec} \mathbb{C}[[t]]$. We study a motivic measure on the space of the twisted ∞-jets on a smooth Deligne–Mumford stack. As an application, we prove that two birational minimal models with Gorenstein quotient singularities have the same orbifold cohomology as a Hodge structure.

1. Introduction

In 1995, Kontsevich introduced the theory of *motivic integration* [Kon95]. Since then, this remarkable idea has become a powerful method for examining both the local and global structures of varieties.

Let X be a variety over $\mathbb{C}$. For $n \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$, an *n-jet* on X is a $\mathbb{C}[[t]]/(t^{n+1})$-point of X, where we have followed the convention $(t^{\infty}) = (0)$. The n-jets of X naturally constitute a variety (or pro-variety if $n = \infty$), denoted $L_n X$. For $m \geqslant n$, the natural surjection $\mathbb{C}[[t]]/(t^{m+1}) \to \mathbb{C}[[t]]/(t^{n+1})$ induces the truncation morphism $L_m X \to L_n X$.

Consider the case where X is smooth and of dimension d. Then $L_n X$ is a locally trivial affine space bundle over X. (Whenever X is singular, it fails. For example, for $n = 1$, $L_1 X$ is the tangent space of X and hence not a locally trivial bundle over X.) The idea of Kontsevich is to give $L_\infty X$ a measure which takes values in the Grothendieck ring $\mathbb{M}$ of k-varieties which is localized by the class $\mathbb{L}$ of the affine line. For each $n \in \mathbb{Z}_{\geqslant 0}$, the family of constructible subsets of $L_n X$ is stable under finite union or finite intersection. In other words, this family is a Boolean algebra. The map

$$\begin{aligned} \{\text{constructible subsets of } L_n X\} &\to \mathbb{M} \\ A &\mapsto \{A\}\mathbb{L}^{-nd} \end{aligned}$$

is a finite additive measure. For each $m > n \in \mathbb{Z}_{\geqslant 0}$, because the truncation morphism $\pi^m_n : L_m X \to L_n X$ is a locally trivial affine space bundle of relative dimension $(m-n)d$, the pull-back

$$(\pi^m_n)^{-1} : \{\text{constructible subsets of } L_n X\} \to \{\text{constructible subsets of } L_m X\},$$

is considered to be an extension of the measure into a bigger Boolean algebra. The *motivic measure* on $L_\infty X$ is defined to be the limit of these extensions. Denef and Loeser generalized the motivic measure to the case where X is singular [DL99].

The integral of a function with respect to the motivic measure produces a new invariant. In particular, when X is smooth and the function is equal to one, then the integral, which is the

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full volume of $L_\infty X$, is the class of X in $\mathbb{M}$. It reduces to the Hodge structure of the cohomology of X if X is complete. By the transformation rule of Kontsevich, for any resolution $Z \to X$, it equals the integral of a function on $L_\infty Z$ determined by the relative canonical divisor $K_{Z/X}$. This implies the following theorem of Kontsevich, which we will generalize.

Theorem 1.1. *Let X and X' be smooth complete varieties. Suppose that there are proper birational morphisms $Z \to X$ and $Z \to X'$ such that $K_{Z/X} = K_{Z/X'}$. Then the rational cohomologies of X and X' have the same Hodge structures.*

The key point is that by the valuative criterion for properness, almost every ∞-jet on X lifts to a unique ∞-jet on Z, and hence the map $L_\infty Z \to L_\infty X$ is bijective outside of measure-zero subsets. Note that Batyrev first proved the equality of Betti numbers in the case where X and X' are Calabi–Yau varieties, with p-adic integration and the Weil conjecture [Bat99a].

Let X be a variety with Gorenstein canonical singularities. Denef and Loeser gave $L_\infty X$ another measure, called the *motivic Gorenstein measure*, denoted by μ_X^{Gor} [DL02]. As in the case when X is smooth, $\mu_X^{\mathrm{Gor}}(L_\infty X)$ is calculated by the relative canonical divisor $K_{Z/X}$ for a resolution $Z \to X$. This implies the following proposition.

Proposition 1.2. *Let X and X' be varieties with Gorenstein canonical singularities. Suppose that there are proper birational morphisms $Z \to X$ and $Z \to X'$ such that $K_{Z/X} = K_{Z/X'}$. Then $\mu_X^{\mathrm{Gor}}(L_\infty X) = \mu_{X'}^{\mathrm{Gor}}(L_\infty X')$*

Quotient singularities form one of the mildest classes of singularities.[1] If X is a variety with quotient singularities, then we can give X an orbifold structure. In the algebro-geometric context, there is a smooth Deligne–Mumford stack $\mathcal{X}$ such that X is the coarse moduli space of $\mathcal{X}$ and the automorphism group of general points of $\mathcal{X}$ is trivial. Although the natural morphism $\mathcal{X} \to X$ is proper and birational, *not* quite every ∞-jet on X lifts to a $\mathbb{C}[[t]]$-point of $\mathcal{X}$ from lack of the *strict* valuative criterion for properness. However, by twisting the source $\operatorname{Spec}\mathbb{C}[[t]]$, we can lift almost every ∞-jet on X to $\mathcal{X}$. More precisely, a *twisted ∞-jet* on $\mathcal{X}$ is a representable morphism $\mathcal{D} \to \mathcal{X}$ such that $\mathcal{D}$ is a smooth Deligne–Mumford stack with the coarse moduli space $\operatorname{Spec}\mathbb{C}[[t]]$ and $\mathcal{D}$ contains $\operatorname{Spec}\mathbb{C}((t))$ as an open substack. (The paper by Abramovich and Vistoli [AV02] was the inspiration for this notion – they introduced the notion of a twisted stable map.) For almost every ∞-jet $\gamma : \operatorname{Spec}\mathbb{C}[[t]] \to X$, there is a unique twisted ∞-jet $\mathcal{D} \to \mathcal{X}$ such that the induced morphism $\operatorname{Spec}\mathbb{C}[[t]] \to X$ of the coarse moduli spaces is γ. If $\overline{\mathcal{L}_\infty \mathcal{X}}$ is the coarse moduli space of the twisted ∞-jets on $\mathcal{X}$, then we define the *motivic measure* $\mu_{\mathcal{X}}$ on $\overline{\mathcal{L}_\infty \mathcal{X}}$ in a similar fashion as on $L_\infty X$, though it takes values in the Grothendieck ring of Hodge structures. We show the following close relationship between μ_X^{Gor} and $\mu_{\mathcal{X}}$.

Theorem 1.3. *The following equality holds:*

$$\mu_X^{\mathrm{Gor}} = \sum_{\mathcal{Y} \subset I(\mathcal{X})} \mathbb{L}^{s(\mathcal{Y})} \mu_{\mathcal{X}}.$$

For the precise meaning, see Theorem 3.15.

Chen and Ruan defined the *orbifold cohomology* for arbitrary orbifold [CR00]. It originates from string theory on orbifolds [DHVW86]. Let X be a variety with Gorenstein quotient singularities and $\mathcal{X}$ as defined earlier. The *inertia stack* of $\mathcal{X}$, denoted $I(\mathcal{X})$, is an object in the algebro-geometric realm that corresponds to the twisted sector. We define the ith orbifold cohomology group of X by

$$H^i_{\mathrm{orb}}(X, \mathbb{Q}) := \bigoplus_{\mathcal{Y} \subset I(\mathcal{X})} H^{i-2s(\mathcal{Y})}(\overline{\mathcal{Y}}, \mathbb{Q}) \otimes \mathbb{Q}(-s(\mathcal{Y})),$$

[1] Here the words 'quotient singularities' mean 'quotient singularities with respect to the étale topology'; see Definition 4.27.

where $\mathcal{Y}$ runs over the connected components of $I(\mathcal{X})$, $\overline{\mathcal{Y}}$ is the coarse moduli space of $\mathcal{Y}$ and $s(\mathcal{Y})$ is an integer which is representation-theoretically determined.

Remark 1.4.

i) We guess but are not sure that our orbifold cohomology is equal to that of Chen and Ruan in [CR00]. The point we wonder about is whether the cohomology groups of $\overline{\mathcal{Y}}$ are isomorphic to those of the analytic orbifold (V-manifold) associated to $\mathcal{Y}$.

ii) The orbifold Hodge numbers (i.e. the Hodge numbers associated with $H^i_{\mathrm{orb}}(X, \mathbb{Q})$) are equal to Batyrev's stringy Hodge numbers [Bat99b]. This follows from Lemma 2.16 and Theorem 1.3.

Theorem 1.3 implies that when X is complete, the invariant $\mu^{\mathrm{Gor}}_X(L_\infty X)$ reduces to the alternating sum of the orbifold cohomology groups of X. Hence we obtain the following theorem as conjectured by Ruan [Rua00].

Theorem 1.5 (Corollary 3.16). *Let X and X' be complete varieties with Gorenstein quotient singularities. Suppose that there are proper birational morphisms $Z \to X$ and $Z \to X'$ such that $K_{Z/X} = K_{Z/X'}$. Then the orbifold cohomologies of X and X' have the same Hodge structures.*

If X and X' are birational minimal models, that is, K_X and K'_X are nef, then for a common resolution Z of X and X', we have the equality $K_{Z/X} = K_{Z/X'}$ (see [KM98, Proposition 3.51]). Hence X and X' have the same orbifold cohomology with a Hodge structure. Note that in the case where X and X' are global quotients, Theorem 1.5 is due to Batyrev [Bat99b] and Denef and Loeser [DL02]. After writing out the initial version of this paper, we learnt via e-mail from Ernesto Lupercio that he and Mainak Poddar had independently proved Theorem 1.5.

1.1 Contents

The paper is organized as follows. In § 2, we review motivic measures. Section 3 is the central part of the paper. Here we introduce the notion of a twisted jet and examine their space. Then we prove the main result. In § 4, we review Deligne–Mumford stacks and prove some general results on Deligne–Mumford stacks, which we need in § 3.

1.2 Conventions and notations

- In §§ 2 and 3, we work over $\mathbb{C}$.
- For a Deligne–Mumford stack $\mathcal{X}$, we denote by $\overline{\mathcal{X}}$ the coarse moduli space of $\mathcal{X}$.
- We denote by (Sch/S) (respectively $(\mathrm{Sch}/\mathbb{C})$) the category of schemes over a scheme S (respectively over $\mathbb{C}$).
- For a $\mathbb{C}$-scheme X (or more generally a stack over $\mathbb{C}$) and a $\mathbb{C}$-algebra R, we denote by $X \otimes R$ the product $X \times_{\mathbb{C}} \mathrm{Spec}\, R$. Then we denote by $X[[t]]$ (respectively $X[[t^{1/l}]]$) the scheme $X \otimes \mathbb{C}[[t]]$ (respectively $X \otimes \mathbb{C}[[t^{1/l}]]$).

2. Motivic measures – a review

In this section, we would like to review the theory of motivic measures, developed by Kontsevich [Kon95], Batyrev [Bat99b], Denef and Loeser [DL99, DL02]. It is also worth mentioning [Cra99] which has a nice introduction and [Loo02, DL01] for surveys.

2.1 Completing Grothendieck rings

Let us first construct the ring in which motivic measures take values. Note that a variety means a reduced scheme of finite type over $\mathbb{C}$.

Definition 2.1. We define the *Grothendieck ring of varieties*, denoted by $K_0(\mathrm{Var})$, to be the abelian group generated by the isomorphism classes $\{X\}$ of varieties with the relations $\{X\} = \{X \backslash Y\} + \{Y\}$ if Y is a closed subvariety of X. The ring structure is defined by $\{X\}\{Y\} = \{X \times Y\}$.

In the same fashion, we can define the Grothendieck ring of separated algebraic spaces of finite types. In reality, it is the same as $K_0(\mathrm{Var})$, since every noetherian algebraic space decomposes into the disjoint union of schemes [Knu71, Proposition 6.6].

Suppose that A is a constructible subset of a variety X, that is, A is a disjont union of the locally closed subvarieties $A_i \subset X$. Then we put $\{A\} := \sum_i \{A_i\} \in K_0(\mathrm{Var})$, which is independent of the stratification choice. We denote the class of $\mathbb{A}^1$ by $\mathbb{L}$ and the localization $K_0(\mathrm{Var})[\mathbb{L}^{-1}]$ by $\mathbb{M}$. For $m \in \mathbb{Z}$, let $F_m\mathbb{M}$ be the subgroup of $\mathbb{M}$ generated by the elements $\{X\}\mathbb{L}^{-i}$ with $\dim X - i \leqslant -m$. The collection $(F_m\mathbb{M})_{m \in \mathbb{Z}}$ is a descending filtration of $\mathbb{M}$ with

$$F_m\mathbb{M} \cdot F_n\mathbb{M} \subset F_{m+n}\mathbb{M}. \tag{1}$$

Definition 2.2. We define the *complete Grothendieck ring of varieties* by

$$\hat{\mathbb{M}} := \varprojlim \mathbb{M}/F_m\mathbb{M}.$$

By condition (1), it has a natural ring structure.

Note that it is not known whether the natural map $\mathbb{M} \to \hat{\mathbb{M}}$ is injective.

Recall that a *Hodge structure* is a finite-dimensional $\mathbb{Q}$-vector space H with a bigrading $H \otimes \mathbb{C} = \bigoplus_{p,q \in \mathbb{Z}} H^{p,q}$ such that $H^{p,q}$ is the complex conjugate of $H^{q,p}$ and each weight summand $\bigoplus_{p+q=m} H^{p,q}$ is defined over $\mathbb{Q}$. The category HS of Hodge structures is an abelian category with a tensor product.

Definition 2.3. We define the *Grothendiek ring of Hodge structures*, denoted $K_0(\mathrm{HS})$, to be the abelian group that consists of the formal differences $\{H\} - \{H'\}$, where $\{H\}$ and $\{H'\}$ are isomorphism classes of Hodge structures. The addition and the multiplication come from $\oplus$ and $\otimes$, respectively.

A *mixed Hodge structure* is a finite-dimensional $\mathbb{Q}$-vector space H with increasing filtration $W_\bullet H$, called the *weight filtration*, such that the associated graded $\mathrm{Gr}^W_\bullet H$ underlies a Hodge structure having $\mathrm{Gr}^W_m H$ as a weight m summand. For a mixed Hodge structure H, we denote by $\{H\}$ the element $\{\mathrm{Gr}^W_\bullet H\}$ of $K_0(\mathrm{HS})$.

The cohomology groups $H^i_c(X, \mathbb{Q})$ with compact supports of a variety X has a natural mixed Hodge structure.

Definition 2.4. We define the *Hodge characteristic* $\chi_h(X)$ of X by

$$\chi_h(X) := \sum_i (-1)^i \{H^i_c(X, \mathbb{Q})\} \in K_0(\mathrm{HS}).$$

Consider the following map:

$$(\text{Varieties}) \to K_0(\mathrm{HS}),\ X \mapsto \chi_h(X).$$

It factors through the map

$$(\text{Varieties}) \to \mathbb{M},\ X \mapsto \{X\},$$

because the following hold:

- $\chi_h(X \times Y) = \chi_h(X)\chi_h(Y)$;
- $\chi_h(X) = \chi_h(X \setminus Y) + \chi_h(Y)$ if $Y \subset X$ is closed;
- the Hodge characteristic of the affine line, $\chi_h(\mathbb{A}^1) = \{H^2_c(\mathbb{A}^1, \mathbb{Q})\}$, is invertible.

