A REMARK ON SMALL VALUES OF ENTIRE FUNCTIONS

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Let $f(z)$ be an entire function, $M(r)$ the maximum of $f(z)$ on $|z| = r$, and $\lambda > 1$. Let $E_\lambda = E_\lambda(f) = \{z\colon \log|f(z)| \leqq (1-\lambda)\log M(|z|)\}$, and denote the density of E_λ by

$$D_R(E_\lambda) = m(z \in E_\lambda\colon |z| \leqq R)/\pi R^2$$

where m is planar measure.

Developing an idea of Lindelöf, Boas, Buck and Erdös (3) prove:

Theorem (A). *For any $\lambda > 1$, there is a positive number $K = K(\lambda)$, the same for all entire functions, such that*

$$\bar{D}(E_\lambda) = \varlimsup_{R\to\infty} D_R(E_\lambda) \leqq K \leqq 1/\lambda;$$

Theorem (B). $\lim_{\lambda\to\infty} \lambda \underline{D}(E_\lambda) = \lim_{\lambda\to\infty} \lambda \varliminf_{R\to\infty} D_R(E_\lambda) = 0;$

and conjecture that perhaps

$$\lim_{\lambda\to\infty} \lambda \bar{D}(E_\lambda) = 0 \tag{*}$$

also holds.

The authors of (3) do not seem to have noticed that their method gives something more, namely inequality (4) below, when the class of entire functions is restricted to those of finite positive order and type, and in fact can be used to show (in support of the conjecture (*)) that, in certain senses, no matter how large λ may be, $\lambda\bar{D}(E_\lambda(f))$ may be arbitrarily small for proper choice of $f(z)$, where $f(z)$ has finite *positive* order ρ. (If f is a polynomial $\bar{D}(E_\lambda) = 0$.)

Precisely, we have:

Proposition I. *Given $\varepsilon > 0$, $\lambda > 1$, there exists a $\rho = \rho(\varepsilon) > 0$ and an entire function $f(z)$ of finite positive order ρ such that*

$$\lambda\bar{D}(E_\lambda(f)) < \varepsilon.$$

As $\varepsilon \to 0$, $\rho \to 0$.

Proposition II. *Given $\varepsilon > 0$, $\rho > 0$, there exists an entire function $f(z)$ of finite positive order ρ and a $\lambda = \lambda(\varepsilon)$ such that*

$$\lambda\bar{D}(E_\lambda(f)) < \varepsilon$$

As $\varepsilon \to 0$, $\lambda \to \infty$.

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Both propositions follow from the same proof which is simply (3) with Jensen's Inequality replaced by Jensen's Theorem; combined with an easy and well-known estimate, and an example of Boas (1).

Proof. Let $H_{r,\lambda} = \{\theta: \log|f(re^{i\theta})| \leqq (1-\lambda)\log M(r)\}$. Let $f(z)$ be an entire function of finite positive order ρ and finite positive type τ, with $f(0) = 1$, and, as usual, let $n(t)$ = the number of zeros of $f(z)$ in $|z| \leqq t$, and

$$N(r) = \int_0^r n(t)/t\,dt.$$

Then as shown in (3),

$$\frac{1}{2\pi}\log M(r)\int_0^\infty m(H_{r,\lambda})d\lambda = \log M(r) - \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta}|\,d\theta$$

$$= \log M(r) - N(r), \qquad (1)$$

by Jensen's Theorem.

Let $l = \varliminf_{r\to\infty} n(r)/r^\rho$. Since τ and ρ are finite, so is l (e.g. (2), p. 16). Given $\eta > 0$, choose R_0 such that for $r \geqq R_0 = R_0(\eta)$,

(a) $M(r) > 1$.
(b) $\log M(r) \leqq (\tau+\eta)r^\rho$.
(c) $n(r) \geqq (l-\eta)r^\rho$.

Then (cf. (2), p. 16, or (1), p. 28), for $r > R_0$,

$$N(r) = \int_0^{R_0} n(t)/t\,dt + \int_{R_0}^r n(t)/t\,dt \geqq \frac{l-\eta}{\rho}r^\rho + O(1),$$

as $r \to \infty$.

Hence, for $r > R_0$,

$$\frac{N(r)}{\log M(r)} \geqq \frac{l-\eta}{\rho(\tau+\eta)} + o(1) \qquad (2)$$

for every $\eta > 0$, as $r \to \infty$.

Also, as in (3), $m(z \in E_\lambda : R_0 \leqq |z| \leqq R) = \int_{R_0}^R m(H_{r,\lambda})r\,dr.$

Integrating this equation with respect to λ and using (1) and (2) gives

$$\int_0^\infty D_R(E_\lambda^*)d\lambda = 1 - R_0^2/R^2 - (2/R^2)\int_{R_0}^R \frac{rN(r)}{\log M(r)}dr \leqq 1 - \frac{l-\eta}{\rho(\tau+\eta)} + o(1) \qquad (3)$$

as $R \to \infty$, for every $\eta > 0$, where $E_\lambda^* = E_\lambda \backslash \{z : |z| \leqq R_0\}$.

But, as observed in (3), $\lambda D_R(E_\lambda^*) \leqq \int_0^\infty D_R(E_\lambda^*)d\lambda$. Hence letting $R \to \infty$, and $\eta \to 0$, (3) becomes

$$1 - \frac{l}{\rho\tau} \geqq \varlimsup_{R\to\infty} \lambda D_R(E_\lambda^*) = \varlimsup_{R\to\infty} \lambda D_R(E_\lambda) = \lambda\bar{D}(E_\lambda). \qquad (4)$$

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The inequality (4) holds for all entire functions of finite positive order and finite positive type.

Consider now the special function

$$g_{\lambda,\rho}(z) = \prod_{n=1}^{\infty} \{1+z^{[\lambda]}([\lambda]n)^{-[\lambda]/\rho}\}$$

where $[\lambda]$ is the greatest integer $\leqq \lambda$, and it is understood that if either $\rho>0$ or $\lambda>1$ is given the other is to be chosen so that $0<\rho<[\lambda]$.

As shown in (1), $g_{\lambda,\rho}(z)$ is an entire function of order ρ, and type

$$(\pi/[\lambda])\csc\pi\rho/[\lambda],$$

with $l = 1$.

Hence we have

$$\lambda\bar{D}(E_\lambda(g_{\lambda,\rho})) \leqq 1 - \frac{\sin \pi\rho/[\lambda]}{\pi\rho/[\lambda]} \tag{5}$$

and so the two propositions.

REFERENCES

(1) R. P. Boas, Jr., Fundamental sets of entire functions, *Annals of Math.* **47** (1946), 21-32.

(2) R. P. Boas, Jr., *Entire Functions* (Academic Press, New York, 1954).

(3) R. P. Boas, Jr., R. C. Buck and P. Erdös, The set on which an entire function is small, *Amer. J. Math.* **70** (1948), 400-402.

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