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ENLARGEMENT OF σ-ALGEBRAS AND COMPACTNESS OF TIME CHANGES

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Introduction. Given a stochastic process adapted to an increasing family of right-continuous σ-algebras, it is often useful for many purposes to enlarge the σ-algebras. In the present paper we shall consider enlargements which involve embedding the process in a larger probability space. The first question investigated is what kinds of enlargements it might be useful to consider. To study stopping times, the least requirement needed to have a complete theory is that convergent sequences of stopping times converge to a function which is also a stopping time, and for this it is necessary to make the enlargement right continuous. There are other constraints on the possible enlargements. For the enlargement to be useful the properties of the process should be retained. For example, if it is strong Markov with respect to the smaller σ-algebras it should remain so with respect to the larger ones. A property of the process which depends on the increasing σ-algebras to which the process is adapted is said to be *preserved under the enlargement* if it holds with respect to the larger σ-algebras. It is quite possible that certain properties are preserved under some types of enlargements but not under some other types. The main result obtained with respect to these questions is that a certain class of enlargements, called *distributional enlargements*, preserves *all* the following properties (i) quasi-left-continuity, (ii) the strong Markov property, (iii) independence of increments with respect to the past, and (iv) the martingale property. This result is dealt with in Section 1.

Next, suppose that there is given a sequence of non-anticipating time changes. It would be very useful to have the result that if the sequence of non-anticipating time changes does not grow too rapidly, then it is possible to extract a subsequence which converges to a non-anticipating time change in the sense that the process time changed with respect to the subsequence converges in distribution to the process time changed at the limit. In the remainder of the paper we deal with the question of to what extent this result holds. The main theorem obtained is that if the process is embedded in a larger probability space and if the increasing σ-fields are given a distributional enlargement, then the conclusion almost holds, in a sense made precise later. In a very ingenious paper, Monroe [**3**] has shown that any right continuous martingale can be embedded in Brownian motion, after enlargement of the σ-algebras. The proof is composed of several parts. In one part, Monroe shows that a

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particular sequence of time changes of Brownian motion has a convergent subsequence. In the proof, essential use is made of two properties of Brownian motion: continuity of sample paths and independence of increments, as well as the particular nature of the time changes. The results in the present paper show that the selection argument given by Monroe is a particular case of a compactness principle, valid even for discontinuous processes which need not be Markov. For another application, to the embedding of potential processes in other Markov processes in addition to Brownian motion, see [**2**].

1. Representation. Let E be a topological space with a countable base. Let $X = (\Omega, \mathscr{A}, \mathscr{M}_t, X_t, P)$ be a stochastic process, where $(\Omega, \mathscr{A}, P)$ is a probability space, $\{\mathscr{M}_t\}$ is a nondecreasing right continuous family of σ-algebras contained in a σ-algebra $\mathscr{M} \subseteq \mathscr{A}$, and X_t is an $\mathscr{M}_t$-measurable map from Ω into E for each t. We will often write X_t as $X(t)$ or $X(t, \omega)$, and so on.

Some terminology is helpful. A *randomized* $\mathscr{M}_t$-stopping time T will mean an $(\mathscr{M}_t \times \mathscr{L})_+$-stopping time T defined on $\Omega \times L$, where $(L, \mathscr{L}, \nu)$ is a probability space, and $\Omega \times L$ is given the product probability $P \times \nu$. T is thus a weighted collection of $\mathscr{M}_t$-stopping times.

We will use the phrase *distributional stopping time* to denote a slightly more complicated object. Let $(H, \mathscr{H})$ be a measurable space. Let $\Lambda = \Omega \times H$, $\mathscr{K} = \mathscr{A} \times \mathscr{H}$. Let Q be some probability on $(\Lambda, \mathscr{K})$ such that Q has projection P on $(\Omega, \mathscr{A})$. Let $\mathscr{F} = \mathscr{M} \times H \equiv$ the collection of sets of the form $A \times H$, A in $\mathscr{M}$. Let $\mathscr{F}_t = \mathscr{M}_t \times H$. Let $\{\mathscr{K}_t\}$ be a nondecreasing family of σ-algebras contained in $\mathscr{K}$. For each bounded $\mathscr{K}_t$-measurable function Y, suppose that

$$(1.1) \qquad E[Y|\mathscr{F}] = E[Y|\mathscr{F}_t].$$

It is easy to see that if (1.1) holds for $\{\mathscr{K}_t\}$ then it holds for $\{\mathscr{K}_{t+}\}$. A right continuous family $\{\mathscr{K}_t\}$ satisfying (1.1) will be referred to as a *distributional enlargement* of $\{\mathscr{M}_t\}$, and a $\mathscr{K}_t$-stopping time will be referred to as a *distributional* $\mathscr{M}_t$*-stopping time*. In this section we shall consider the extent to which the notions of randomized and distributional stopping times coincide.