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We also denote by χ_h the induced homomorphism $\mathbb{M} \to K_0(\mathrm{HS})$.

For $m \in \mathbb{Z}$, let $F_m K_0(\mathrm{HS})$ be the subgroup generated by the elements $\{H\}$ such that the maximum weight of H is less than or equal to $-m$.

Definition 2.5. We define the *complete Grothendieck ring of Hodge structures*, denoted by $\hat{K}_0(\mathrm{HS})$, as follows:

$$\hat{K}_0(\mathrm{HS}) := \varprojlim K_0(\mathrm{HS})/F_m K_0(\mathrm{HS}).$$

We can see that the natural map $K_0(\mathrm{HS}) \to \hat{K}_0(\mathrm{HS})$ is injective. As the maximal weight of $H^i(X, \mathbb{Q})$ does not exceed $2\dim X$, χ_h extends to $\chi_h : \hat{\mathbb{M}} \to \hat{K}_0(\mathrm{HS})$.

2.2 Jets on schemes

For the sake of convenience, we denote by (t^∞) the ideal (0) of the power series ring $\mathbb{C}[[t]]$. For $n \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$, we denote by D_n the affine scheme $\operatorname{Spec} \mathbb{C}[[t]]/(t^{n+1})$.

Definition 2.6. Let X be a scheme. For $n \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$, we define the *scheme of n-jets*[2] of X, denoted $L_n X$, to be the scheme representing the functor

$$(\mathrm{Sch}/\mathbb{C}) \to (\mathrm{Sets})$$
$$U \mapsto \mathrm{Hom}_{(\mathrm{Sch}/\mathbb{C})}(U \times D_n, X).$$

Greenberg [Gre61] proved the representability of the functor for $n < \infty$. For $m, n \in \mathbb{Z}_{\geqslant 0}$ with $m < n$, a canonical closed immersion $D_m \to D_n$ induces a canonical projection $L_n X \to L_m X$. Since all these projections are affine morphisms, the projective limit $L_\infty X = \varprojlim L_n X$ exists in the category of schemes.

If X is of finite type, then, for $n < \infty$, so is $L_n X$. If X is smooth and of pure dimension d, then, for each $n \in \mathbb{Z}_{\geqslant 0}$, the natural projection $L_{n+1}X \to L_n X$ is a Zariski locally trivial $\mathbb{A}^d$-bundle. If $f : Y \to X$ is a morphism of schemes, then for each n, there is a canonical morphism $f_n : L_n Y \to L_n X$.

2.3 Motivic measure

Let X be a scheme of pure dimension d. By abuse of notation, we also denote the set of points of $L_\infty X$ by $L_\infty X$. Let $\pi_n : L_\infty X \to L_n X$ be the canonical projection.

Definition 2.7. A subset A of $L_\infty X$ is *stable at level n* if we have:

i) $\pi_n(A)$ is a constructible subset in $L_n X$;
ii) $A = \pi_n^{-1}\pi_n(A)$;
iii) for any $m \geqslant n$, the projection $\pi_{m+1}(A) \to \pi_m(A)$ is a piecewise trivial $\mathbb{A}^d$-bundle.

(A morphism $f : Y \to X$ of schemes is called a *piecewise trivial $\mathbb{A}^d$-bundle* if there is a stratification $X = \coprod X_i$ such that $f|_{f^{-1}(X_i)} : f^{-1}(X_i) \to X_i$ is isomorphic to $X_i \times \mathbb{A}^d \to X_i$ for each i.) A subset A of $L_\infty X$ is *stable* if it is stable at level n for some $n \in \mathbb{Z}_{\geqslant 0}$.

The stable subsets of $L_\infty X$ constitute a Boolean algebra. If $A \subset L_\infty X$ is a stable subset, then $\{\pi_m(A)\}\mathbb{L}^{-md} \in \hat{\mathbb{M}}$ is constant for $m \gg 0$. We denote it[3] by $\mu_X(A)$. The map

$$\mu_X : \{\text{stable subsets of } L_\infty X\} \to \hat{\mathbb{M}}$$
$$A \mapsto \mu_X(A)$$

is a finite additive measure. Let us extend this measure to a bigger family of subsets of $L_\infty X$.

[2] In [DL99], an ∞-jet is called an *arc*. As it is more convenient, we prefer our terminology.
[3] This differs from the definition in [Cra99], [Bat99b] and [DL99] by a factor $\mathbb{L}^d$.

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Definition 2.8. A subset $A \subset L_\infty X$ is called *measurable* if, for every $m \in \mathbb{Z}$, there are stable subsets $A_m \subset L_\infty X$ and $C_i \subset L_\infty X$, $i \in \mathbb{Z}_{>0}$ such that the symmetric difference $(A \cup A_m) \setminus (A \cap A_m)$ is contained in $\cup_i C_i$ and we have $\mu_X(C_i) \in F_m\mathbb{M}$ for all i, and $\lim_{i\to\infty} \mu_X(C_i) = 0$ in $\hat{\mathbb{M}}$.

The measurable subsets of $L_\infty X$ also constitute a Boolean algebra. Suppose that $A \subset L_\infty X$ is measurable and $A_m \subset L_\infty X$, $m \in \mathbb{Z}$ are stable subsets as in Definition 2.8. We put $\mu_X(A) := \lim_{m\to\infty} \mu_X(A_m)$. It is independent of the choice of A_m, see [Loo02, Proposition 2.2], [DL02, Theorem A.6]. The map

$$\begin{aligned}\mu_X : \{\text{measurable subsets of } L_\infty X\} &\to \hat{\mathbb{M}}\\ A &\mapsto \mu_X(A)\end{aligned}$$

is a finite additive measure.

Definition 2.9. We call this the *motivic measure* on $L_\infty X$.

Definition 2.10. Let $A \subset L_\infty X$ be a measurable subset and $\nu : A \to \mathbb{Z} \cup \{\infty\}$ a function. We say that ν is a *measurable function* if the fibers are measurable and $\mu_X(\nu^{-1}(\infty)) = 0$. For a measurable function ν, we formally define the *motivic integration* of $\mathbb{L}^\nu$ by

$$\int_A \mathbb{L}^\nu \, d\mu_X := \sum_{n\in\mathbb{Z}} \mu_X(\nu^{-1}(n))\mathbb{L}^n.$$

We say that $\mathbb{L}^\nu$ is *integrable* if this series converges in $\hat{\mathbb{M}}$.

Example 2.11. Let $\mathcal{I}$ be an ideal sheaf on X. A point $\gamma \in L_\infty X$ corresponds to a morphism $\operatorname{Spec}\kappa \to L_\infty X$ for the residue field κ of γ and hence to a morphism $\gamma' : \operatorname{Spec}\kappa[[t]] \to X$. The function

$$\begin{aligned}\operatorname{ord}\mathcal{I} : L_\infty X &\to \mathbb{Z}_{\geqslant 0} \cup \{\infty\}\\ \gamma &\mapsto n \quad \text{if } (\gamma')^{-1}\mathcal{I} = (t^n)\end{aligned}$$

is a measurable function by the following lemma.

Lemma 2.12 [Loo02, Proposition 3.1], [DL99, Lemma 4.4]. *For a subvariety $Y \subset X$ of positive codimension, the subset $L_\infty Y \subset L_\infty X$ is of measure zero.*

Example 2.13 [Bat99b, Theorem 3.6], [Cra99, Theorem 2.15]. Let X be a smooth variety of dimension d and $E = \sum_{i=1}^r d_i E_i$ an effective divisor on X with simple normal crossing support. For a subset $J \subset \{1, \dots, r\}$ we define

$$E_J^\circ := \bigcap_{i\in J} E_i \Big\backslash \bigcup_{i\in\{1,\dots,r\}\setminus J} E_i.$$

If $\mathcal{I}_E$ is the ideal sheaf associated to E, then we have the following formula:

$$\int_{L_\infty X} \mathbb{L}^{-\operatorname{ord}\mathcal{I}_E} \, d\mu_X = \sum_{J\subset\{1,\dots,r\}} \{E_J^\circ\} \prod_{i\in J} \frac{\mathbb{L}-1}{\mathbb{L}^{d_i+1}-1}.$$

2.4 The transformation rule

Let X and Y be varieties of pure dimension d and $f : Y[[t]] \to X[[t]]$ a morphism over D_∞. We define the morphisms $f_n : L_nY \to L_nX$ as follows. For a scheme U, a U-point γ of L_nY is a morphism $\gamma' : D_n \times U \to Y$. If ψ is the composition of natural morphisms $D_n \times U \xrightarrow{\mathrm{pr}} D_n \to D_\infty$, then we define $f_n(\gamma)$ to be the U-point of L_nX corresponding to the composition

$$D_n \times U \xrightarrow{(\gamma',\psi)} Y \times D_\infty = Y[[t]] \xrightarrow{f} X[[t]] \to X.$$

Assume that Y is smooth. We define the *jacobian ideal sheaf* $\mathcal{J}_f$ of f to be the zeroth Fitting ideal sheaf of $\Omega_{Y[[t]]/X[[t]]}$ (see [EH99, V.1.3] or [Eis95, 20.2]). This is the ideal sheaf such that $\mathcal{J}_f \Omega^d_{Y[[t]]/D_\infty} = f^*(\Omega^d_{X[[t]]/D_\infty}/(\mathrm{tors}))$ as subsheaves of $\Omega^d_{Y[[t]]/D_\infty}$, where (tors) is the torsion. The following, called the *transformation rule*, is the most basic theorem in the theory.

Theorem 2.14. [DL02, Theorem 1.16], [Loo02, Theorem 3.2]. *Let A be a measurable set in $L_\infty Y$ and $\nu : f_\infty(A) \to \mathbb{Z} \cup \{\infty\}$ a measurable function. Suppose that $f_\infty|_A$ is injective. Then we have the following equality:*

$$\int_{f_\infty(A)} \mathbb{L}^\nu \, d\mu_X = \int_A \mathbb{L}^{\nu \circ f_\infty - \mathrm{ord}\, \mathcal{J}_f} \, d\mu_Y.$$

We will generalize this later (Theorem 3.18).

2.5 The motivic Gorenstein measure

Let X be a variety with 1-Gorenstein and canonical singularities, that is, the canonical sheaf ω_X is invertible and all discrepancies are greater than or equal to zero (see [KMM87, §§ 0–2]). Then there exists a natural morphism $\Omega^d_X \to \omega_X$. The kernel of this morphism is the torsion. We define an ideal sheaf $\mathcal{I}_X$ on X by the equation

$$\mathcal{I}_X \omega_X = \mathrm{Im}(\Omega^d_X \to \omega_X).$$

Then $\mathbb{L}^{\mathrm{ord}\, \mathcal{I}_X}$ is integrable by Example 2.13 and Lemma 2.16.

Definition 2.15. We define the *motivic Gorenstein measure* μ^{Gor}_X on $L_\infty X$ as follows:

$$\mu^{\mathrm{Gor}}_X : \{\text{measurable subsets of } L_\infty X\} \to \mathbb{M}$$
$$A \mapsto \int_A \mathbb{L}^{\mathrm{ord}\, \mathcal{I}_X} \, d\mu_X.$$

Lemma 2.16. *Let X and X' be complete varieties with 1-Gorenstein canonical singularities.*

i) *Let A be a measurable subset of $L_\infty X$ and $f : Z \to X$ be a resolution. Then*

$$\int_A \mathbb{L}^{\mathrm{ord}\, \mathcal{I}_X} \, d\mu_X = \int_{f_\infty^{-1}(A)} \mathbb{L}^{-\mathrm{ord}\, \mathcal{K}} \, d\mu_Z,$$

where $\mathcal{K}$ is the ideal sheaf associated with $K_{Z/X}$.

ii) *Suppose that there exist proper birational morphisms $f : Z \to X$ and $g : Z \to X'$ with $K_{Z/X} = K_{Z/X'}$. Then we have $\mu^{\mathrm{Gor}}_X(L_\infty X) = \mu^{\mathrm{Gor}}_{X'}(L_\infty X')$.*

Proof. i) By Theorem 2.14, we have

$$\int_A \mathbb{L}^{\mathrm{ord}\, \mathcal{I}_X} \, d\mu_X = \int_{f_\infty^{-1}(A)} \mathbb{L}^{\mathrm{ord}\, \mathcal{I}_X \circ f_\infty - \mathrm{ord}\, \mathcal{J}_f} \mu_Z.$$

We have to show

$$\mathrm{ord}\, \mathcal{I}_X \circ f_\infty - \mathrm{ord}\, \mathcal{J}_f = -\mathrm{ord}\, \mathcal{K}. \tag{2}$$

Pulling back $\mathcal{I}_X \omega_X \cong \Omega^d_X/(\mathrm{tors})$, we have $(f^{-1}\mathcal{I}_X)(f^*\omega_X) \cong \mathcal{J}_f \Omega^d_Z$. On the other hand, we have $f^*\omega_X \cong \mathcal{K}\omega_Z \cong \mathcal{K}\Omega^d_Z$. Hence $(f^{-1}\mathcal{I}_X) \cdot \mathcal{K} = \mathcal{J}_f$. This shows equation (2).

ii) is a direct consequence of item i. □

Remark 2.17. The invariant $\mu^{\mathrm{Gor}}_X(L_\infty X)$ has been already introduced in [Kon95] using a resolution of singularities. Denef and Loeser constructed the invariant more directly with the motivic Gorenstein measure [DL02].

Suppose X is complete. The question is whether $\chi_h \circ \mu_X^{\mathrm{Gor}}(L_\infty X)$ is the alternating sum of *a kind of cohomology groups*, as in the case where X is smooth. It is known that when X is a global quotient, the answer is yes [Bat99b, DL02]. Our result, Theorem 3.15, states that when X has only quotient singularities, the answer is also yes.

3. Twisted jets

In this section, we deal with the theory of Deligne–Mumford stacks. See § 4 for the generalities about Deligne–Mumford stacks.

3.1 Non-twisted jets on stacks

The following is a direct generalization of the notion of jets on schemes.

Definition 3.1. Let $\mathcal{X}$ be a Deligne–Mumford stack. For $n \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$, we define the *stack of non-twisted n-jets* of $\mathcal{X}$, denoted by $L_n\mathcal{X}$, as follows. An object of $L_n\mathcal{X}$ over $U \in (\mathrm{Sch}/\mathbb{C})$ is an object of $\mathcal{X}$ over $U \times D_n$. For a morphism $\varphi : V \to U$ in $(\mathrm{Sch}/\mathbb{C})$, a morphism in $L_n\mathcal{X}$ over φ is a morphism in $\mathcal{X}$ over $\varphi \times \mathrm{id}_{D_n}$.

Lemma 3.2. *For every $n \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$, $L_n\mathcal{X}$ is a stack.*

Proof. It is clear that they satisfy the axioms of a category fibered in groupoids. If $(U_i \to U)_i$ is an étale covering in $(\mathrm{Sch}/\mathbb{C})$, then so is $(U_i \times D_n \to U \times D_n)_i$. As a result, they also satisfy the axioms of a stack. ☐

Let $f : Y \to X$ be a morphism of schemes and $Z \subset Y$ a closed subscheme with an ideal sheaf $\mathfrak{a}$. We say that f is *Z-étale* if for any ring A and any nilpotent ideal $J \subset A$, and for any commutative diagram of solid arrows

$$\begin{array}{ccc} \mathrm{Spec}\, A/J & \xrightarrow{\varphi} & Y \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}\, A & \longrightarrow & X \end{array}$$

such that $\varphi^{-1}\mathfrak{a}$ is nilpotent, there is a unique broken arrow which makes the whole diagram commutative. Z-étaleness is also defined for a representable morphism of stacks in the evident fashion.

Lemma 3.3.

i) *Let M be a scheme and $N \subset M$ a closed subscheme. We denote by $(L_n M)_N$ the subscheme of $L_n M$ parametrizing the jets with the base point in N. Let $p : M \to \mathcal{X}$ be an N-étale morphism. Then, for every $n \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$, we have a natural isomorphism as follows:*

$$(L_n M)_N \cong L_n\mathcal{X} \times_{\mathcal{X}} N.$$

ii) *For every $n \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$, $L_n\mathcal{X}$ is a Deligne–Mumford stack.*

Proof. i) Let us first show that $L_n\mathcal{X} \times_{\mathcal{X}} N$ is an algebraic space. Let $\pi : L_n\mathcal{X} \to \mathcal{X}$ be the canonical projection. An object of $L_n\mathcal{X} \times_{\mathcal{X}} N$ is a triple (γ, f, α) where $\gamma : U \times D_n \to \mathcal{X}$, $f : U \to N$ and $\alpha : \pi(\gamma) \to p(f)$ is a morphism in $\mathcal{X}$ over U. By definition, an automorphism of (γ, f, α) is an automorphism θ of γ such that $\alpha \circ \pi(\theta) = p(\mathrm{id}_f) \circ \alpha$. Hence θ must be the identity. We have thus proved that the automorphism of every object of $L_n\mathcal{X} \times_{\mathcal{X}} N$ is trivial and hence that $L_n\mathcal{X} \times_{\mathcal{X}} N$ is an algebraic space (see [LM00, Corollaire 8.1.1]).

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The diagram of solid arrows

$$\begin{array}{ccccc} U & \xrightarrow{f} & N & \hookrightarrow & M \\ \big\downarrow & & & \overset{\tau}{\nearrow} & \big\downarrow{\scriptstyle p} \\ U \times D_n & & \xrightarrow[\gamma]{} & & \mathcal{X} \end{array}$$

is commutative. We can see that there is a unique broken arrow τ in the diagram. If $n < \infty$, since p is N-étale, this is trivial from the definition. If $n = \infty$, since D_∞ is the direct limit of D_n, $0 \leqslant n < \infty$, this follows from the case $n < \infty$. Sending (γ, f, α) to τ defines a morphism $\Phi : L_n\mathcal{X} \times_{\mathcal{X}} M \to L_n M$.

The inverse of the morphism Φ is given by $(p_n, \pi_M) : (L_n M)_N \to L_n\mathcal{X} \times_{\mathcal{X}} N$ where $p_n : L_n M \to L_n\mathcal{X}$ and $\pi_M : (L_n M)_N \to N = (L_0 M)_N$ are the natural morphisms. We have thus proved item i.

ii) Now suppose that p is étale and surjective. Consider the following cartesian diagram:

$$\begin{array}{ccc} L_n M & \longrightarrow & L_n\mathcal{X} \\ {\scriptstyle \pi_M}\big\downarrow & \square & \big\downarrow{\scriptstyle \pi} \\ M & \xrightarrow[p]{} & \mathcal{X} \end{array}$$

As π_M is representable and p is étale and surjective, π is representable (see [LM00, Lemme 4.3.3]). This completes the proof (see [LM00, Proposition 4.5]). $\square$

3.2 Twisted jets

For a positive integer l, we put $\zeta_l := \exp(2\pi\sqrt{-1}/l)$. Let $\mu_l = \langle \zeta_l \rangle$ be the group of the lth roots of 1. μ_l acts on D_n by $\zeta_l : t \mapsto \zeta_l t$. We denote by $\mathcal{D}_n^l$ the quotient stack $[D_m/\mu_l]$ with $m = nl$. The stack $\mathcal{D}_n^l$ has the canonical atlas $D_m \to \mathcal{D}_n^l$ and the closed point $\operatorname{Spec}\mathbb{C} \to \mathcal{D}_n^l$. We fix a morphism $\mathcal{D}_n^l \to D_n$ such that D_n is the coarse moduli space of $\mathcal{D}_n^l$ for this morphism and such that the composition

$$D_m \to \mathcal{D}_n^l \to D_n$$

is given by the ring homomorphism $\mathbb{C}[[t]]/(t^{n+1}) \to \mathbb{C}[[t]]/(t^{m+1})$, $t \mapsto t^l$.

Definition 3.4. Let $\mathcal{X}$ be a Deligne–Mumford stack. A *twisted n-jet of order l* on $\mathcal{X}$ is a representable morphism $\mathcal{D}_n^l \otimes \Omega \to \mathcal{X}$ for an algebraically closed field $\Omega \supset \mathbb{C}$.

For a Deligne–Mumford stack $\mathcal{X}$, the *inertia stack* of $\mathcal{X}$, denoted by $I(\mathcal{X})$, is the stack parametrizing the pairs (ξ, α) such that $\xi \in \operatorname{ob}\mathcal{X}$ and $\alpha \in \operatorname{Aut}(\xi)$. For details on inertia stacks, see § 4.3. There is a natural forgetting morphism $I(\mathcal{X}) \to \mathcal{X}$. For $l \in \mathbb{Z}_{>0}$, let $I^l(\mathcal{X}) \subset I(\mathcal{X})$ denote the substack parametrizing the pairs (ξ, α) with $\operatorname{ord}(\alpha) = l$.

Let $\gamma : \mathcal{D}_n^l \otimes \Omega \to \mathcal{X}$ be a twisted n-jet of order l on $\mathcal{X}$. The canonical morphism

$$\tilde{\gamma} : D_m \otimes \Omega \to \mathcal{D}_n^l \otimes \Omega \to \mathcal{X}$$

is considered to be an Ω-point of $L_m\mathcal{X}$ and the canonical morphism

$$\overline{\gamma} : \operatorname{Spec}\Omega \to D_m \otimes \Omega \to \mathcal{D}_n^l \otimes \Omega \to \mathcal{X}$$

is considered to be an Ω-point of $\mathcal{X}$. Since the automorphism group of the closed point of $\mathcal{D}_n^l \otimes \Omega$ is identified with μ_l, γ induces an injection $\mu_l \to \operatorname{Aut}(\overline{\gamma})$. If $b \in \operatorname{Aut}(\overline{\gamma})$ is the image of ζ_l, then the pair $(\overline{\gamma}, b)$ is regarded as an Ω-point of $I^l(\mathcal{X})$ and the triple $(\tilde{\gamma}, (\overline{\gamma}, b), \mathrm{id}_{\overline{\gamma}})$ is regarded as an Ω-point of $L_m\mathcal{X} \times_{\mathcal{X}} I^l(\mathcal{X})$. We define a map Ψ by

$$\begin{aligned} \Psi : \{\text{twisted } n\text{-jets of order } l \text{ on } \mathcal{X}\} &\to |L_m\mathcal{X} \times_{\mathcal{X}} I^l(\mathcal{X})| \\ \gamma &\mapsto (\tilde{\gamma}, (\overline{\gamma}, b), \mathrm{id}_{\overline{\gamma}}). \end{aligned}$$

LEMMA 3.5. *The subset* $\mathrm{Im}(\Psi) \subset |L_m\mathcal{X} \times_{\mathcal{X}} I^l(\mathcal{X})|$ *is closed for Zariski topology.*

Proof. Fix an atlas $p : M \to \mathcal{X}$ with M separated.

We will first characterize the points in $\mathrm{Im}(\Psi)$. On account of the arguments on groupoid spaces in § 4.1, we can see that the following points are equivalent.

i) To give a commutative diagram

$$\begin{array}{ccc} D_m \otimes \Omega & \xrightarrow{\eta} & M \\ \downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{D}^l_n \otimes \Omega & \xrightarrow[\gamma]{} & \mathcal{X} \end{array}$$

such that γ is a twisted n-jet of order l.

ii) To give a morphism of groupoid spaces

$$\begin{array}{ccc} (D_m \otimes \Omega) \times \mu_l & \xrightarrow{\delta'} & M \times_{\mathcal{X}} M \\ {\scriptstyle \mathrm{pr}}\downarrow \ \downarrow{\scriptstyle \mu_l-\text{action}} & & {\scriptstyle \mathrm{pr}_1}\downarrow \ \downarrow{\scriptstyle \mathrm{pr}_2} \\ D_m \otimes \Omega & \xrightarrow[\eta]{} & M \end{array}$$

such that the composition

$$\operatorname{Spec}\Omega \hookrightarrow (D_m \otimes \Omega) \times \{\zeta_l\} \hookrightarrow (D_m \otimes \Omega) \times \mu_l \xrightarrow{\delta'} M \times_{\mathcal{X}} M$$

corresponds to an automorphism of order l of the following Ω-point of $\mathcal{X}$:

$$\operatorname{Spec}\Omega \hookrightarrow D_m \otimes \Omega \xrightarrow{\eta} M \xrightarrow{p} \mathcal{X}.$$

iii) To give a morphism $\delta : D_m \otimes \Omega \to M \times_{\mathcal{X}} M$ such that $\mathrm{pr}_1 \circ \delta \circ \zeta_l = \mathrm{pr}_2 \circ \delta$ and the composition

$$\operatorname{Spec}\Omega \hookrightarrow D_m \otimes \Omega \xrightarrow{\delta} M \times_{\mathcal{X}} M$$

corresponds to an automorphism of order l of the following Ω-point of $\mathcal{X}$:

$$\operatorname{Spec}\Omega \hookrightarrow D_m \otimes \Omega \xrightarrow{\delta} M \times_{\mathcal{X}} M \xrightarrow{\mathrm{pr}_1} M \xrightarrow{p} \mathcal{X}.$$

Any point of $|L_m\mathcal{X} \times_{\mathcal{X}} I^l(\mathcal{X})|$ is represented by the triple $(\psi, (\overline{\psi}, b), \mathrm{id}_{\overline{\psi}})$ such that ψ is an Ω-point of $L_m\mathcal{X}$ with an algebraically closed field Ω, $\overline{\psi}$ is an Ω-point of $\mathcal{X}$ corresponding to the composition

$$\operatorname{Spec}\Omega \hookrightarrow D_m \otimes \Omega \xrightarrow{\psi} \mathcal{X}$$

and b is an automorphism of $\overline{\psi}$. Then the equivalence above implies that:

> $(\psi, (\overline{\psi}, b), \mathrm{id}_{\overline{\psi}})$ is in $\mathrm{Im}(\Psi)$ if and only if for a lift $\eta : D_m \otimes \Omega \to M$ of ψ, there exists a morphism $\delta : D_m \otimes \Omega \to M \times_{\mathcal{X}} M$ such that $\mathrm{pr}_1 \circ \delta = \eta$ and $\mathrm{pr}_2 \circ \delta = \eta \circ \zeta_l$ and the composition $\operatorname{Spec}\Omega \to D_m \otimes \Omega \xrightarrow{\delta} M \times_{\mathcal{X}} M$ corresponds to b. (♣)

Let ξ and σ be points of $L_m\mathcal{X} \times_{\mathcal{X}} I^l(\mathcal{X})$. Suppose that σ is in $\mathrm{Im}(\Psi)$ and ξ is a specialization of σ. It suffices to show that ξ is in $\mathrm{Im}(\Psi)$. By [LM00, Proposition 7.2.1], there is a complete discrete valuation ring R with an algebraically closed residue field κ and a quotient field K such that there is a commutative diagram as follows:

$$\begin{array}{ccc}
\operatorname{Spec} K & & \\
\downarrow & \overset{\sigma}{\searrow} & \\
\operatorname{Spec} R & \xrightarrow{\theta} & L_m\mathcal{X} \times_{\mathcal{X}} I^l(\mathcal{X}) \\
\uparrow & \underset{\xi}{\nearrow} & \\
\operatorname{Spec} \kappa & &
\end{array}$$

(Here by abusing the notation, the arrows σ and ξ in the diagram are representatives of the points σ and ξ, respectively.) If θ corresponds to a triple $(\lambda_R, (\overline{\lambda_R}, b_R), \mathrm{id}_{\overline{\lambda_R}})$, then, the pull-backs $(\lambda_K, (\overline{\lambda_K}, b_K), \mathrm{id}_{\overline{\lambda_K}})$ and $(\lambda_\kappa, (\overline{\lambda_\kappa}, b_\kappa), \mathrm{id}_{\overline{\lambda_\kappa}})$ correspond to σ and ξ, respectively. By extending R, we can assume that $\overline{\lambda_R} : \operatorname{Spec} R \to \mathcal{X}$ lifts to $\nu_R : \operatorname{Spec} R \to M$. As p is étale, $\lambda_R : D_m \otimes R \to \mathcal{X}$ uniquely lifts to $\tilde{\nu} : D_m \otimes R \to M$ such that the diagram

$$\begin{array}{ccc}
\operatorname{Spec} R & \xrightarrow{\nu} & M \\
\downarrow & \overset{\tilde{\nu}}{\nearrow} & \downarrow \\
D_m \otimes R & \xrightarrow[\lambda_R]{} & \mathcal{X}
\end{array}$$

is commutative. Let $\overline{K}$ be the algebraic closure of K, let η be the composition $D_m \otimes \overline{K} \to D_m \otimes R \xrightarrow{\tilde{\nu}} X$ and let $b'_R : \operatorname{Spec} R \to M \times_{\mathcal{X}} M$ be the lift of ν_R which corresponds to b_R. From (♣) and the previous assumption, there is a morphism $\delta : D_m \otimes \overline{K} \to M \times_{\mathcal{X}} M$ such that $\mathrm{pr}_1 \circ \delta = \eta$ and $\mathrm{pr}_2 \circ \delta = \eta \circ \zeta_l$, and the composition

$$\operatorname{Spec} \overline{K} \hookrightarrow D_m \otimes \overline{K} \xrightarrow{\delta} M \times_{\mathcal{X}} M$$

equals the composition

$$\operatorname{Spec} \overline{K} \to \operatorname{Spec} R \xrightarrow{b'_R} M \times_{\mathcal{X}} M.$$

We can replace $\overline{K}$ with a finite extension K' of K. Moreover, replacing R with its normalization in K', we can assume that $K' = K$ and that b'_R and δ induce the same morphism $\operatorname{Spec} K \to M \times_{\mathcal{X}} M$.

$$\begin{array}{ccccc}
\operatorname{Spec} K & \longrightarrow & \operatorname{Spec} R & \xrightarrow{b'_R} & M \times_{\mathcal{X}} M \\
\downarrow & \overset{\delta}{\nearrow} & \downarrow & & \downarrow\downarrow \\
D_m \otimes K & \longrightarrow & D_m \otimes R & \xrightarrow[\tilde{\nu}]{} & M
\end{array}$$

Consider the unique morphism $\tau : D_m \otimes R \to M \times_{\mathcal{X}} M$ such that $\mathrm{pr}_1 \circ \tau = \tilde{\nu}$. Then the two morphisms, $\mathrm{pr}_2 \circ \tau$ and $\tau \circ \zeta_l$, are the same morphism because of the separatedness of M. Then the composition $D_m \otimes \kappa \to D_m \otimes R \xrightarrow{\tau} M \times_{\mathcal{X}} M$ satisfies the condition in (♣). Hence $\xi \in \operatorname{Im}(\Psi)$. The proof is now complete. □

Definition 3.6. We define the *stack of twisted n-jets of order l* on $\mathcal{X}$, denoted by $\mathcal{L}_n^l\mathcal{X}$, to be the reduced closed substack of $L_m\mathcal{X} \times_{\mathcal{X}} I^l(\mathcal{X})$ with support $\operatorname{Im}(\Psi)$. We define the *stack of twisted n-jets* on $\mathcal{X}$, denoted by $\mathcal{L}_n\mathcal{X}$, to be the disjoint sum $\coprod_{l \geqslant 0} \mathcal{L}_n^l\mathcal{X}$. In particular, $\mathcal{L}_0\mathcal{X}$ is the inertia stack $I(\mathcal{X})$.