Let $\{\mathscr{K}_t\}$ be any family satisfying (1.1) and let $\{T(n)\}$ be a nondecreasing sequence of $\mathscr{K}_t$-stopping times. We will show that these stopping times can be *simultaneously* represented by weighted collections of $\mathscr{M}_t$-stopping times, in the following sense. There exist probability spaces $(W, \mathscr{W}, \gamma)$ and $(L, \mathscr{L}, \nu)$, a map $\varphi: \Lambda \times W \rightarrow \Omega \times L$, and a nondecreasing sequence $\{S(n)\}$ of $(\mathscr{M}_t \times \mathscr{L})_+$-stopping times, such that

$$(1.2) \qquad (Q \times \gamma)\varphi^{-1} = P \times \nu, \quad \text{and}$$

$$(1.3) \qquad T(n) = S(n) \circ \varphi \quad (\text{mod } Q \times \gamma) \quad \text{for each } n.$$

Conditions (1.2) and (1.3) show that the $T(n)$ may (for most purposes) be identified with the $S(n)$. We state the result as

(1.4) Proposition. *A nondecreasing sequence of stopping times for a family of*

σ-algebras satisfying (1.1) *can be simultaneously represented in the above manner as randomized stopping times.*

In the proof of (1.4), L can be chosen to be $[0, 1]^{\mathcal{N}}$, where $\mathcal{N}$ = the natural numbers, with $\nu = \lambda^{\mathcal{N}}$, where λ is Lebesgue measure on $[0, 1]$. L may be thought of as a space of labels for collections of $\mathcal{M}_t$-stopping times. $(W, \mathcal{W}, \gamma)$ can be chosen equal to $(L, \mathcal{L}, \nu)$. W may be thought of as a way of fattening up Λ to ensure the existence of independent random variables. The proof of (1.4) is essentially given in the next two lemmas.

(1.5) LEMMA. *Let $(\Lambda, \mathcal{K}, Q)$ be a probability space. Let $\mathcal{F}$ be a σ-algebra contained in $\mathcal{K}$, and let $\{\mathcal{F}_t\}$ be a nondecreasing right continuous family of σ-algebras with $\mathcal{F}_t \subseteq \mathcal{F}$. Let $T: \Lambda \to [0, \infty]$ be a $\mathcal{K}$-measurable map such that*

(1.6) $\quad Q(T \leqq t|\mathcal{F})$ *is $\mathcal{F}_t$-measurable* mod Q *for all t.*

Then there exists a map $\xi: \Lambda \times [0, 1] \to [0, 1]$ and a map $S: \Lambda \times [0, 1] \to [0, \infty]$ such that

(1.7) $\quad$ *ξ is $\mathcal{K} \times \mathcal{B}$ measurable (where $\mathcal{B}$ is the collection of Borel subsets of $[0, 1]$), ξ has distribution λ (where λ is Lebesgue measure on $[0, 1]$), and ξ is independent of $\mathcal{F} \times [0, 1]$ (where $\mathcal{F} \times [0, 1]$ is the collection of sets $A \times [0, 1]$, $A \in \mathcal{F}$),*

(1.8) $\quad$ *S is an $\mathcal{F}_t \times \mathcal{B}$-stopping time,*

(1.9) $\quad T(\omega) = S(\omega, \xi(\omega, u))$ *for $Q \times \lambda$-almost every (ω, u) in $\Lambda \times [0, 1]$.*

Proof. For every rational t let $Y(t)$ be an $\mathcal{F}_t$-measurable function such that $Y(t) = Q(T \leqq t|\mathcal{F})$ mod Q. We may choose $Y(t)$ such that $0 \leqq Y(t) \leqq 1$ and $Y(s) \leqq Y(t)$ for all $s \leqq t$. Let $Z(t) = \inf \{Y(s)|s \text{ rational}, s > t\}$ for all real t. Let $Z(\infty) \equiv 1$. For each t, $Z(t)$ is $\mathcal{F}_t$-measurable and $Z(t) = Q(T \leqq t|\mathcal{F})$ mod Q. $Z(t)$ is right continuous and nondecreasing as a function of t. Let $Z_-(t) = \sup \{Z(s)|s < t\}$. Define $\xi: \Lambda \times [0, 1] \times [0, 1]$ by

(1.10) $\quad \xi(\omega, u) = Z_-(T(\omega))(\omega) + u(Z(T(\omega))(\omega) - Z_-(T(\omega))(\omega)).$

It is easy to see that ξ is $\mathcal{K} \times \mathcal{B}$-measurable.

Fix a in $[0, 1]$. Let $b = \sup \{t|Z(t) \leqq a\}$. b is an $\mathcal{F}$-measurable function on Λ. $\{\xi \leqq a\} = \{(\omega, u)|T(\omega) < b\} \cup \{(\omega, u)|T(\omega) = b \text{ and } u(Z(T(\omega)) - Z_-(T(\omega))) \leqq a - Z_-(T(\omega))\}$. Hence $Q \times \lambda(\xi \leqq a|\mathcal{F} \times [0, 1]) = Z_-(b) + (Z(b) - Z_-(b))(a - Z_-(b))/(Z(b) - Z_-(b)) = a$. Thus (1.7) holds.

For ω in Λ and v in $[0, 1]$, let

(1.11) $\quad S(\omega, v) = \sup \{t|Z(t, \omega) < v\}.$

Then $\{S > c\} = \{(\omega, u)|Z(c, \omega) < v\}$ = union over all rational a of $\{\omega|Z(c, \omega) < a\} \times [a, 1]$, a set in $\mathcal{F}_c \times \mathcal{B}$. Thus (1.8) holds.

Let $A = \{(\omega, u)|S(\omega, \xi(\omega, u)) > t\}$. Then $A = \{(\omega, u)|Z(t, \omega) < \xi(\omega, u)\} = \{(\omega, u)|Z(t, \omega) < Z_-(T(\omega)) + u(Z(T(\omega)) - Z_-(T(\omega)))\}$. Thus $A \subseteq B \equiv$

$\{(\omega, u)|T(\omega) > t\}$. On the other hand, $Q \times \lambda(A) = E[Q \times \lambda[A|\mathscr{F} \times [0, 1]]] = E[1 - Z(t)] = Q(T > t) = Q(B)$. This proves (1.9) and completes the proof of the lemma.

The map ξ constructed in Lemma (1.5) has one additional property, that will be useful in the next lemma. Let $\mathscr{H}_t$ denote the σ-algebra generated by $\mathscr{F}_t$ and all sets of the form $\{T \leqq s\}$, $s \leqq t$. It follows easily from (1.10) that

(1.12) ξ restricted to $\{T \leqq t\}$ is $\mathscr{H}_t \times \mathscr{B}$-measurable.

(1.13) LEMMA. *Let $(\Lambda, \mathscr{K}, Q)$ be a probability space. Let $\mathscr{F}$ be a σ-algebra, $\mathscr{F} \subseteq \mathscr{K}$. Let $\{\mathscr{F}_t\}$ be a nondecreasing right continuous family of σ-algebras with $\mathscr{F}_t \subseteq \mathscr{F}$. Let $T(n)$: $\Lambda \to [0, \infty]$ be a non-decreasing family of $\mathscr{K}$-measurable maps such that*

(1.14) *$E[Y|\mathscr{F}] = E[Y|\mathscr{F}_t]$ for each bounded function Y which is measurable with respect to the σ-algebra generated by all sets of the form $\{T(n) \leqq s\}$, $n = 1, 2, \ldots, s \leqq t$.*

Then there exist maps $\xi(n)$: $\Lambda \times [0, 1]^n \to [0, 1]$ and $S(n)$: $\Lambda \times [0, 1]^n \to [0, \infty]$ such that the $S(n)$ are nondecreasing (when regarded as functions on $\Lambda \times [0, 1]^{\mathscr{N}}$) and

(1.15) *$\xi(n)$ is $\mathscr{K} \times \mathscr{B}^n$ measurable, $\xi(n)$ has distribution λ, and (when regarded as functions on $\Lambda \times [0, 1]^n$) $\xi(1), \ldots, \xi(n)$ are independent of each other, and are together independent of $\mathscr{F} \times [0, 1]^n$,*

(1.16) *$S(n)$ is an $(\mathscr{F}_t \times \mathscr{B}^n)_+$-stopping time, and*

(1.17) *$T(n)(\omega) = S(n)(\omega, \xi(1)(\omega, u_1), \ldots, \xi(n)(\omega, u_1, \ldots, u_n))$ for $Q \times \lambda^n$-almost every $(\omega, u_1, \ldots, u_n)$ in $\Lambda \times [0, 1]^n$.*

Proof. We may take $\xi(1) = \xi$ and $S(1) = S$, where ξ and S are the maps constructed in Lemma (1.5). Now suppose that for some k the maps $\xi(1), \ldots, \xi(k)$ and $S(1), \ldots, S(k)$ have been constructed. We shall construct $\xi(k + 1)$ and $S(k + 1)$. Before doing this, a further inductive assumption will be placed on the $\xi(i)$, $i = 1, \ldots, k$. Let $\mathscr{H}_t$ denote the σ-algebra generated by $\mathscr{F}_t$ and all sets of the form $\{T(n) \leqq s\}$, $n = 1, 2, \ldots, s \leqq t$. It will be assumed that

(1.18) each $\xi(i)$ restricted to $\{T(i) \leqq t\}$ is $(\mathscr{H}_t \times \mathscr{B}^i)_+$-measurable, for $i = 1, \ldots, k$.

By (1.12) this assumption holds for $k = 1$.

Let θ: $\Lambda \times [0, 1]^k \to \Lambda \times [0, 1]^k$ be defined by $\theta(\omega, u_1, \ldots, u_k) = (\omega, \xi(1)(\omega, u_1), \ldots, \xi(k)(\omega, u_1, \ldots, u_k))$. Let $\Lambda' = \Lambda \times [0, 1]^k$, $\mathscr{K}' = \mathscr{K} \times \mathscr{B}^k$. Define Q' on $\mathscr{K}'$ by $Q'(A) = Q \times \lambda^k(\theta^{-1}(A))$. Let $\mathscr{F}' = \mathscr{F} \times \mathscr{B}^k$. Let $\mathscr{G}_t = \mathscr{F}_t \times \mathscr{B}^k$ on $\{S(k) \leqq t\}$, $\mathscr{G}_t = \mathscr{F}_t \times [0, 1]^k$ on $\{S(k) > t\}$. Let $\mathscr{F}_t' = \mathscr{G}_{t+}$.

As usual, $S(1), \ldots, S(k)$ and $T(k + 1)$ may be considered to be defined on

Λ'. We wish to show that

(1.19) $Q'(T(k+1) \leqq t|\mathscr{F}')$ is $\mathscr{F}_t'$-measurable mod Q' for all t.