If we set

$$l_0 := \max\{l \mid l = \operatorname{ord}(\alpha),\ \text{for some } \xi \in \operatorname{ob}\mathcal{X} \text{ and for some } \alpha \in \operatorname{Aut}(\xi)\},$$

then for any $l > l_0$, $\mathcal{L}_n^l\mathcal{X} = \emptyset$. So the disjoint sum is indeed a finite sum.

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3.3 The formal neighborhood of $I(\mathcal{X})$ and its canonical automorphism

Let $\mathcal{X}$ be a smooth Deligne–Mumford stack with $x : \operatorname{Spec} k \to \mathcal{X}$ its closed point. Then the tangent space $T_x\mathcal{X}$ is defined to be T_vM for an atlas $M \to \mathcal{X}$ and a lift $v : \operatorname{Spec} k \to \mathcal{X}$ of x, uniquely determined up to unique isomorphism. Then $\operatorname{Aut}(x)$ naturally acts on $T_x\mathcal{X}$. We now globalize it.

Let $\mathcal{Y}$ be a connected component of the inertia stack $I(\mathcal{X})$ with $F : \mathcal{Y} \to \mathcal{X}$ being the forgetting map. Let $\mathcal{X}_0$ be the image of $\mathcal{Y}$ by F. The completion $\hat{\mathcal{O}}_{\mathcal{X}}$ of $\mathcal{O}_{\mathcal{X}}$ along $\mathcal{X}_0$ is considered to be an $\mathcal{O}_{\mathcal{X}_0}$-algebra. Then we define a coherent sheaf $\mathcal{A}$ on $\mathcal{Y}$ to be the pull-back of $\hat{\mathcal{O}}_{\mathcal{X}}$ by $F : \mathcal{Y} \to \mathcal{X}_0$.

Definition 3.7. We define $\mathcal{N} := \mathcal{S}\mathrm{pec}\,\mathcal{A}$ and call it the *formal neighborhood* of $\mathcal{Y}$.

If we set $\hat{\mathcal{X}} := \mathcal{S}\mathrm{pec}\,\hat{\mathcal{O}}_{\mathcal{X}}$ where we consider $\hat{\mathcal{O}}_{\mathcal{X}}$ to be an $\mathcal{O}_{\mathcal{X}}$-algebra, there is a natural morphism $\hat{\mathcal{X}} \to \mathcal{X}$ which is $\mathcal{X}_0$-étale. Since $\mathcal{Y} \to \mathcal{X}_0$ is unramified and flat, it is étale. Hence the natural morphism $\mathcal{N} \to \hat{\mathcal{X}}$ is also étale and the composition $\mathcal{N} \to \hat{\mathcal{X}} \to \mathcal{X}$ is $\mathcal{Y}$-étale.

Let U and V be varieties. Let $\xi : U \to \mathcal{X}$ be an étale morphism, $\nu : V \to U$ a morphism and α an automorphism of $\xi \circ \nu$. Suppose that $(\xi \circ \nu, \alpha) : V \to \mathcal{Y}$ is étale.

$$\begin{array}{ccc} V & \xrightarrow{\nu} & U \\ {\scriptstyle (\xi\circ\nu,\alpha)}\downarrow & & \downarrow{\scriptstyle \xi} \\ \mathcal{Y} & \longrightarrow & \mathcal{X} \end{array}$$

Then we obtain the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\nu} & U \\ {\scriptstyle \nu}\downarrow & \overset{\alpha}{\swarrow\!\!\swarrow} & \downarrow{\scriptstyle \xi} \\ U & \xrightarrow[\xi]{} & \mathcal{X} \end{array}$$

where '$\Rightarrow$' denotes a two-morphism. Let $\tilde{\alpha} : V \to U \times_{\mathcal{X}} U$ be the corresponding morphism.

$$\begin{array}{ccc} & & U \times_{\mathcal{X}} U \\ & \overset{\tilde{\alpha}}{\nearrow} & {\scriptstyle \mathrm{pr}_1}\downarrow\;\downarrow{\scriptstyle \mathrm{pr}_2} \\ V & \xrightarrow[\nu]{} & U \end{array}$$

If $\hat{\mathcal{O}}_U$ is the completion of $\mathcal{O}_U$ along $\nu(V)$, then $\mathcal{A}_V = \nu^*\hat{\mathcal{O}}_U$. We have a canonical automorphism of $\mathcal{A}_V$ as follows:

$$\mathcal{A}_V \xrightarrow{\mathrm{pr}_2^*} \tilde{\alpha}^*\hat{\mathcal{O}}_{U\times_{\mathcal{X}}U} \xrightarrow{(\mathrm{pr}_1^*)^{-1}} \mathcal{A}_V$$

and hence a canonical automorphism of $\mathcal{A}$ and $\mathcal{N}$. Now this automorphism of $\mathcal{N}$ is considered to be a globalization of the action on $T_x\mathcal{X}$ mentioned above.

3.4 Shift number and orbifold cohomology

Suppose that $\mathcal{Y}$ is contained in $I^l(\mathcal{X})$ for an integer $l \geqslant 1$. Let (x, α) be a closed point of $\mathcal{Y}$ where x is a closed point of $\mathcal{X}$ and $\alpha \in \operatorname{Aut}(x)$. Then α acts on the tangent space $T_x\mathcal{X}$. For a suitable basis, this automorphism is given by the diagonal matrix

$$\operatorname{diag}(\zeta_l^{a_1}, \ldots, \zeta_l^{a_d})$$

with $1 \leqslant a_j \leqslant l$ and $d = \dim \mathcal{X}$.

Definition 3.8. We define the *shift number* of $\mathcal{Y}$ by

$$s(\mathcal{Y}) := \dim \mathcal{X} - \frac{1}{l}\sum_{j=1}^{d} a_j = \frac{1}{l}\sum_{j=1}^{d}(l - a_j).$$

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This is determined by the rank of the eigenbundles of $\mathcal{N}$ for the canonical action. Hence it depends only on $\mathcal{Y}$.

Suppose that the coarse moduli space $X = \overline{\mathcal{X}}$ is a variety with Gorenstein quotient singularities and $\mathcal{X}$ has no reflections. Then the matrix $\mathrm{diag}(\zeta_l^{a_1}, \ldots, \zeta_l^{a_d})$ is in $\mathrm{SL}_d(\mathbb{C})$ (see [Wat74]). Hence $s(\mathcal{Y})$ is an integer.

Now, let us define the orbifold cohomology.

Definition 3.9. Assume X is complete. Then we define the *ith orbifold cohomology group* with a Hodge structure as follows:

$$H^i_{\mathrm{orb}}(X, \mathbb{Q}) := \bigoplus_{\mathcal{Y}} H^{i-2s(\mathcal{Y})}(\overline{\mathcal{Y}}, \mathbb{Q}) \otimes \mathbb{Q}(-s(\mathcal{Y})),$$

where $\mathcal{Y}$ runs over the connected components of $I(\mathcal{X})$ and $\mathbb{Q}(-s(\mathcal{Y}))$ is a Tate twist $\mathbb{Q}(-1)^{\otimes s(\mathcal{Y})}$.

Since the natural morphism $\overline{\mathcal{Y}} \to X$ is quasi-finite, $\overline{\mathcal{Y}}$ is a scheme (see [LM00, Théorème A.2]). Because of this and Corollary 4.23, $\overline{\mathcal{Y}}$ is a complete variety with quotient singularities. Therefore the rational cohomology groups of $\mathcal{Y}$ have *pure* Hodge structures. For the projective case see [Dan78, Corollary 14.4]. For the general case, it follows from the following two facts: one is that the intersection cohomology of every complete variety has pure Hodge structures [Sai90], the other is that the rational cohomology of a variety with quotient singularities equals the intersection cohomology.

3.5 The motivic measure on twisted ∞-jets

Let $\mathcal{X}$ be a smooth Deligne–Mumford stack of pure dimension d. By abusing the notation, we also denote by $\overline{\mathcal{L}_\infty \mathcal{X}}$ the set of points $|\mathcal{L}_\infty \mathcal{X}| = |\overline{\mathcal{L}_\infty \mathcal{X}}|$. We denote by π_n the natural morphism $\overline{\mathcal{L}_\infty \mathcal{X}} \to \overline{\mathcal{L}_n \mathcal{X}}$.

Definition 3.10. A subset A of $\overline{\mathcal{L}_\infty \mathcal{X}}$ is *stable at level n* if we have:

i) $\pi_n(A)$ is a constructible subset in $\overline{\mathcal{L}_n \mathcal{X}}$;
ii) $A = \pi_n^{-1}\pi_n(A)$.

A subset $A \subset \overline{\mathcal{L}_\infty \mathcal{X}}$ is *stable* if it is stable at level n for some $n \in \mathbb{Z}_{\geqslant 0}$.

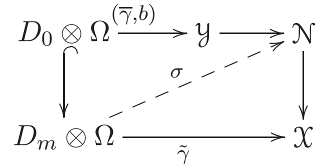
We define the notion of the *measurable subset* similarly. Then we define *motivic measure* $\mu_{\mathcal{X}}$ on $\overline{\mathcal{L}_\infty \mathcal{X}}$ by

$$\mu_{\mathcal{X}}(A) := \mathbb{L}^{-nd}\chi_h(\pi_n(A)) \in \hat{K}_0(\mathrm{HS}),\ n \gg 0,$$

where $\mathbb{L} = \{\mathbb{Q}(-1)\} = \chi_h(\mathbb{A}^1)$. It is well defined by the following proposition.

Proposition 3.11. *Let $n \in \mathbb{Z}_{\geqslant 0}$, let $B \subset \overline{\mathcal{L}_n \mathcal{X}}$ be a constructible subset and let C be the inverse image of B by the natural morphism $\overline{\mathcal{L}_{n+1} \mathcal{X}} \to \overline{\mathcal{L}_n \mathcal{X}}$. Then we have the equality $\chi_h(C) = \mathbb{L}^d \chi_h(B)$.*

Proof. Let $\mathcal{Y}$ be a connected component of $I^l(\mathcal{X})(= \mathcal{L}_0^l \mathcal{X})$ and put $(\mathcal{L}_n \mathcal{X})_{\mathcal{Y}} := \mathcal{L}_n \mathcal{X} \times_{I(\mathcal{X})} \mathcal{Y}$. Let $\gamma = (\tilde{\gamma}, (\overline{\gamma}, b), \mathrm{id}_{\overline{\gamma}}) \in (\mathcal{L}_n \mathcal{X})_{\mathcal{Y}} \subset L_m \mathcal{X} \times_{\mathcal{X}} \mathcal{Y}$ be an Ω-point. Then we have the following commutative diagram of solid arrows:



Since $\mathcal{N} \to \mathcal{X}$ is $\mathcal{Y}$-étale, there is a unique broken arrow σ that fits into the diagram. Sending γ to σ determines a closed immersion

$$\iota : (\mathcal{L}_n \mathcal{X})_{\mathcal{Y}} \hookrightarrow L_m \mathcal{N}.$$

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Let $\mathfrak{g}$ be the canonical automorphism of $\mathcal{N}$. In view of the definition of $\mathcal{N}$ and $\clubsuit$ in the proof of Lemma 3.5, we see that for $\sigma \in L_m\mathcal{N}$, $\sigma \in \mathrm{Im}(\iota)$ if and only if $\sigma \circ \zeta_l = \mathfrak{g} \circ \sigma$. From Lemma 3.12, there exists an atlas $h : V \to \mathcal{Y}$ such that $\mathcal{N}_V := \mathcal{N} \times_{\mathcal{Y}} V \cong V \otimes \mathbb{C}[[v_1, \dots, v_c]]$ ($c = \dim \mathcal{X} - \dim \mathcal{Y}$), and the pull-back of $\mathfrak{g}$ is given by $\mathrm{diag}(\zeta_l^{a_1}, \dots, \zeta_l^{a_c})$, $1 \leqslant a_i \leqslant l$. So for an m-jet δ on $\mathcal{N}_V$, $h \circ \delta$ is in $\mathrm{Im}(\iota)$ if and only if the image of δ by $L_m(\mathcal{N}_V) \to L_m V$ is ζ_l-invariant and $\delta^*(v_i)$ is of the following form:

$$r_0 t^{a_i} + r_1 t^{a_i+l} + r_2 t^{a_i+2l} + \cdots .$$

Therefore, we have $(\mathcal{L}_n\mathcal{X})_{\mathcal{Y}} \times_{\mathcal{Y}} V \cong L_n V \times \mathbb{A}^{nc}$ and the projection $(\mathcal{L}_{n+1}\mathcal{X})_{\mathcal{Y}} \times_{\mathcal{Y}} V \to (\mathcal{L}_n\mathcal{X})_{\mathcal{Y}} \times_{\mathcal{Y}} V$ is a Zariski locally trivial $\mathbb{A}^d$-bundle.

We may assume that a finite group, say G, acts on each connected component V' of V, $V' \to \overline{\mathcal{Y}}$ is G-invariant and the induced morphism $V'/G \to \overline{\mathcal{Y}}$ is étale (see Lemma 4.26). Then we have $\overline{(\mathcal{L}_n\mathcal{X})_{\mathcal{Y}}} \times_{\overline{\mathcal{Y}}} (V'/G) \cong ((L_n V')/G) \times \mathbb{A}^{nc}$. Therefore, from Lemma 3.14, to prove the proposition, it suffices to show that there is a stratification of $(L_n V')/G$ such that the natural morphism $(L_{n+1}V')/G \to (L_n V')/G$ is, over each stratum, an analytically locally trivial fibration of the quotient of an affine space by a linear finite group action.