Let $A = \{T(k+1) \leqq t\}$. Let Z be a bounded $\mathscr{F} \times [0, 1]^k$ measurable function. Let f be a bounded Borel function on $[0, 1]^k$. Define g on Λ' by $g(\omega, u_1, \ldots, u_k) = f(u_1, \ldots, u_k)$. It must be shown that (expectation with respect to Q')

(1.20) $E[Zg\chi_A] = E[ZgE[\chi_A|\mathscr{F}_t']]$.

We first show (expectation with respect to Q')

(1.21) $E[g\chi_A|\mathscr{F} \times [0, 1]^k] = E[g\chi_A|\mathscr{F}_t \times [0, 1]^k]$.

(1.21) is equivalent to

(1.22) $E[Wg\chi_A] = E[Vg\chi_A]$ for all bounded $\mathscr{F}$-measurable W, V (defined on Λ' in the obvious way) with $V = E[W|\mathscr{F}_t]$.

But $E[Wg\chi_A] = E[Wg \circ \theta\chi_A]$, where the first expectation is with respect to Q' and the second with respect to $Q \times \lambda^k$. By (1.18) the function $g \circ \theta\chi_A$ is $\mathscr{H}_t \times \mathscr{B}^k$-measurable. It follows from (1.14) that $E[Wg \circ \theta\chi_A] = E[Vg \circ \theta\chi_A]$. Thus (1.22) is proved, and hence (1.21). Returning to the proof of (1.20), $E[Zg\chi_A] = E[ZE[g\chi_A|\mathscr{F} \times [0, 1]^k]] = E[ZE[g\chi_A|\mathscr{F}_t \times [0, 1]^k]] = E[E[Z|\mathscr{F}_t \times [0, 1]^k]g\chi_A] = E[E[Z|\mathscr{F}_t']g\chi_A] = E[ZE[g\chi_A|\mathscr{F}_t']] = E[ZgE[\chi_A|\mathscr{F}_t']]$, so (1.20) is proved.

By Lemma (1.5), there exist maps ξ': $\Lambda \times [0, 1]^{k+1} \to [0, 1]$ and S': $\Lambda \times [0, 1]^{k+1} \to [0, \infty]$ such that ξ' is $\mathscr{K} \times \mathscr{B}^{k+1}$ measurable, ξ' has distribution λ, ξ' is independent of $\mathscr{F} \times \mathscr{B}^k \times [0, 1]$ with respect to Q', S' is an $\mathscr{F}_t' \times \mathscr{B}$-stopping time, hence an $(\mathscr{F}_t \times \mathscr{B}^{k+1})_+$-stopping time, and $T(k+1)(\omega) = S'(\omega, u_1, \ldots, u_k, \xi'(\omega, u_1, \ldots, u_{k+1}))$ mod $Q' \times \lambda$.

Define θ': $\Lambda \times [0, 1]^{k+1} \to \Lambda \times [0, 1]^{k+1}$ by $\theta'(\omega, u_1, \ldots, u_{k+1}) = (\theta(\omega, u_1, \ldots, u_k), u_{k+1})$. Let $\xi(k+1) = \xi' \circ \theta'$ and $S(k+1) = S(k) \vee S'$. Clearly (1.15), (1.16) and (1.17) hold with $n = k+1$. By Lemma (1.5), ξ' restricted to $\{T(k+1) \leqq t\}$ is $(\mathscr{H}_t \times \mathscr{B}^{k+1})_+$-measurable, so $\xi(k+1)$ restricted to $\{T(k+1) \leqq t\}$ is $(\mathscr{H}_t \times \mathscr{B}^{k+1})_+$-measurable. Thus (1.18) holds with k replaced by $k+1$. This proves Lemma (1.13).

Proof of Proposition (1.14). As noted earlier, we choose $L = [0, 1]^{\mathscr{N}}$, where $\mathscr{N}$ = the natural numbers, $\nu = \lambda^{\mathscr{N}}$, $\mathscr{L} = \mathscr{B}^{\mathscr{N}}$. We choose $(W, \mathscr{W}, \gamma) = (L, \mathscr{L}, \nu)$. Given $\Lambda = \Omega \times H, \mathscr{K} = \mathscr{M} \times \mathscr{D}$, $\{\mathscr{K}_t\}$, Q such that (1.1) holds, and a nondecreasing sequence $\{T(n)\}$ of $\mathscr{K}_t$-stopping times, let $\xi(n)$ and $S(n)$ be the maps described in Lemma (1.13). It is easy to see that we may take $S(n)$ to be defined on $\Omega \times [0, 1]^{\mathscr{N}}$ for all n and $\xi(n)$ to be defined on $\Lambda \times [0, 1]^{\mathscr{N}}$ for all n. The map φ: $\Lambda \times W \to \Omega \times L$ is then defined by

(1.23) $\varphi(\omega, h, y) = (\omega, \xi(1)(\omega, h, y), \xi(2)(\omega, h, y), \ldots)$ for all ω in Ω, h in H, y in $[0, 1]^{\mathscr{N}}$.

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Properties (1.2) and (1.3) follow from Lemma (1.13) so the proposition is proved.

The effect of the map (1.23) may be visualized more clearly by considering a fibre $\{\omega\} \times H \times [0, 1]^{\mathscr{N}}$ in $\Omega \times H \times [0, 1]^{\mathscr{N}}$, for some fixed ω in Ω. Q induces a certain conditional probability $Q(\omega)$ on this fibre. The map $\bar{\xi} \equiv (\xi(1), \xi(2), \ldots)$ restricted to this fibre sets up a measure-preserving correspondence between the fibre and the standard probability space $([0, 1]^{\mathscr{N}}, \mathscr{B}^{\mathscr{N}}, \lambda^{\mathscr{N}})$. φ thus maps $\Omega \times H \times [0, 1]^{\mathscr{N}}$ fibre by fibre onto $\Omega \times [0, 1]^{\mathscr{N}}$ in a measure preserving way.

An $\mathscr{M}_t$*-time change* T is a map $T: \Omega \times [0, \infty] \rightarrow [0, \infty]$, such that for each fixed a in $[0, \infty]$, $T(\cdot, a)$ is an $\mathscr{M}_t$-stopping time on Ω, and for each fixed ω in Ω, $T(\omega, \cdot)$ is nondecreasing and right continuous on $[0, \infty]$. The notions of a randomized time change and a distributional time change follow immediately from the definitions for stopping times. It would be interesting to know whether the two notions are essentially equivalent.

By Proposition (1.4) we can say that there is no essential difference between a randomized and a distributional stopping time, nor between increasing sequences of these objects. Nevertheless the question remains open, whether one can simultaneously represent all the stopping times making up a time change in the manner of Proposition (1.4). Indeed, it is not obvious that a decreasing sequence can be so represented. For practical purposes, however, the conceptual distinction between distributional and randomized time changes may not be important. It follows readily from Proposition (1.4) that (i) quasi-left-continuity, (ii) the strong Markov property, (iii) independence of increments with respect to the past, and (iv) the martingale property each carry over from the process $(\Omega, \mathscr{A}, \mathscr{M}_t, X_t, P)$ to the process $(\Lambda, \mathscr{K}, \mathscr{K}_t, X_t, Q)$ whenever (1.1) holds.

The following lemma will be useful in Section 3.

(1.24) Lemma. *Any finite sequence of distributional stopping times can be represented in the manner of Proposition* (1.4).

Proof. This lemma is a corollary of Proposition (1.4). One simply embeds the given sequence in a larger nondecreasing sequence. Inductively, having embedded $T(1), \ldots, T(k)$ in $U(1), \ldots, U(l)$, let $U(0) = 0$, $U(l + 1) = \infty$ and set $V(i) = (T(k + 1) \wedge U(i)) \vee U(i - 1)$ for $i = 1, \ldots, l + 1$. Embed $T(1), \ldots, T(k + 1)$ in the sequence $V(1), U(1), \ldots, V(l), U(l), V(l + 1)$. We can recover $T(k + 1)$ from the nondecreasing sequence by noting that $T(k + 1) = V(j)$, where j is the first i such that $V(i) < U(i)$ or $i = l + 1$. It is easy to see that this embedding carries over to the representation given by Proposition (1.4), so the lemma follows.

2. Compactness. Let $X = (\Omega, \mathscr{A}, \mathscr{M}_t, X_t, P)$ be as in Section 1. Let H denote the set of nondecreasing right continuous maps h from $[0, \infty]$ into

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$[0, \infty]$. H is a compact metric space with the usual topology (pointwise convergence on a dense set including ∞). Let $\mathscr{H}$ be the set of Borel sets on H. Let $\mathscr{H}_t$ be the σ-algebra generated by $\{h | h(a) \leqq s\}$, for all a in $[0, \infty)$, all $s \leqq t$. By right continuity, the coordinate maps $h \to h(a)$ are $\mathscr{H}$-measurable; hence $\mathscr{H}_t \subseteq \mathscr{H}$. $\{\mathscr{H}_t\}$ is clearly nondecreasing and right continuous.

Let $\Lambda, \mathscr{K}, \mathscr{F}, \{\mathscr{F}_t\}$ be as in Section 1. Any function on Ω will be extended to Λ in the obvious way. Let $\mathscr{K}_t = (\mathscr{M}_t \times \mathscr{H}_t)_+$. Let Q be a probability on $(\Lambda, \mathscr{H})$ such that Q has projection P on $(\Omega, \mathscr{A})$ and (1.1) holds. Define the time change T on $\Lambda \times [0, \infty]$ by

$$(2.1) \qquad T(\omega, h, a) = h(a).$$

Then T is a *distributional* $\mathscr{M}_t$-time change, by definition. It should be noted that any *ordinary* $\mathscr{M}_t$-time change S on $\Omega \times [0, \infty]$ can be represented in this way by defining the map ψ: $\Omega \to \Lambda$ by

$$(2.2) \qquad \psi(\omega) = (\omega, S(\omega, \cdot))$$

and letting

$$(2.3) \qquad Q = P\psi^{-1}.$$

We would like to prove compactness of distributional time changes. (Theorems (2.9) and (2.11)). As defined here, the time change is specified completely by the probability measure Q on $(\Lambda, \mathscr{K})$. We shall thus identify the time change with Q and define an appropriate topology on the set of such Q's.

Let Π denote the set of probabilities Q on $(\Lambda, \mathscr{K})$ such that Q has projection P on $(\Omega, \mathscr{M})$ and (1.1) holds. (Thus Π depends on $\{\mathscr{M}_t\}$.) Let $\mathscr{G}$ be a fixed σ-algebra contained in $\mathscr{A}$. For each Y in $\mathscr{L}_1(\Omega, \mathscr{G}, P)$ and each f in $\mathscr{C}(H)$ (where $\mathscr{C}(H)$ denotes the bounded continuous functions on H), define the map $\varphi(Y, f)$: $\Pi \to \mathbf{R}$ by

$$(2.4) \qquad \varphi(Y, f)(Q) = \int YfdQ.$$

(As noted earlier, we define Y on Λ in the obvious way.)