Let H be a subgroup of G and $W \subset V'$ a connected component of the locus of the points with stabilizer H. Let $w \in W$ be a close point. As is well known, there is a representation $\rho : H \subset \mathrm{GL}_{\dim \mathcal{Y}}(\mathbb{C})$ which describes the H-action on an analytic neighborhood of w. Let $(L_n V)_w \subset L_n V$ be the subset of the jets which maps the only point of D_n to w. Then the induced H-action on $(L_n V)_w \cong \mathbb{A}^{m \dim \mathcal{Y}}$ is given by $\rho^{\oplus m}$. Therefore $(L_{n+1}V')_W/H \to (L_n V')_W/H$ is an analytically locally trivial fibration of $\mathbb{A}^{\dim \mathcal{Y}}/H$. Let $G' \subset G$ be the subgroup of the elements keeping W stable. Then H is a normal subgroup of G'. It is easy to see that the image of $(L_n V')_W/H$ in $(L_n V')/G$ is naturally isomorphic to $((L_n V')_W/H)/(G'/H)$. Since G'/H freely acts on $(L_n V')_W/H$, the assertion follows. $\square$

Lemma 3.12. *Let G be a finite group, V a smooth G-variety and W a smooth closed subvariety consisting of G-invariant points.*

i) *Assume that V and W are affine, say $V = \mathrm{Spec}\, R$ and $W = \mathrm{Spec}\, R/\mathfrak{p}$. Moreover assume that $\mathfrak{p}$ is generated by $c = \mathrm{codim}(W, V)$ elements. Then the completion of V along W is isomorphic as G-schemes to $\mathrm{Spec}(R/\mathfrak{p})[[x_1, \dots, x_c]]$, $G \subset \mathrm{GL}_c(R/\mathfrak{p})$.*

ii) *Assume that G is a finite cyclic group. Then there is an affine open covering $\cup V_i$ of V such that for every i, if we write $V_i = \mathrm{Spec}\, R$ and $W \cap V_i = \mathrm{Spec}\, R/\mathfrak{p}$, the completion of V_i along $W \cap V_i$ is isomorphic as G-schemes to $\mathrm{Spec}(R/\mathfrak{p})[[x_1, \dots, x_c]]$, $G \subset \mathrm{GL}_c(\mathbb{C})$.*

Proof. i) We denote by $\hat{R}$ the completion of R with respect to an ideal $\mathfrak{p}$ and by $\widehat{R_{\mathfrak{p}}}$ the completion of the local ring $R_{\mathfrak{p}}$ with respect to the maximal ideal. Let K be the quotient field of $R/\mathfrak{p}$ and f_i generators of $\mathfrak{p}$. It is well known that there is an isomorphism $\widehat{R_{\mathfrak{p}}} \to K[[x_1, \dots, x_c]]$ sending f_i to x_i. We have a natural injection $\hat{R} \to \widehat{R_{\mathfrak{p}}} \cong K[[x_1, \dots, x_c]]$. Clearly the image contains the subring $(R/\mathfrak{p})[[x_1, \dots, x_c]]$. Consider the injection $\iota : (R/\mathfrak{p})[[x_1, \dots, x_c]] \to \hat{R}$. Since the induced map $R/\mathfrak{p} \to \hat{R}/\hat{\mathfrak{p}}$ is the identity and the images f_i of x_i generate $\hat{\mathfrak{p}}$, ι is a surjection and hence an isomorphism (see [Eis95, Theorem 7.16.]).

Consider the induced G-action on $\hat{R} = (R/\mathfrak{p})[[x_1, \dots, x_c]]$. For $g \in G$, write $g(x_i) = \sum a_{ij} x_j +$ (higher terms), $a_{ij} \in R/\mathfrak{p}$. Denote by $\bar{g}$ the endomorphism of $\hat{R}$ associated to the invertible matrix (a_{ij}). Working in characteristic zero, we can isomorphically replace x_i with $x_i' = \sum_{g \in G} \bar{g}^{-1} g(x_i)$. We find that, with respect to the new coordinates, the G-action is linear.

ii) In the previous situation, $\mathrm{Spec}\, \hat{R}$ is naturally isomorphic to the completion of the normal bundle $N_{W/V}$ along the zero section. Here let us assume G is a cyclic group with generator g. Then $N_{W/V}$ decomposes to eigenbundles. On each eigenbundle, the g-action is uniquely represented by a scalar matrix aI where I is the identity matrix and $a \in \mathbb{C}$. Therefore, by shrinking V to an open subset where the eigenbundles are free, we conclude that the G-actions on $N_{W/V}$ and $\operatorname{Spec} \hat{R}$ are realizable in $\mathbb{C}$. □

Remark 3.13. The author guesses that even in the case of a general finite group, the action on $N_{W/V}$ is *étale* locally realizable in $\mathbb{C}$. From facts on splitting fields of finite groups (see [CR88]), this is true at least over the generic point of W.

Lemma 3.14. *Let T and S be varieties and $f : T \to S$ an analytically locally trivial fibration of $\mathbb{A}^d/G$ for a finite group $G \subset \mathrm{GL}_d(\mathbb{C})$. Then $\chi_h(T) = \chi_h(S)\mathbb{L}^d$.*

Proof. Since the fiber is a quotient of an affine space, the higher direct images of $\mathbb{Q}_T$ vanishes

$$R^i f_* \mathbb{Q}_T \cong \begin{cases} \mathbb{Q}_S & (i = 0), \\ 0 & (i > 0). \end{cases}$$

Hence the spectral sequence is degenerate and it follows that $H^i(T, \mathbb{Q}) \cong H^i(S, \mathbb{Q})$ for every i.

Taking a stratification of S, we may assume that S is smooth. Since S and T have at most quotient singularities (in the analytic sense), by Poincaré duality we conclude that $H_c^{\dim T - i}(T, \mathbb{Q}) \cong H_c^{\dim S - i}(S, \mathbb{Q})$.

By regarding the sheaves as mixed Hodge modules, as studied by Saito [Sai90] (see also [Sai89]), we can regard these isomorphisms of cohomology groups as those of mixed Hodge structures. This implies the assertion. □

3.6 Main theorem

Let X be a variety with Gorenstein quotient singularities. Then X has canonical singularities. Let $\mathcal{X}$ be a smooth Deligne–Mumford stack without reflections such that X is the coarse moduli space of $\mathcal{X}$. We denote by λ the canonical morphism $\mathcal{X} \to X$. If $\gamma : \mathcal{D}_n^l \otimes \Omega \to \mathcal{X}$ is a twisted n-jet on $\mathcal{X}$ of order l, then it induces a morphism $\gamma' : D_n \otimes \Omega \to X$ of the coarse moduli spaces. We define the map $\lambda_{(n)} : \overline{\mathcal{L}_n \mathcal{X}} \to L_n X$ by $\gamma \mapsto \gamma'$. The following theorem is our main result.

Theorem 3.15. *Let $B \subset \overline{\mathcal{L}_\infty \mathcal{X}}$ be a measurable subset and put $A := \lambda_{(\infty)}(B)$. Then we have the following equation in $\hat{K}_0(\mathrm{HS})$:*

$$\chi_h \mu_X^{\mathrm{Gor}}(A) = \sum_{\mathcal{Y} \subset I(\mathcal{X})} \mathbb{L}^{s(\mathcal{Y})} \mu_{\mathcal{X}}(\pi_0^{-1}(\overline{\mathcal{Y}}) \cap B),$$

where $\mathcal{Y}$ runs over the connected components of $I(\mathcal{X})$.

The proof is postponed until the end of the section.

Corollary 3.16. *Let X and X' be complete varieties with Gorenstein quotient singularities. Suppose that there are proper birational morphisms $Z \to X$ and $Z \to X'$ such that $K_{Z/X} = K_{Z/X'}$. Then the orbifold cohomology groups of X and X' have the same Hodge structure.*

Proof. By Theorem 3.15 and Proposition 3.11, we have

$$\begin{aligned} \chi_h \mu_X^{\mathrm{Gor}}(L_\infty X) &= \sum_{\mathcal{Y} \subset I(\mathcal{X})} \mathbb{L}^{s(\mathcal{Y})} \chi_h(\overline{\mathcal{Y}}) \\ &= \sum_i (-1)^i \{H_{\mathrm{orb}}^i(X, \mathbb{Q})\}. \end{aligned}$$

From Lemma 2.16, we have $\sum_i (-1)^i \{H_{\mathrm{orb}}^i(X, \mathbb{Q})\} = \sum_i (-1)^i \{H_{\mathrm{orb}}^i(X', \mathbb{Q})\}$. Since $H_{\mathrm{orb}}^i(X, \mathbb{Q})$ and $H_{\mathrm{orb}}^i(X', \mathbb{Q})$ have a pure Hodge structure of weight i, $\{H_{\mathrm{orb}}^i(X, \mathbb{Q})\} = \{H_{\mathrm{orb}}^i(X', \mathbb{Q})\}$ for every i. □

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Lemma 3.17. *Let X_{sing} denote the singular locus of X with a reduced subscheme structure.*

i) *The subset $\lambda_{(\infty)}^{-1}(L_\infty(X_{\mathrm{sing}}))$ of $\overline{\mathcal{L}_\infty \mathcal{X}}$ is of measure zero.*

ii) *The map $\lambda_{(\infty)}$ is bijective over $L_\infty X \setminus L_\infty(X_{\mathrm{sing}})$.*

Proof. i) It suffices to show that for every n, $\pi_n(\overline{\mathcal{L}_\infty \mathcal{X}} \setminus \lambda_{(\infty)}^{-1}(L_\infty(X_{\mathrm{sing}}))) = \overline{\mathcal{L}_n \mathcal{X}}$. But this is clear by the local description of $\overline{\mathcal{L}_n \mathcal{X}}$ in the proof of Proposition 3.11.

ii) *Surjectivity.* Let $\eta : \operatorname{Spec}\Omega[[t]] \to X$ be an Ω-point of $L_\infty X \setminus L_\infty(X_{\mathrm{sing}})$ with an algebraically closed field Ω. We define $\mathcal{D}$ to be the normalization of the fiber product $\mathcal{X} \times_X \operatorname{Spec}\Omega[[t]]$ (see Definition 4.13). Then the Deligne–Mumford stack $\mathcal{D}$ contains the scheme $\operatorname{Spec}\Omega((t))$ as an open substack. Therefore, the coarse moduli space $\overline{\mathcal{D}}$ of $\mathcal{D}$ contains $\operatorname{Spec}\Omega((t))$ as an open subscheme. The scheme $\overline{\mathcal{D}}$ must be the spectrum of a local ring $S \subset \Omega((t))$. From the universality of coarse moduli space, there is a natural morphism $\overline{\mathcal{D}} \to \operatorname{Spec}\Omega[[t]]$. So we have $\Omega[[t]] \subset S \subset \Omega((t))$. As $\Omega[[t]]$ and $\Omega((t))$ are the only intermediate rings between $\Omega[[t]]$ and $\Omega((t))$, the ring S must be $\Omega[[t]]$. Suppose that $E = \operatorname{Spec}\tilde{S}$ is an atlas of $\mathcal{D}$ and $\tilde{S}$ is a regular local ring. Since $S = \Omega[[t]]$ is henselian, the natural morphism $E \to \overline{\mathcal{D}}$ is finite [Gro, Theorem 18.5.11]. Hence $\tilde{S}$ is complete (see [Eis95, Corollary 7.6]). So $\tilde{S} \cong \Omega[[t]]$. Consider the groupoid space $E \times_{\mathcal{D}} E \rightrightarrows E$. The scheme $E \times_{\mathcal{D}} E$ must be the disjoint sum of spectra of complete regular local rings. Since the first projection $\mathrm{pr}_1 : E \times_{\mathcal{D}} E \to E$ is étale, there is an isomorphism

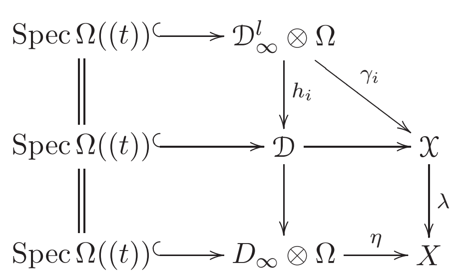
$$E \coprod \cdots \coprod E \cong E \times_{\mathcal{D}} E$$

such that the composition

$$E \coprod \cdots \coprod E \cong E \times_{\mathcal{D}} E \xrightarrow{\mathrm{pr}_1} E$$

is isomorphic on each component. If l denotes the number of the components in $E \times_{\mathcal{D}} E$, then the second projection $\mathrm{pr}_2 : E \times_{\mathcal{D}} E \to E$ determines the action of some group G on E with $|G| = l$. Since this action is effective, the group G is isomorphic to μ_l for some l. For a suitable isomorphism $\mu_l \cong G$, the action is given by $t \mapsto \zeta_l t$. Hence the stack $\mathcal{D}$ is isomorphic to $\mathcal{D}_\infty^l \otimes \Omega$ and the morphism $\mathcal{D}_\infty^l \otimes \Omega \cong \mathcal{D} \to \mathcal{X}$ is a twisted ∞-jet on $\mathcal{X}$. The image of this twisted ∞-jet by $\lambda_{(\infty)}$ is η.

Injectivity. Let $\gamma_1, \gamma_2 : \mathcal{D}_\infty^l \otimes \Omega \to \mathcal{X}$ be two twisted ∞-jets on $\mathcal{X}$ of order l. We suppose that $\eta := \lambda_{(\infty)}(\gamma_1) = \lambda_{(\infty)}(\gamma_1)$ and $\eta \in L_\infty X \setminus L_\infty(X_{\mathrm{sing}})$. Construct $\mathcal{D}$ from η as above. Then for each $i \in \{1, 2\}$, there is a unique morphism $h_i : \mathcal{D}_\infty^l \otimes \Omega \to \mathcal{D}$ such that the following diagram is commutative:



Let E be an atlas of $\mathcal{D}$ as above. Then the natural morphism $E \times_{\mathcal{D}, h_i} (\mathcal{D}_\infty^l \otimes \Omega) \to E$ is a birational morphism of smooth one-dimensional schemes. Therefore it is an isomorphism and so is h_i (see [LM00, Proposition 3.8.1]). Then we can easily see that γ_1 and γ_2 have the same image in $|L_\infty \mathcal{X} \times_{\mathcal{X}} I^l(\mathcal{X})|$. □

To prove Theorem 3.15, we now need to generalize the transformation rule. Let $\mathcal{V}$ be a Deligne–Mumford stack over D_∞ of pure relative dimension d. For each $n \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$, we define $\mathcal{V}_n$ to be the moduli stack of the D_∞-morphisms $D_n \to \mathcal{V}$. Then for $m \geqslant n$, there is a natural projection $\mathcal{V}_m \to \mathcal{V}_n$. So we can define the *motivic measure* $\mu_{\mathcal{V}}$ over $\overline{\mathcal{V}_\infty}$ which takes values in $\hat{K}_0(\mathrm{HS})$, in a similar fashion as before. (We should replace condition iii in Definition 2.7 with the condition that $\chi_h(\pi_{m+1}A) = \mathbb{L}^d \chi_h(\pi_m A)$. It makes sense because of Lemma 3.14.)

Let $\mathcal{W}$ be another Deligne–Mumford stack over D_∞ of pure relative dimension d and let $h : \mathcal{W} \to \mathcal{V}$ be a D_∞-morphism. We put $\Omega'_{\mathcal{W}/\mathcal{V}} := \mathrm{Im}(\Omega_{\mathcal{W}/D_\infty} \setminus (\mathrm{tors}) \to \Omega_{\mathcal{W}/\mathcal{V}})$, where $(\mathrm{tors}) \subset \Omega_{\mathcal{W}/D_\infty}$ is the torsion. Then we define the *jacobian ideal sheaf* $\mathcal{J}_h$ of h to be the zeroth Fitting ideal of $\Omega'_{\mathcal{W}/\mathcal{V}}$.

Theorem 3.18. *Let $A \subset \overline{\mathcal{W}_\infty}$ be a measurable set. Suppose that $h_\infty|_A : A \to \overline{\mathcal{V}_\infty}$ is injective. Let ν be a measurable function on $h_\infty(A)$. Then*

$$\int_{h_\infty(A)} \mathbb{L}^{\nu}\, d\mu_{\mathcal{V}} = \int_A \mathbb{L}^{\nu \circ h_\infty - \mathrm{ord}\, \mathcal{J}_h}\, d\mu_{\mathcal{W}}.$$

Proof. It is a direct consequence of Lemma 3.19. □

We denote by $\mathcal{J}(\mathcal{V}/D_\infty)$ (respectively $\mathcal{J}(\mathcal{W}/D_\infty)$) the dth Fitting ideal sheaf of $\Omega_{\mathcal{V}/D_\infty}$ (respectively $\Omega_{\mathcal{W}/D_\infty}$).