Let Φ be the set of all such φ. Let $\mathscr{T}$ be the topology on Π generated by all φ in Φ. It is easy to see that $\mathscr{T}$ is also generated by those maps $\varphi(\chi_G, f)$ obtained as G runs over an $\mathscr{L}_1$-dense subset of $\mathscr{G}$ and f runs over a sup-norm dense subset of $\mathscr{C}(H)$. In particular if $\mathscr{G}$ is countably generated mod P then $\mathscr{T}$ has a *countable base*. If $Q_n \to Q(\mathscr{T})$ then we write $Q_n \Rightarrow Q(\mathscr{G})$. This form of convergence is clearly similar to weak convergence on Λ, but the fact that P is fixed allows one to dispense with a topology on Ω+and at the same time to strengthen the definition of convergence.

Let a be in $(0, \infty)$. For any f in $\mathscr{C}([0, \infty])$, define g on H by $g(h) = f(h(a))$.

For sufficiently small $\delta > 0$ define $\varphi(\delta)$ and $\theta(\delta)$ on H by

$$(2.5) \qquad \varphi(\delta)(h) = \int_a^{a+\delta} f(h(s))ds\,(1/\delta), \quad \theta(\delta)(h) = \int_{a-\delta}^{a} f(h(s))ds(1/\delta).$$

Then $\varphi(\delta)$ and $\theta(\delta)$ are in $\mathscr{C}(H)$ but g is not. Clearly $\varphi(\delta) \to g$ everywhere on H as $\delta \to 0$.

(2.6) LEMMA. *There exists a set A (depending on Q) such that $A \subseteq [0, \infty]$, $[0, \infty]$-A has Lebesgue measure* 0, *and $h(s) \to h(a)$ pointwise* mod Q *as $s \to a$, for all a in A. In particular, $\theta(\delta) \to g$* mod Q *as $\delta \to 0$, for all f in $\mathscr{C}[0, \infty]$), a in A.*

Proof. Each function h has at most countably many points of discontinuity. Thus the lemma follows from Fubini's Theorem.

(2.7) LEMMA. *There exists a set A (depending on Q) such that $A \subseteq [0, \infty]$, $[0, \infty]$-A has Lebesgue measure* 0, *with the property that if $Q_n \Rightarrow Q(\mathscr{G})$, if $a_1, \ldots, a_k$ are in A, f is in $\mathscr{C}([0, \infty]^k)$, and Y is in $\mathscr{L}_1(\Omega, \mathscr{G}, P)$, then*

$$(2.8) \qquad \int Yf(h(a_1), \ldots, h(a_k))dQ_n \to \int Yf(h(a_1), \ldots, h(a_k))dQ \text{ as } h \to \infty.$$

A can be chosen to contain the point ∞.

Proof. Let A be chosen as in Lemma (2.6), adding ∞ if necessary. By the Stone-Weierstrass Theorem it is enough to prove (2.8) when $f = f_1 \ldots f_n$, where f_i in $\mathscr{C}([0, \infty])$ is nondecreasing. Let g_i, $\varphi_i(\delta)$, and $\theta_i(\delta)$ be defined as above. If $a_i = \infty$ let $\varphi_i = \theta_i = g_i = f(h(\infty))$. Assume $Y \geqq 0$. Then

$$\int Y\varphi_1(\delta) \ldots \varphi_k(\delta)dQ \geqq \overline{\lim} \int Yg_1 \ldots g_k dQ_n$$

and

$$\underline{\lim} \int Yg_1 \ldots g_k dQ_n \geqq \int Y\theta_1(\delta) \ldots \theta_k(\delta)dQ.$$

Letting $\delta \to 0$ proves the lemma.

Let Π' be the set of all probabilities Q on $(\Lambda, \mathscr{K})$ such that Q has projection P on $(\Omega, \mathscr{A})$. (That is, $\Pi' = \Pi$ with $\mathscr{M}_t = \mathscr{M}$.)

(2.9) THEOREM. *If $\mathscr{M} \subseteq \mathscr{G}$* mod P, *then Π is a closed subset of Π'.*

Proof. Let Y be a fixed element of $\mathscr{L}_1(\Omega, \mathscr{M}, P)$. Let $Z = E[Y|\mathscr{M}_t]$. We must show that if Q is a limit point of Π in Π' then

$$(2.10) \qquad \int YUdQ = \int ZUdQ \quad \text{for all bounded } \mathscr{K}_t\text{-measurable } U.$$

For each $\epsilon > 0$ let $Z_\epsilon = E[Y|\mathscr{M}_{t+\epsilon}]$. Let $\mathscr{G}'$ be the σ-algebra generated by

Y and the Z_ϵ. Since $\mathscr{G}'$ is countably generated mod P we may choose a sequence Q_n such that $Q_n \Rightarrow Q(\mathscr{G}')$.