Lemma 3.19. *Let $A \subset \overline{\mathcal{W}_\infty}$ be a stable subset of level l. Assume that $h_\infty|_A$ is injective, that $\mathrm{ord}\, \mathcal{J}_h$ is constant and equal to $e < \infty$ and that $\mathrm{ord}\, \mathcal{J}(\mathcal{V}/D_\infty)$ and $\mathrm{ord}\, \mathcal{J}(\mathcal{W}/D_\infty)$ are bounded from above on $h_\infty(A)$ and A, respectively. Then for $n \gg 0$, $h_n : \pi_n A \to h_n \pi_n A$ is a piecewise trivial $\mathbb{A}^e$-bundle.*

Proof. Looijenga's proof [Loo02, Lemma 9.2] also works in this setting.

Take a non-twisted ∞-jet $\gamma : \mathrm{Spec}\, \Omega[[t]] \to \mathcal{W}$ in A, and put $\mathfrak{m} := (t) \subset \Omega[[t]]$. Let q be the image of the closed point by γ. Take another $\theta \in A$ such that $\pi_{n-e}(\gamma) = \pi_{n-e}(\theta)$. Then the morphism

$$\theta^* - \gamma^* : \mathcal{O}_{\mathcal{W},q} \to \mathfrak{m}^{n-e+1}/\mathfrak{m}^{2(n-e+1)}$$

is a $\mathbb{C}[[t]]$-derivation. So it defines an $\Omega[[t]]$-module homomorphism

$$\partial\theta : \gamma^* \Omega_{\mathcal{W}/D_\infty} \to \mathfrak{m}^{n-e+1}/\mathfrak{m}^{2(n-e+1)}.$$

The length of the torsion of $\gamma^* \Omega_{\mathcal{W}/D_\infty}$ equals $(\mathrm{ord}\, \mathcal{J}(\mathcal{W}/D_\infty))(\gamma)$ and hence it is bounded. So, since $n \gg 0$, the composition map

$$\overline{\partial\theta} : \gamma^* \Omega_{\mathcal{W}/D_\infty} \xrightarrow{\partial\theta} \mathfrak{m}^{n-e+1}/\mathfrak{m}^{2(n-e+1)} \to \mathfrak{m}^{n-e+1}/\mathfrak{m}^{n+1}$$

annihilates the torsion. Conversely, every $\Omega[[t]]$-module homomorphism $\gamma^* \Omega_{\mathcal{W}/D_\infty} \to \mathfrak{m}^{n-e+1}/\mathfrak{m}^{n+1}$ that annihilates the torsion is $\overline{\partial\theta}$ for some θ.

After some work, we can see that if $\theta \in A$ is such that $h_n \pi_n(\gamma) = h_n \pi_n(\theta)$ then $\pi_{n-e}(\gamma) = \pi_{n-e}(\theta)$ (see [Loo02, Lemma 9.2]). So $\overline{\partial\theta}$ is defined. It is easy to see that $\pi_n(\theta_1) = \pi_n(\theta_2)$ if and only if $\overline{\partial\theta_1} = \overline{\partial\theta_2}$ and that $h_n\pi_n(\theta_1) = h_n\pi_n(\theta_2)$ if and only if $\overline{\partial\theta_1}$ and $\overline{\partial\theta_2}$ have the same image in $\mathrm{Hom}_{\Omega[[t]]}((h\gamma)^* \Omega_{\mathcal{V}/D_\infty}, \mathfrak{m}^{n-e+1}/\mathfrak{m}^{n+1})$. Hence $h_n^{-1} h_n \pi_n(\gamma)$ is isomorphic to an affine space

$$\begin{aligned}\mathrm{Hom}_{\Omega[[t]]}(\gamma^* \Omega'_{\mathcal{W}/\mathcal{V}}, \mathfrak{m}^{n-e+1}/\mathfrak{m}^{n+1}) \cong \mathrm{Ker}(&\mathrm{Hom}_{\Omega[[t]]}(\gamma^* \Omega_{\mathcal{W}/D_\infty} \setminus (\mathrm{tors}), \mathfrak{m}^{n-e+1}/\mathfrak{m}^{n+1}) \\ &\to \mathrm{Hom}_{\Omega[[t]]}((h\gamma)^* \Omega_{\mathcal{V}/D_\infty}, \mathfrak{m}^{n-e+1}/\mathfrak{m}^{n+1})).\end{aligned}$$

The length of $\gamma^* \Omega'_{\mathcal{W}/\mathcal{V}}$ equals $e = \mathrm{ord}\, \mathcal{J}_h(\gamma)$. So $h_n^{-1} h_n \pi_n(\gamma)$ is isomorphic to an affine space of dimension e.

The rest is easy. □

Proof of Theorem 3.15. Let $\mathcal{Y}$ be a connected component of $I^l(\mathcal{X})$ and $\mathcal{N}$ its formal neighborhood. We may assume that B is contained in $\pi_0^{-1}(\overline{\mathcal{Y}})$. Let $\tilde{\mathcal{N}}$ be the quotient of $\mathcal{N}$ by the canonical automorphism $\mathfrak{g}$, that is, $\mathcal{S}\mathrm{pec}\, \mathcal{A}^{\mathfrak{g}}$ where $\mathcal{A}^{\mathfrak{g}} \subset \mathcal{A}$ is the subsheaf of the $\mathfrak{g}$-invariant sections. Then the natural morphism $\mathcal{N} \to X$ factors as

$$\mathcal{N} \to \tilde{\mathcal{N}} \xrightarrow{f} X.$$

In the proof of Proposition 3.11, we saw that for each n, m with $m = nl$, there is a closed immersion $\iota : (\mathcal{L}_n\mathcal{X})_{\mathcal{Y}} \hookrightarrow L_m\mathcal{N}$. Let $[l] : D_m \to D_n$ be the morphism associated to the ring homomorphism defined by $t \mapsto t^l$. If $\gamma : \mathcal{D}_n^l \otimes \Omega \to \mathcal{X}$ is a twisted n-jet in $\pi_n(B)$, then $\iota(\gamma)$ fits into the diagram

$$\begin{array}{ccc} D_m \otimes \Omega & \xrightarrow{\iota(\gamma)} & \mathcal{N} \\ {\scriptstyle [l]}\downarrow & & \downarrow \\ D_n \otimes \Omega & \xrightarrow[\sigma]{} & \tilde{\mathcal{N}} \end{array} \tag{3}$$

Then $\lambda_{(n)}(\gamma) = f_n(\sigma)$. We define a subset $\tilde{B} \subset \overline{L_\infty\tilde{\mathcal{N}}}$ to be the image of B by the map $\gamma \mapsto \sigma$. Then $A = f_\infty(\tilde{B})$ and $f_\infty|_{\tilde{B}}$ is bijective outside of measure zero subsets. Let $\mathcal{I}_X$ (respectively $\mathcal{I}_{\tilde{\mathcal{N}}}$) be the ideal sheaf on X defined by

$$\mathcal{I}_X\omega_X = \mathrm{Im}(\Omega_X^d \to \omega_X)$$
$$(\text{respectively } \mathcal{I}_{\tilde{\mathcal{N}}}\omega_{\tilde{\mathcal{N}}} = \mathrm{Im}(\Omega_{\tilde{\mathcal{N}}}^d \to \omega_{\tilde{\mathcal{N}}}))$$

and define μ_X^{Gor} and $\mu_{\tilde{\mathcal{N}}}^{\mathrm{Gor}}$ to be $\mathbb{L}^{\mathrm{ord}\,\mathcal{I}_X}\mu_X$ and $\mathbb{L}^{\mathrm{ord}\,\mathcal{I}_{\tilde{\mathcal{N}}}}\mu_{\tilde{\mathcal{N}}}$ respectively. Since the morphism f has no ramification divisor, by a similar argument as in the proof of Lemma 2.16 we see that $f^{-1}\mathcal{I}_X = \mathcal{J}_f \cdot \mathcal{I}_{\tilde{\mathcal{N}}}$, where $\mathcal{J}_f$ is the jacobian ideal sheaf. So, by Theorem 3.18, we obtain

$$\mu_{\tilde{\mathcal{N}}}^{\mathrm{Gor}}(\tilde{B}) = \mu_X^{\mathrm{Gor}}(A).$$

We have thus reduced the problem to the case of a cyclic quotient; it suffices to show the following lemma. □

LEMMA 3.20. *Let the notation be as above. We have* $\mathbb{L}^{s(\mathcal{Y})}\mu_{\mathcal{X}}(B) = \chi_h\mu_{\tilde{\mathcal{N}}}^{\mathrm{Gor}}(\tilde{B})$.

Proof. The proof is essentially by a trick used in [DL02]. We first consider an easy case where $\mathcal{X}$ is a quotient stack $[\mathbb{A}_R^c/G]$ of an affine space over a ring R whose spectrum is a smooth variety of dimension $d-c$, and $G \subset \mathrm{SL}_c(\mathbb{C})$ a finite cyclic group of order l generated by $g = \mathrm{diag}(\zeta_l^{a_1}, \dots, \zeta_l^{a_c})$, $1 \leqslant a_i < l$. Suppose that $\mathcal{Y}$ is the component associated to g. Then $\mathcal{N} = \hat{\mathbb{A}}_R^c(= \mathrm{Spec}\, R[[x_1, \dots, x_c]])$, its canonical automorphism is $g = \mathrm{diag}(\zeta_l^{a_1}, \dots, \zeta_l^{a_c})$ and $\tilde{\mathcal{N}} = \hat{\mathbb{A}}_R^c/G$. Since the natural morphisms $\hat{\mathbb{A}}_R^c \to \mathbb{A}_R^c$ and $\hat{\mathbb{A}}_R^c/G \to \mathbb{A}_R^c/G$ are $(\mathrm{Spec}\, R)$-étale, and since we consider only jets which send the only closed point into $\mathrm{Spec}\, R$, it is easy to replace $\hat{\mathbb{A}}_R^c$, $\hat{\mathbb{A}}_R^c/G$ with $\mathbb{A}_R^c$, $\mathbb{A}_R^c/G$.

Consider three R-algebra homomorphisms:

i) $u^* : R[[t]][\underline{x}] \to R[[t]][\underline{x}]$, $x_i \mapsto t^{a_i}x_i$;

ii) $\alpha^* : R[[t]][\underline{x}] \to R[[t]][\underline{x}]$, $x_i \mapsto x_i$, $t \mapsto t^l$;

iii) $\beta^* : R[[t]][\underline{x}]^G \to R[[t]][\underline{x}]$, the composition of α^* and the inclusion $R[[t]][\underline{x}]^G \hookrightarrow R[[t]][\underline{x}]$.

Since $R[\underline{x}]^G$ is generated by the monomials $x_1^{m_1} \dots x_c^{m_c}$ with $\sum a_i m_i \equiv 0 \pmod{l}$, there is a $R[[t]]$-homomorphism $v^* : R[[t]][\underline{x}]^G \to R[[t]][\underline{x}]$ with $u^* \circ \beta^* = \alpha^* \circ v^*$, i.e.

$$\begin{array}{ccc} R[[t]][\underline{x}] & \xrightarrow{u^*} & R[[t]][\underline{x}] \\ {\scriptstyle \beta^*}\uparrow & & \uparrow{\scriptstyle \alpha^*} \\ R[[t]][\underline{x}]^G & \xrightarrow[v^*]{} & R[[t]][\underline{x}] \end{array}$$

Here the horizontal arrows are $R[[t]]$-algebra homomorphisms and the vertical ones send $t \mapsto t^l$.

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Write the diagram of the associated schemes as follows:

$$\begin{array}{ccc} \mathcal{N}[[t]] & \xleftarrow{u} & E_2[[t]] \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\alpha} \\ \tilde{\mathcal{N}}[[t]] & \xleftarrow[v]{} & E_1[[t]] \end{array}$$

where E_i are copies of $\mathbb{A}^c_R$.

Let B_0 be the image of B by $\iota : (\mathcal{L}_\infty \mathcal{X})_\mathcal{Y} \hookrightarrow L_\infty \mathcal{N}$. Then for $\gamma' \in B_0$, $(\gamma')^*(x_i)$ is of the form

$$(\gamma')^*(x_i) = r_0 t^{a_i} + r_1 t^{a_i+l} + r_2 t^{a_i+2l} + \cdots . \tag{4}$$

If we put $\eta := u_\infty^{-1}(\gamma')$, then we have

$$\eta^*(x_i) = r_0 + r_1 t^l + r_2 t^{2l} + \cdots .$$

Therefore if we define $\delta \in L_\infty E_1$ by

$$\delta^*(x_i) = r_0 + r_1 t + r_2 t^2 + \cdots , \tag{5}$$

then we have the following commutative diagram:

$$\begin{array}{ccc} E_2[[t]] & \xleftarrow{\eta} & D_\infty \otimes \Omega \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle[l]} \\ E_1[[t]] & \xleftarrow[\delta]{} & D_\infty \otimes \Omega \end{array}$$

Here $[l]$ is the morphism defined by $t \mapsto t^l$. Let $B_1 \subset L_\infty(E_1)$ be the image of B_0 by the map $\gamma' \mapsto \delta$. It is easy to see that if B is stable at level n, then so is B_1, and that $\{\pi_n(B)\} = \{\pi_{nl}(B_0)\} = \{\pi_n(B_1)\}\mathbb{L}^{-c}$, where π_i are truncation morphisms of $\overline{\mathcal{L}_\infty \mathcal{X}}$, $L_\infty \mathcal{N}$ and $L_\infty E_1$, respectively. Therefore

$$\chi_h \mu_{E_1}(B_1) = \mu_\mathcal{X}(B)\mathbb{L}^c. \tag{6}$$

Put $\sigma := v_\infty(\delta)$. The chain of the correspondences, $\gamma \mapsto \gamma' \mapsto \delta \mapsto \sigma$, defines a map $(\mathcal{L}_\infty \mathcal{X})_\mathcal{Y} \to L_\infty \tilde{\mathcal{N}}$, which is the same as in the proof of the theorem (see diagram (3) and compare it with the last two diagrams).