Let $a_1, \ldots, a_k$ in $[0, \infty]$ and $s_1, \ldots, s_k \leqq t$ be given. For any $\delta > 0$, choose $f_1, \ldots, f_k$ bounded, continuous and nonincreasing on $[0, \infty]$ such that $f_1 = 1$ on $[0, s_i - \delta)$ and $f_i = 0$ on $[s_i, \infty]$. Let $b_1, \ldots, b_k$ be in the set A described in Lemma (2.7). Let $g = f_1(h(b_1)), \ldots, f_k(h(b_k))$. Then $g \in \mathscr{K}_t$. Since Q_n is in Π, $\int YgdQ_n = \int ZgdQ_n$. Letting $n \to \infty$, $\int YgdQ = \int ZgdQ$. Letting $\epsilon \to 0$, $\int Y\chi dQ = \int Z\chi dQ$, where χ is the characteristic function of $\{h(b_1) < s_1\} \cap \ldots \cap \{h(b_k) < s_k\}$. Approximating $a_1 \ldots a_k$ from above by $b_1, \ldots, b_k$ we have that $\int Y\chi dQ = \int Z\chi dQ$, where χ is the characteristic function of $\{h(a_1) < s_1\} \cap \ldots \cap \{h(b_k) < s_k\}$. Hence $\int YUdQ = \int Z_\epsilon UdQ$ for any bounded H_t-measurable U and any $\epsilon > 0$. But then $\int YUdQ = \int Z_{2\epsilon}UdQ$ for any bounded $\mathscr{M}_{t+\epsilon} \times H_{t+\epsilon}$-measurable U. Letting $\epsilon \to 0$ proves the theorem.

(2.11) THEOREM. *Π' is compact.*

Proof. Clearly we may take $\mathscr{G} = \mathscr{A}$. It is easy to check that a probability Q on $(\Lambda, \mathscr{K})$ with projection P on $(\Omega, \mathscr{A})$ is equivalent to a map β: $\mathscr{L}_1(\Omega, \mathscr{A}, P) \times \mathscr{C}(H) \to \mathbf{R}$ which is bilinear, positive and satisfies $\beta(1, 1) = 1$, $|\beta(Y, f)| \leqq ||Y|| \, ||f||$, where the first factor is an $\mathscr{L}_1$-norm and the second is a sup-norm. The space of such β is compact, by the same argument that shows that the unit ball of the dual of a normed linear space is compact in the weak* topology. This proves the theorem.

3. Convergence. Let $\mathscr{G}$ be a σ-algebra contained in $\mathscr{A}$ such that each X_t is $\mathscr{G}$-measurable mod P. Let Q_n and Q be probabilities in Π such that $Q_n \Rightarrow Q(\mathscr{G})$. Define the distributional $\mathscr{M}_t$-time change T on Λ by $T(\omega, h, a) = h(a)$. For a given a in $[0, \infty]$, it is unfortunately not true that the Q_n-distribution of $X(T(\cdot, a))$ converges to the Q-distribution of $X(T(\cdot, a))$. Indeed, the distribution of $T(\cdot, a)$ itself does not converge. However, with some assumptions on X, a slightly weaker convergence can be proved to hold. From now on in this section, let X be right continuous and quasi-left-continuous for all times less than ∞. Let $a_i < b_i$ be in $[0, \infty)$, $i = 1, \ldots, k$. Let f_i be in $\mathscr{C}(E)$, $i = 1, \ldots, k$. Define φ_i by

$$(3.1) \qquad \varphi_i(h) = \int_{a_i}^{b_i} f_i(X(h(s)))ds.$$

(3.2) THEOREM. *Suppose that*

$$(3.3) \qquad \lim_{c \to \infty} Q_n\left(\max_i h(b_i) > c\right) = 0$$

uniformly over all n. Then the joint Q_n-distribution of the φ_i converges to the joint Q-distribution.

Variants of Theorem (3.2) could be proved for more general assumptions on X, but for simplicity we shall confine ourselves to the case just given.

In the proof, the following result is needed.

(3.4) THEOREM. *Let* $Y = (\Omega, \mathscr{A}, \mathscr{M}_t, Y_t, P)$ *be right continuous and quasi-left-continuous for all times less than* ∞, *taking values in a compact metric space. Let* $S(1, n), \ldots, S(k, n)$, $n = 1, 2, \ldots,$ *and* $S(1), \ldots, S(k)$ *be randomized* $\mathscr{M}_t$*-stopping times on some* $\Omega \times L$ *such that*

$$(3.5) \qquad \lim_{c \to \infty} P \times \nu \left(\max_i S(i, n) > c \right) = 0$$

uniformly over all n. *Suppose that for every set* G *in* $\mathscr{G}$, *the joint distribution of the* $S(i, n)$ *restricted to* $G \times L$ *converges to the joint distribution of* $S(i)$, $i = 1, \ldots, k$, *restricted to* $G \times L$. *Then for every set* G *in* $\mathscr{G}$ *the joint distribution of the* $Y(S(i, n))$, $i = 1, \ldots, k$, *restricted to* $G \times L$ *converges to the joint distribution of the* $Y(S(i))$ *restricted to* $G \times L$.

Theorem (3.4) is proved under more general conditions for the case $k = 1$ in [**1**] (Theorems (1.9) and (1.11) of that paper). The proof for arbitrary k is almost identical to that for $k = 1$ and will be omitted.

Proof of Theorem (3.2). It will be shown first that

$$(3.6) \qquad \lim_{n \to \infty} \int \varphi_1 \ldots \varphi_k \, dQ_n = \int \varphi_1 \ldots \varphi_k \, dQ.$$

Let $\Lambda' = \Lambda \times [a_1, b_1] \times \ldots \times [a_k, b_k]$, $\mathscr{K}' = \mathscr{K} \times \mathscr{B}_1 \times \ldots \times \mathscr{B}_k$, where $\mathscr{B}_i$ is the collection of Borel sets on $[a_i, b_i]$. Let $\mathscr{K}_t'$ be $(\mathscr{K}_t \times \mathscr{B}_1 \times \ldots \times \mathscr{B}_k)_+$, $Q_n' = Q_n \times \lambda_1 \times \ldots \times \lambda_k$, $Q' = Q \times \lambda_1 \times \ldots \times \lambda_k$ where λ_i is normalized Lebesgue measure on $[a_i, b_i]$. Let $\mathscr{F}' = \mathscr{M} \times H \times [a_1, b_1] \times \ldots \times [a_k, b_k]$, $\mathscr{F}_t' = \mathscr{M}_t \times H \times [a_1, b_1] \times \ldots \times [a_k, b_k]$. It is easy to see that the primed version of (1.1) holds for Q_n and Q. Define $T(i)$ on Λ' by $T(i)(\omega, h, s_1, \ldots, s_k) = h(s_i)$. Then for each Q_n and Q, $T(i)$ is a $\mathscr{K}_t'$-stopping time, for $i = 1, \ldots, k$.

Since $Q_n \Rightarrow Q(\mathscr{G})$ it follows that for each set G in $\mathscr{G}$, the joint Q_n'-distribution of the $T(i)$ restricted to $G \times H \times [a_1, b_1] \times \ldots \times [a_k, b_k]$ converges to the corresponding Q'-distribution as $n \to \infty$.

By Lemma (1.24), if $L = [0, 1]^{\mathscr{N}}$, $\mathscr{L} = \mathscr{B}^{\mathscr{N}}$, $\nu = \lambda^{\mathscr{N}}$, we can find $(\mathscr{M}_t \times \mathscr{L})_+$-stopping times $S(1, n), \ldots, S(k, n), S(1), \ldots, S(k)$ on $\Omega \times L$ such that for each set G in $\mathscr{G}$, and each n, the $P \times \nu$-distribution of the $S(i, n)$ restricted to $G \times L$ is equal to the joint Q_n'-distribution of the $T(i)$ restricted to $G \times H \times [a_1, b_1] \times \ldots \times [a_k, b_k]$. A similar statement holds for $S(i)$ and Q'. Let $Y = (f_1(X), \ldots, f_k(X))$, a process taking its values in a compact subset of $\mathbf{R}^k$. By Theorem (3.4), the joint distribution of the $f_i(Y(T(i)))$ restricted to $G \times H \times [a_1, b_1] \times \ldots \times [a_k, b_k]$ converges to the corresponding Q-distribution as $n \to \infty$. This at once implies (3.6). But since the pairs a_i, b_i and the functions f_i were not assumed to be distinct, (3.6) implies the seemingly stronger equation

$$(3.7) \quad \lim_{n\to\infty} \int \varphi_1{}^{j(1)} \ldots \varphi_k{}^{j(k)} dQ_n = \int \varphi_1{}^{j(1)} \ldots \varphi_k{}^{j(k)} dQ,$$

for all nonnegative integral powers $j(1), \ldots, j(k)$. This proves Theorem (3.2).

4. Continuous time changes. When studying continuous time changes the appropriate choice for the space H of Section 1 is the collection of all non-decreasing continuous maps from $[0, \infty)$ into $[0, \infty)$ with the usual complete metric. In this case the maps $h \to h(a)$ are continuous on H, so that the proofs become simpler. The analogue of theorem (3.2) holds with φ_i of the form $\varphi_i(h) = f(h(a_i))$, a stronger result. Let Π' be the set of all probability measures on $(\Lambda, \mathscr{K})$ with projection P on $(\Omega, \mathscr{A})$. Let Π denote the subset of Π' for which (1.1) holds. The analogue of Theorem (2.9) is easily seen to hold. Only in the proof of compactness is a little more care needed. Since H is not compact, some tightness condition must be assumed.

(4.1) THEOREM. *Let H_n be a sequence of compact subsets of H. Let α_n be a sequence of numbers, $\alpha_n \nearrow 1$. Let Π_0' be the set of measures Q in Π' such that $Q(\Omega \times H_n) \geqq \alpha_n$ for all n. Then Π_0' is compact.*

Proof. Let β be a map from $\mathscr{L}_1(\Omega, \mathscr{A}, P) \times \mathscr{C}(H)$ which is bilinear, positive, and satisfies $\beta(1, 1) = 1$, $|\beta(Y, f)| \leqq ||Y|| \, ||f||$. Suppose $\beta(1, f) \geqq \alpha_n$ for each f in $\mathscr{C}(H)$ with $f \geqq 0$ on H, $f \geqq 1$ on H_n. It is a straightforward task to show that there exists a probability Q in Π_0' such that $\int YfdQ = \beta(Y, f)$ for all Y, f. Thus, to prove compactness of Π_0' it is enough to prove compactness for the set of all β. Just as in Theorem (2.11), this set is compact by the same argument that proves the unit ball in the dual of a normed linear space is compact in the weak* topology. This proves Theorem (4.1).

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