Shrinking $\operatorname{Spec} R$ to an open subset, suppose that the canonical sheaf $\omega_{\operatorname{Spec} R}$ of $\operatorname{Spec} R$ is generated by a section e'. Consider a d-form $e = dx_1 \wedge \cdots \wedge dx_c \wedge e'$ on $\mathcal{N}$. This is stable under the G-action. If r denotes the natural morphism $\tilde{\mathcal{N}} \to \mathcal{N}$, the canonical sheaf $\omega_{\tilde{\mathcal{N}}}$ of $\tilde{\mathcal{N}}$ is generated by a d-form $\tilde{e}$ with $r^*\tilde{e} = e$. Direct computation gives $v^*\tilde{e} = t^{\sum a_i/l}(dx_1 \wedge \cdots \wedge dx_c \wedge e')$. Hence we have the following equations for subsheaves of $\Omega^d_{E_1[[t]]/D_\infty}$:

$$\begin{aligned} (t^{\sum a_i/l}) v^{-1}\mathcal{J}_{\tilde{\mathcal{N}}} \cdot v^*\omega_{\tilde{\mathcal{N}}/D_\infty} &= (t^{\sum a_i/l})(v^*\Omega^d_{\tilde{\mathcal{N}}[[t]]/D_\infty})/(\text{tors}) \\ &= (t^{\sum a_i/l})\mathcal{J}_v \cdot \Omega^d_{E_1[[t]]/D_\infty} \\ &= \mathcal{J}_v \cdot v^*\omega_{\tilde{\mathcal{N}}/D_\infty}. \end{aligned}$$

This means that $\operatorname{ord} \mathcal{J}_{\tilde{\mathcal{N}}} \circ v_\infty - \operatorname{ord} \mathcal{J}_v \equiv -\sum a_i/l$. From the transformation rule, we obtain that $\chi_h \mu^{\mathrm{Gor}}_{\tilde{\mathcal{N}}}(A) = \mathbb{L}^{-\sum a_i/l}\mu_{E_1}(B_1)$, and using equation (6), that $\chi_h \mu^{\mathrm{Gor}}_{\tilde{\mathcal{N}}}(A) = \mathbb{L}^{s(\mathcal{Y})}\mu_\mathcal{X}(B)$. We have proved the assertion in this case.

As for the general case, the proof follows along almost the same lines: we take the fiber product $\mathcal{N} \times_\mathcal{Y} V$ for an atlas $V \to \mathcal{Y}$ to linearize the canonical automorphism. Define B_V, $B_{V,1}$ and $\tilde{B}_V$ in the evident fashion. Using the argument for the preceding case and replacing B, we find that $B_{V,1}$ and $\tilde{B}_V$ are stable at level n and a morphism $v_n : \pi_n B_{V,1} \to \pi_n \tilde{B}_V$ is a trivial affine space bundle of the expected relative dimension. Here we have used Lemma 3.19 instead of the transformation rule.

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The natural morphism $\pi_{n+1}B_V \to \pi_n B_{V,1}$ is an affine space bundle of relative dimension $d-c$ which is Zariski locally trivial on V. (Recall equations (4) and (5). This bundle results from the truncation $L_{n+1}(\operatorname{Spec} R) \to L_n(\operatorname{Spec} R)$ and the identity of $(r_i)_{0\leqslant i\leqslant n}$.) Hence $\pi_{n+1}B_V \to \pi_n\tilde{B}_V$ is also a Zariski locally trivial affine space bundle of the expected relative dimension. By the same argument as in the proof of Lemma 3.11, we can conclude that $\pi_{n+1}B \to \pi_n\tilde{B}$ is an analytically locally trivial fibration of a quotient of an affine space. The lemma follows from Lemma 3.14. □

4. General results on Deligne–Mumford stacks

In this section, we give some general results on Deligne–Mumford stacks which we need in § 3. There are currently some good references for stacks (e.g. [DM69, Vis89, Góm01, LM00]).

We fix a base scheme S.

4.1 Deligne–Mumford stacks

A *stack* is a category fibered in groupoids over (Sch/S) such that every Isom functor is a sheaf and every descent datum is effective. A morphism of stacks $\mathcal{X} \to \mathcal{Y}$ is *representable*[4] if for any $U \in (\mathrm{Sch}/S)$ and any morphism $U \to \mathcal{Y}$, the fiber product $U \times_{\mathcal{Y}} \mathcal{X}$ is represented by a scheme.

Definition 4.1. Let $\mathbf{P}$ be a property of morphisms $f : Y \to X$ of S-schemes, stable under base change and local in the étale topology on X (e.g. surjective, proper, etc.). We say that a representable morphism $f : \mathcal{Y} \to \mathcal{X}$ of stacks has property $\mathbf{P}$ if for every S-scheme U and every morphism $U \to \mathcal{X}$, the projection $U \times_{\mathcal{X}} \mathcal{Y} \to U$ has property $\mathbf{P}$.

Definition 4.2. A *(separated) Deligne–Mumford stack* is a stack $\mathcal{X}$ which satisfies the following:

i) the diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable and finite;

ii) there exists a scheme M and a morphism $M \to \mathcal{X}$ (necessarily representable after condition i), which is étale and surjective.

A scheme M in condition ii is called an *atlas* of $\mathcal{X}$. A Deligne–Mumford stack $\mathcal{X}$ is *of finite type* if there is an atlas of finite type.

Definition 4.3. Let $\mathbf{P}$ be a property of morphisms $f : Y \to X$ of S-schemes, stable under *étale* base change and local in the étale topology on X (e.g. birational, being an open immersion with dense image, etc.). We say that a representable morphism $f : \mathcal{Y} \to \mathcal{X}$ of stacks has property $\mathbf{P}$ if for every scheme U and every étale morphism $U \to \mathcal{X}$, the projection $U \times_{\mathcal{X}} \mathcal{Y} \to U$ has property $\mathbf{P}$.

Definition 4.4. Let $\mathbf{Q}$ be a property of schemes that is local in the étale topology (e.g. reduced, smooth, normal, locally integral, etc.). Let $\mathcal{X}$ be a Deligne–Mumford stack. We say that $\mathcal{X}$ has property $\mathbf{Q}$ if an atlas of $\mathcal{X}$ has property $\mathbf{Q}$.

Definition 4.5. A (not necessarily representable) morphism $f : \mathcal{Y} \to \mathcal{X}$ of finite-type Deligne–Mumford stacks is *proper* if there is a S-scheme Z and a proper surjective morphism $g : Z \to \mathcal{Y}$ such that $f \circ g$ is (necessarily representable and) proper.

A Deligne–Mumford stack $\mathcal{X}$ of finite type is *complete* if it is proper over S.

Although our condition appears weaker than that of [DM69, Definition 4.11], the two conditions are actually equivalent by Chow's lemma (see [DM69, Definition 4.12], [LM00, Théorème 16.6] and [Vis89, Proposition 2.6]).

[4] In [LM00], this is called *schématique*.

Example 4.6. Let Z be a S-scheme and G a finite group acting on X. The *quotient stack* $[Z/G]$ is defined as follows: an object over $U \in (\mathrm{Sch}/S)$ is a G-torsor $P \to U$ with a G-equivariant morphism $P \to Z$, and a morphism over $U' \to U$ is a cartesian diagram

$$\begin{array}{ccc} P' & \longrightarrow & P \\ \downarrow & \square & \downarrow \\ U' & \longrightarrow & U \end{array}$$

which is compatible with the G-equivariant morphisms $P' \to Z$ and $P \to Z$. It is a Deligne–Mumford stack with a canonical atlas $Z \to [Z/G]$.

Here we define the points of a Deligne–Mumford stack. For details see [LM00, ch. 5].

Definition 4.7. Let $\mathcal{X}$ be a Deligne–Mumford stack. A *point* of $\mathcal{X}$ is a S-morphism $\operatorname{Spec} K \to \mathcal{X}$ for a field K with a morphism $\operatorname{Spec} K \to S$.

Let $x_i : \operatorname{Spec} K_i \to \mathcal{X}\,(i = 1, 2)$ be points of $\mathcal{X}$. We say that x_1 and x_2 are *equivalent* if there is a field K_3 such that $K_3 \supset K_1, K_2$ and the diagram

$$\begin{array}{ccc} \operatorname{Spec} K_3 & \longrightarrow & \operatorname{Spec} K_2 \\ \downarrow & & \downarrow \\ \operatorname{Spec} K_1 & \longrightarrow & \mathcal{X} \end{array}$$

commutes.

Definition 4.8. We define the *set of points* of $\mathcal{X}$, denoted by $|\mathcal{X}|$, to be the set of the equivalence classes of points of $\mathcal{X}$.

The *Zariski topology* on $|\mathcal{X}|$ is defined as follows: an open subset is $|\mathcal{U}| \subset |\mathcal{X}|$ for an open substack $\mathcal{U} \subset \mathcal{X}$. There is a one-to-one correspondence between the closed subsets of $|\mathcal{X}|$ and the reduced closed substacks of $\mathcal{X}$.

We now introduce the notion of (étale) groupoid space which is equivalent to Deligne–Mumford stacks. Further details can be found in [Vis89, p. 668], [LM00, (2.4.3), (3.4.3), Proposition 3.8, (4.3)] and [Góm01, § 2.4].

Definition 4.9. An *(étale) groupoid space* consists of the following data:

i) two S-schemes X_0 and X_1;
ii) five morphisms: *source* and *target* $q_i : X_1 \to X_0$ $(i = 1, 2)$, *origin* $\varepsilon : X_0 \to X_1$, *inverse* $\tau : X_1 \to X_1$ and *composition* $m : X_1 \times_{q_1, X_0, q_2} X_1 \to X_1$ which satisfies the following:
    a) q_1 and q_2 are étale and $(q_1, q_2) : X_1 \to X_0 \times X_0$ is finite;
    b) the axioms of *associativity*, *identity element* and *inverse*.

We denote this groupoid space by $X_1 \rightrightarrows X_0$.

Given a groupoid space $X_1 \rightrightarrows X_0$, we define the category fibered in the groupoids $[X_1 \rightrightarrows X_0]'$ as follows: an object over $U \in (\mathrm{Sch}/S)$ is a morphism $U \to X_0$ of S-schemes and a morphism of $a : U \to X_0$ to $b : V \to X_0$ is a pair of morphisms $f : U \to V$ and $h : V \to X_1$ such that $q_1 \circ h \circ f = a$ and $q_2 \circ h = b$. Then $[X_1 \rightrightarrows X_0]'$ is a *prestack* (see [LM00, § 3.1]).

Definition 4.10. We define the *stack associated* to a groupoid space $X_1 \rightrightarrows X_0$, denoted by $[X_1 \rightrightarrows X_0]$, to be the stack associated with the prestack $[X_1 \rightrightarrows X_0]'$ ([LM00, Lemme 3.2]).

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The stack $\mathcal{X} = [X_1 \rightrightarrows X_0]$ is a Deligne–Mumford stack with the canonical atlas $X_0 \to \mathcal{X}$. We can identify the fiber product $X_0 \times_{\mathcal{X}} X_0$ with X_1. Conversely, given a Deligne–Mumford stack $\mathcal{X}$ and an atlas $X_0 \to \mathcal{X}$, then the schemes X_0 and $X_1 = M \times_{\mathcal{X}} M$ underlies a natural groupoid space structure with $q_i = \mathrm{pr}_i$ and $\varepsilon = \Delta_{M/\mathcal{X}}$. The associated stack $[M \times_{\mathcal{X}} M \rightrightarrows M]$ is canonically isomorphic to $\mathcal{X}$. In summary, giving a groupoid space $X_1 \rightrightarrows X_0$ is equivalent to giving a Deligne–Mumford stack $\mathcal{X}$ and an atlas $X_0 \to \mathcal{X}$.

Let $\xi : U \to \mathcal{X}$ be a morphism, which is considered as an object of $\mathcal{X}$. If ξ lifts to $\xi' : U \to X_0$, then the automorphism group of ξ is identified with the set of morphisms $\eta : U \to X_1$ with $q_1 \circ \eta = q_2 \circ \eta = \xi'$.

Definition 4.11. A *morphism* $f : (Y_1 \rightrightarrows Y_0) \to (X_1 \rightrightarrows X_0)$ *of groupoid spaces* is a pair of morphisms $f_i : Y_i \to X_i$ $(i = 0, 1)$ which respects the groupoid space structures.

Given a morphism $f : (Y_1 \rightrightarrows Y_0) \to (X_1 \rightrightarrows X_0)$, then we have a natural morphism of prestacks $[f]' : [Y_1 \rightrightarrows Y_0]' \to [X_1 \rightrightarrows X_0]'$ and hence a natural morphism of stacks $[f] : [Y_1 \rightrightarrows Y_0] \to [X_1 \rightrightarrows X_0]$ from [LM00, Lemme 3.2].

Conversely, consider a commutative diagram

$$\begin{array}{ccc} Y_0 & \xrightarrow{f_0} & X_0 \\ \downarrow & & \downarrow \\ \mathcal{Y} & \xrightarrow[g]{} & \mathcal{X} \end{array}$$

such that $\mathcal{X}$, $\mathcal{Y}$ are Deligne–Mumford stacks and the vertical arrows are atlases. If we define $Y_1 := Y_0 \times_{\mathcal{Y}} Y_0$ and $X_1 := X_0 \times_{\mathcal{X}} X_0$, and if $f_1 : Y_1 \to X_1$ is the natural morphism, then the pair of (f_0, f_1) determines a morphism $f : (Y_1 \rightrightarrows Y_0) \to (X_1 \rightrightarrows X_0)$ of groupoid spaces. Evidently $[f] = g$.

Example 4.12. Let $\mathcal{X} = [Z/G]$ be a quotient stack with G finite. There is a canonical atlas $Z \to \mathcal{X}$. Then the groupoid space $Z \times_{\mathcal{X}} Z \rightrightarrows Z$ is isomorphic to the groupoid space

$$Z \times G \underset{\mathrm{pr}_1}{\overset{G-\mathrm{action}}{\rightrightarrows}} Z$$

whose origin, inverse and composition are induced by the group structure of G.

Let $\mathcal{X}$ be a locally integral Deligne–Mumford stack, associated to a groupoid space $X_1 \rightrightarrows X_0$. Let X_i^{nor} be the normalization of X_i, respectively. Then the lifts of the structure morphisms of $X_1 \rightrightarrows X_0$ induce a groupoid space $X_1^{\mathrm{nor}} \rightrightarrows X_0^{\mathrm{nor}}$.

Definition 4.13 [Vis89, Definition 1.18]. We define the *normalization* $\mathcal{X}^{\mathrm{nor}}$ of $\mathcal{X}$ to be the stack associated with $X_1^{\mathrm{nor}} \rightrightarrows X_0^{\mathrm{nor}}$.

It is easy to show the uniqueness and the universality of the normalization.

4.2 Quasi-coherent sheaves

Definition 4.14. A *quasi-coherent sheaf* $\mathcal{F}$ on a Deligne–Mumford stack $\mathcal{X}$ consists of the following data:

i) for each étale morphism $U \to \mathcal{X}$ with a scheme U there is a quasi-coherent sheaf $\mathcal{F}_U$ on U;
ii) for each diagram of étale morphisms

$$\begin{array}{ccccc} V & & \xrightarrow{\varphi} & & U \\ & \searrow & & \swarrow & \\ & & \mathcal{X} & & \end{array}$$

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with V and U schemes there is an isomorphism $\Theta_\varphi : \mathcal{F}_V \to \varphi^*\mathcal{F}_U$ that satisfies the cocycle condition.

Example 4.15.

i) The *structure sheaf* $\mathcal{O}_\mathcal{X}$ on $\mathcal{X}$ is defined by $(\mathcal{O}_\mathcal{X})_U := \mathcal{O}_U$.
ii) The *sheaf of differentials* $\Omega_{\mathcal{X}/S}$ is defined by $(\Omega_{\mathcal{X}/S})_U := \Omega_{U/S}$ and by the canonical isomorphism.
iii) Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism of Deligne–Mumford stacks. We define the *sheaf of relative differentials* $\Omega_{\mathcal{Y}/\mathcal{X}}$ of $\mathcal{Y}$ over $\mathcal{X}$ to be the unique sheaf such that for each commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\text{étale}} & \mathcal{Y} \\ \downarrow & & \downarrow \\ U & \xrightarrow[\text{étale}]{} & \mathcal{X}, \end{array}$$

where $(\Omega_{\mathcal{Y}/\mathcal{X}})_V := \Omega_{V/U}$. Then we have the following exact sequence:

$$f^*\Omega_{\mathcal{X}/S} \to \Omega_{\mathcal{Y}/S} \to \Omega_{\mathcal{Y}/\mathcal{X}} \to 0.$$

Definition 4.16. In Definition 4.14, if every $\mathcal{F}_U$ is an $\mathcal{O}_U$-algebra and every Θ_φ is a homomorphism of $\mathcal{O}_V$-algebras, then we say that $\mathcal{F}$ is an $\mathcal{O}_\mathcal{X}$-algebra.

As in the case of schemes, to an $\mathcal{O}_\mathcal{X}$-algebra $\mathcal{F}$ we can associate a representable and affine morphism $\mathcal{S}\mathrm{pec}\,\mathcal{F} \to \mathcal{X}$. For details, see [LM00, Equation (14.2)].

4.3 Inertia stacks

In this section, we study the inertia stack. It is an algebro-geometric object corresponding to the *twisted sector* introduced by Kawasaki [Kaw78] and used by Chen and Ruan [CR00] to define the orbifold cohomology.

Definition 4.17. For a Deligne–Mumford stack $\mathcal{X}$, its *inertia stack*, denoted by $I(\mathcal{X})$, is the stack defined as follows: an object over $U \in (\mathrm{Sch}/S)$ is a pair (ξ, α) where $\xi \in \mathrm{ob}\,\mathcal{X}_U$ and $\alpha \in \mathrm{Aut}(\xi)$, and a morphism $(\xi, \alpha) \to (\eta, \beta)$ is a morphism $\gamma : \xi \to \eta$ in $\mathcal{X}$ such that $\gamma \circ \alpha = \beta \circ \gamma$.

There is a natural forgetting morphism $I(\mathcal{X}) \to \mathcal{X}$. The forgetting morphism $I(\mathcal{X}) \to \mathcal{X}$ is isomorphic to

$$\mathrm{pr}_1 : \mathcal{X} \times_{\Delta, \mathcal{X}\times\mathcal{X}, \Delta} \mathcal{X} \to \mathcal{X}.$$

Hence, if $\mathcal{X}$ is complete, then so is $I(\mathcal{X})$.

The following lemma may be well known.

Lemma 4.18. *Let Z be a scheme with an action of a finite group G. Then we have an isomorphism*

$$I([Z/G]) \cong \coprod_{g \in \mathrm{Conj}(G)} [Z^g/C(g)],$$

where $\mathrm{Conj}(G)$ is a set of representatives of the conjugacy classes, Z^g is the locus of the points fixed under the g-action and $C(g)$ is the centralizer of g.

Proof. Let $U \in (\mathrm{Sch}/S)$ be a connected scheme. An object of $[Z/G]$ over U is a G-torsor $P \to U$ with a G-equivariant morphism $P \to Z$. Its automorphism α is an automorphism of a G-torsor $P \to U$ that is compatible with $P \to Z$. For some étale surjective $V \to U$, the fiber product

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$P_V := P \times_U V$ is isomorphic to $G \times V$ as G-torsors over V. Here $G \times V$ is a G-torsor for the *right action* of G. The pull-back α_V is represented by the *left action* of some $g \in G$. g is determined up to conjugacy and we can assume $g \in \mathrm{Conj}(G)$.

Let $\psi : G \times V \cong P_V \to P$ be the natural morphism. Now let us show that if $a \in C(g)$ and $b \notin C(g)$, then we have $\psi(\{a\} \times V) \cap \psi(\{b\} \times V) = \emptyset$. Let x (respectively y) be a geometric point of $\{a\} \times V$ (respectively $\{b\} \times V$) and assume that $\psi(x) = \psi(y)$. Then we have

$$\psi(x) = \psi(gxg^{-1}) = \alpha\psi(x)g^{-1} = \alpha\psi(y)g^{-1} = \psi(gyg^{-1}) \neq \psi(y).$$

This is a contradiction. So P decomposes into $C(g)$-torsors as $P \cong P' \times J$ where $P' := \psi(C(g) \times V)$ and J is a finite set. Let f_V denote the composition $f \circ \psi$. In the following diagram:

$$\begin{array}{ccc} C(g) \times V & \xrightarrow{f_V} & Z \\ {\scriptstyle \alpha_V}\downarrow & \nearrow{\scriptstyle f_V} & \downarrow{\scriptstyle g} \\ C(g) \times V & \xrightarrow[f_V]{} & Z \end{array}$$

we have $g \circ f_V = f_V \circ \alpha_V$ since α_V equals the right action of g on $C(g) \times V$ and f_V is G-equivariant. We also have $f_V = f_V \circ \alpha_V$ and hence $g \circ f_V = f_V$. This implies that $f_V(C(g) \times V)$ is in Z^g and hence so is $f(P')$. Thus the $C(g)$-torsor $P' \to V$ with $f : P' \to Z^g$ is an object of $[Z^g/C(g)]$. For a non-connected $U \in (\mathrm{Sch}/S)$ and an object of $I([Z/G])$ over U, we can assign it to be an object of $\coprod_{g \in \mathrm{Conj}(G)} [Z^g/C(g)]$ in the obvious way. We leave the rest for the reader. □

Definition 4.19. A morphism $f : \mathcal{Y} \to \mathcal{X}$ of stacks is *barely faithful* if, for every object ξ of $\mathcal{Y}$, the map $\mathrm{Aut}(\xi) \to \mathrm{Aut}(f(\xi))$ is bijective.

Clearly, all barely faithful morphisms are faithful functors. From [LM00, Proposition 2.3 and Corollaire 8.1.2], all barely faithful morphisms of Deligne–Mumford stacks are *representable* in the sense of [LM00, Définition 3.9]. Because all separated and quasi-finite morphisms of algebraic spaces are schématique [LM00, Théorème A.2], all barely faithful and quasi-finite morphisms of Deligne–Mumford stacks are *representable* for our definition.

Example 4.20. All immersions are barely faithful. All morphisms of schemes are barely faithful.

Lemma 4.21. *Barely faithful morphisms are stable under base change.*

Proof. Let $f : \mathcal{Y} \to \mathcal{X}$ be a barely representable morphism of stacks and $a : \mathcal{X}' \to \mathcal{X}$ any morphism of stacks. An object of the fiber product $\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}'$ is a triple (ξ, η, α), where ξ is an object of $\mathcal{Y}$, η is an object of $\mathcal{X}'$ and $\alpha : f(\xi) \to a(\eta)$ is a morphism in $\mathcal{X}_U$ for some $U \in (\mathrm{Sch}/S)$. Its automorphism is a pair of automorphisms $\varphi \in \mathrm{Aut}(\xi)$ and $\psi \in \mathrm{Aut}(\eta)$ with $\alpha \circ f(\varphi) = a(\psi) \circ \alpha$. Since the map $f : \mathrm{Aut}(\xi) \to \mathrm{Aut}(f(\xi))$ is bijective, for each ψ there is one and only one φ with $\alpha \circ f(\varphi) = a(\psi) \circ \alpha$. We have thus proved the lemma. □

Proposition 4.22. *Let $f : \mathcal{Y} \to \mathcal{X}$ be a barely faithful morphism of Deligne–Mumford stacks. Then the inertia stack $I(\mathcal{Y})$ is naturally isomorphic to the fiber product $\mathcal{Y} \times_{\mathcal{X}} I(\mathcal{X})$.*

Proof. The natural morphism $\Psi : I(\mathcal{Y}) \to \mathcal{Y} \times_{\mathcal{X}} I(\mathcal{X})$ is defined as follows: for an object ξ of $\mathcal{Y}$ and its automorphism α, the pair (ξ, α), which is an object of $I(\mathcal{Y})$, is mapped to the triple $(\xi, (f(\xi), f(\alpha)), \mathrm{id}_{f(\xi)})$.

We will show initially that Ψ is a fully faithful functor. Let ξ be an object of $\mathcal{Y}$. The automorphism group of $(\xi, (f(\xi), f(\alpha)), \mathrm{id}_{f(\xi)})$ is a pair of automorphisms $\beta \in \mathrm{Aut}(\xi)$ and $\nu \in \mathrm{Aut}((f(\xi), f(\alpha))) = C(f(\alpha))$ such that $f(\beta) = \nu$. Hence Ψ is barely faithful. Let η be another object of $\mathcal{Y}$ and τ an automorphism of η. It suffices to show that if $\mathrm{Hom}_{I(\mathcal{Y})}((\xi, \alpha), (\eta, \tau)) = \emptyset$, then

$$\mathrm{Hom}_{\mathcal{Y} \times_{\mathcal{X}} I(\mathcal{X})}(\Psi(\xi, \alpha), \Psi(\eta, \tau)) = \emptyset.$$

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Suppose that there is an element (σ, ε) of $\mathrm{Hom}_{\mathcal{Y}\times_{\mathcal{X}} I(\mathcal{X})}(\Psi(\xi,\alpha),\Psi(\eta,\tau))$, where σ is a morphism $\xi \to \eta$ and ε is a morphism $f(\xi) \to f(\eta)$ such that the diagram

$$\begin{array}{ccc} f(\xi) & \xrightarrow{\varepsilon} & f(\eta) \\ {\scriptstyle f(\alpha)}\downarrow & & \downarrow{\scriptstyle f(\tau)} \\ f(\xi) & \xrightarrow{\varepsilon} & f(\eta) \end{array}$$

is commutative and $f(\sigma) = \varepsilon$. Since f is barely faithful, the diagram

$$\begin{array}{ccc} \xi & \xrightarrow{\sigma} & \eta \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \tau} \\ \xi & \xrightarrow{\sigma} & \eta \end{array}$$

is commutative, that is, $\mathrm{Hom}_{I(\mathcal{Y})}((\xi,\alpha),(\eta,\tau)) \neq \emptyset$.

Now, let us show Ψ is an isomorphism, that is, an equivalence of categories. Let $(\xi,(\theta,\beta),\upsilon)$ be an object of $\mathcal{Y}\times_{\mathcal{X}} I(\mathcal{X})$ where υ is an isomorphism $f(\xi)\to\theta$. Then there is a natural bijection $\mathrm{Aut}(\xi)\to\mathrm{Aut}(f(\xi))\to\mathrm{Aut}(\theta)$. Let $\alpha\in\mathrm{Aut}(\xi)$ be an automorphism corresponding to $\beta\in\mathrm{Aut}(\theta)$. Then we can see that $\Phi(\xi,\alpha)$ is isomorphic to $(\xi,(\theta,\beta),\upsilon)$. We have thus completed the proof. □

COROLLARY 4.23. *If $S = \mathrm{Spec}\,\mathbb{C}$ and $\mathcal{X}$ is a smooth Deligne–Mumford stack, then $I(\mathcal{X})$ is also smooth.*

Proof. From Lemmas 4.26 and 4.21, there is an étale, surjective and barely faithful morphism $\coprod_i [M_i/G_i] \to \mathcal{X}$ such that each M_i is smooth and each G_i is a finite group. Then the assertion follows from Lemma 4.18 and Proposition 4.22. □

4.4 Coarse moduli space

DEFINITION 4.24. Let $\mathcal{X}$ be a Deligne–Mumford stack. The *coarse moduli space* of $\mathcal{X}$ is an algebraic space X with a morphism $\mathcal{X}\to X$ such that:

i) for any algebraically closed field Ω with a morphism $\mathrm{Spec}\,\Omega\to S$, $\mathcal{X}(\Omega)\to X(\Omega)$ is a bijection;

ii) for any algebraic space Y, any morphism $\mathcal{X}\to Y$ uniquely factors as $\mathcal{X}\to X\to Y$.

Keel and Mori proved that the coarse moduli space always exists [KM97, Corollary 1.3].

Example 4.25. Let Z be an algebraic space and G a finite group acting on Z. Then the coarse moduli space of the quotient stack $[Z/G]$ is the quotient algebraic space Z/G.

The following lemma is well-known.

LEMMA 4.26 (see, e.g., [AV02, Lemma 2.2.3]). *Let $\mathcal{X}$ be a Deligne–Mumford stack and X its coarse moduli space. Then there is an étale covering $(X_i\to X)_i$ such that $\mathcal{X}\times_X X_i$ is isomorphic to a quotient stack $[Z_i/G_i]$ with a scheme Z_i and a finite group G_i. Hence the canonical morphism $\mathcal{X}\to X$ is proper.*

Now we assume $S = \mathrm{Spec}\,\mathbb{C}$.

DEFINITION 4.27. Let X be a variety. We say that X has *quotient singularities* if there is an étale covering $(U_i/G_i\to X)_i$ with a smooth variety U_i and a finite group G_i.

Lemma 4.26 shows that for a variety X, X has quotient singularities if it is the coarse moduli space of some smooth Deligne–Mumford stack. In fact, 'only if' also holds (Lemma 4.29).

Let $\mathcal{X}$ be a smooth Deligne–Mumford stack and $x:\mathrm{Spec}\,\mathbb{C}\to\mathcal{X}$ a closed point. Then $\mathrm{Aut}(x)$ acts on the tangent space $T_x\mathcal{X}$.

Definition 4.28. We say that $\alpha \in \mathrm{Aut}(x)$ is a *reflection* if the subspace of the α-fixed points $(T_x\mathcal{X})^\alpha$ is of codimension 1.

Lemma 4.29. *Let X be a k-variety with quotient singularities. Then there is a smooth Deligne–Mumford stack $\mathcal{X}$ without reflections such that the automorphism group of general geometric points is trivial and X is the coarse moduli space of $\mathcal{X}$.*

Proof. We will give only a sketch. There is a finite set of pairs $(V_i \to X, G_i)_i$ such that:

i) V_i is a smooth variety;

ii) G_i is a finite group acting effectively on V_i without reflections;

iii) $V_i \to X$ is a morphism étale in codimension 1 which factors as $V_i \to V_i/G_i \to X$ with $V_i/G_i \to X$ étale.

Let V_{ij} be the normalization of $V_i \times_X V_j$. Then the natural morphisms $V_{ij} \to V_i$ and $V_{ij} \to V_j$ are étale in codimension 1. From the purity of branch locus, they are actually étale, and hence V_{ij} is smooth. The diagonal $\Delta : V_i \to V_i \times_X V_i$ factors through $\Delta' : V_i \to V_{ij}$. Then, with a suitable multiplication morphism, the diagram

$$\coprod V_{i,j} \overset{\Delta'}{\underset{\longrightarrow}{\overset{\longleftarrow}{\longrightarrow}}} \coprod V_i$$

has the structure of groupoid space. We set $\mathcal{X}$ as the associated stack. Clearly $\mathcal{X}$ has no reflections. The canonical morphism $\mathcal{X} \to X$ makes X the coarse moduli space of $\mathcal{X}$ (see [Gil84, Proposition 9.2]). Any geometric point x of $\mathcal{X}$ has a lift $\tilde{x} : \operatorname{Spec}\Omega \to V_i$ with Ω being an algebraically closed field. The automorphism group of x is identified with $\{y : \operatorname{Spec}\Omega \to V_{ii} \mid p_1 \circ y = p_2 \circ y = \tilde{X}\}$. If y is over the smooth locus of X, then this group is trivial. □

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