ON THE SYMMETRY OF CUBIC GRAPHS

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1. Introduction. Let G be a connected finite graph in which each edge has two distinct ends and no two distinct edges have the same pair of ends. We suppose further that G is *cubic*, that is, each vertex is incident with just three edges.

An *s-path* in G, where s is any positive integer, is a sequence $S = (v_0, v_1, \ldots, v_s)$ of $s + 1$ vertices of G, not necessarily all distinct, which satisfies the following two conditions:

(i) Any three consecutive terms of S are distinct.

(ii) Any two consecutive terms of S are the two ends of some edge of G.

If these conditions hold we call v_0 the *head* and v_s the *tail* of S.

An *automorphism* of G is a 1-1 mapping f of the set $V(G)$ of verticew of G onto itself such that fv and fw are the two ends of an edge of G if and only if v and w are the two ends of an edge of G. The automorphisms of G constitute a group $\mathbf{A}(G)$.

If $S = (v_0, v_1, \ldots, v_s)$ is any s-path of G we write fS for the s-path $(fv_0, fv_1, \ldots, fv_s)$, for each $f \in \mathbf{A}(G)$. We say G is *s-regular* if for each ordered pair $\{S, T\}$ of s-paths of G, not necessarily distinct, there is a unique element f of $\mathbf{A}(G)$ such that $T = fS$. Our main object in this paper is the proof of the following

THEOREM. *Suppose all the oriented edges of G are equivalent under* $\mathbf{A}(G)$. *Then there exists a positive integer s such that G is s-regular.*

As is explained in **(3)** the s-regular cubic graphs can be divided into two classes according to the nature of the group $\mathbf{A}(G)$. The first of these classes is discussed in the main paper of **(3)** and the second in the Addendum. It is shown that s is at most 5 for the first class and at most 4 for the second. Examples of graphs, of the first class only, are given for the values 2, 3, 4, and 5 of s. Other examples of the first class are given in **(1)**. In **(2)**, Frucht describes a 1-regular graph, without determining to which of the two classes it belongs. In §3 of the present paper we show that all 1-regular cubic graphs belong to the second class. Accordingly Frucht's graph is the first known member of this class.

2. Proof of the theorem. Let $S = (v_0, v_1, \ldots, v_s)$ be any s-path of G. The edges incident with v_0 join it to just two vertices, w and w' say, other than v_1.

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Similarly the edges incident with v_s join it to just two vertices, x and x' say, other than v_{s-1}. We call the s-paths $(w, v_0, v_1, \ldots, v_{s-1})$ and $(w', v_0, v_1, \ldots, v_{s-1})$ the *successors* of S. Similarly we call $(v_1, v_2, \ldots, v_s, x)$ and $(v_1, v_2, \ldots, v_s, x')$ the *predecessors* of S.

An s-path S' of G is *accessible* from S if there is a finite sequence $(S_1, S_2, \ldots, S_k)$ of s-paths of G satisfying the following conditions.

(i) $S_1 = S$ and $S_k = S'$.

(ii) For $1 \leqslant i < k$, S_{i+1} is either a predecessor or a successor of S_i in G.

(2.1) *Any s-path of G is accessible from any other (with the same value of s).*

Proof. Let S be any s-path of G. Let $\mathbf{W}$ be the class of all s-paths of G accessible from S. Then $S \in \mathbf{W}$.

Let V be the class of all vertices of G belonging to at least one member of $\mathbf{W}$. If $V(G) - V$ is not null then, by the connection of G we can find an edge E of G having one end p in $V(G) - V$ and one end q in V. Now q belongs to some $U_1 \in \mathbf{W}$. Starting with U_1 and taking predecessors as often as necessary we obtain $U_2 \in \mathbf{W}$ having q as its head. But then p is the head of a successor of U_2, contrary to the definition of p. We deduce that $V = V(G)$.

Let $S' = (w_0, w_1, \ldots, w_s)$ be any s-path of G. By the result just proved there exists $Z_1 \in \mathbf{W}$ such that w_s is a vertex of Z_1. From Z_1 by taking successors we obtain $Z_2 \in \mathbf{W}$ having w_s as its tail. From Z_2 by taking predecessors we can obtain $Z_3 \in \mathbf{W}$ with the following properties: The head of Z_3 is w_s and the second term of Z_3 is not w_{s-1}. Accordingly S' can be obtained from Z_3 by taking successors. Hence $S' \in \mathbf{W}$.

(2.2) *Suppose there is a positive integer s such that all the s-paths of G are equivalent under $\mathbf{A}(G)$ but G is not s-regular. Then for each s-path S of G there exists $f \in \mathbf{A}(G)$ such that $fS = S$ and f interchanges the two successors of S.*

Proof. Let S be any s-path of G, with successors T and T'. Since G is not s-regular there are s-paths Z_1 and Z_2 and distinct elements x and y of $\mathbf{A}(G)$ such that $Z_1 = xZ_2$ and $Z_1 = yZ_2$. There exists $z \in \mathbf{A}(G)$ such that $Z_1 = zS$. Write $g = z^{-1}xy^{-1}z$. Then $gS = S$ but g is not the identical automorphism of G.

We now show that there is an s-path S_1 of G such that S_1, but no successor of S_1, is invariant under g. For suppose not. Then if Z is any s-path invariant under g one of the successors of Z is also invariant under g. Hence both successors of Z are invariant under g. Let Z^{-1} be the s-path obtained from Z by reversing the order of the vertices. Then Z^{-1} and its successors are invariant under g. Hence the predecessors of Z are invariant under g, since they are obtained from the successors of Z^{-1} by reversing the order of vertices. Hence each s-path of G is invariant under g, by (2.1). This is impossible since g is not the identical automorphism of G.

Let the successors of S_1 be T_1 and T_1'. Let h be one of the elements of $\mathbf{A}(G)$ satisfying $hS = S_1$. Then h maps the successors of S onto the successors of S_1. We may adjust the notation so that $hT = T_1$ and $hT' = T_1'$. Since no successor of S_1 is invariant under g we have $gT_1 = T_1'$ and $gT_1' = T_1$. Write $f = h^{-1}gh$. Then $fS = S$, $fT = T'$ and $fT' = T$. This completes the proof of (2.2).

We complete the proof of the main theorem as follows. We are given that all the oriented edges of G are equivalent under $\mathbf{A}(G)$. Hence there is a greatest positive integer s such that all the s-paths of G are equivalent under $\mathbf{A}(G)$. Assume G is not s-regular.

Consider any $(s+1)$-path $X = (v_0, v_1, \ldots, v_{s-1})$ of G. Let its successors be $Y = (y, v_0, v_1, \ldots, v_s)$ and $Y' = (y', v_0, v_1, \ldots, v_s)$. Let k be one of the elements of $\mathbf{A}(G)$ satisfying $k(v_1, v_2, \ldots, v_{s-1}) = (v_0, v_1, \ldots, v_s)$. Then kv_0 is either y or y', and we may adjust the notation so that $kv_0 = y$. This implies $Y = kX$. By (2.2) there exists $f \in \mathbf{A}(G)$ such that $f(v_0, v_1, \ldots, v_s) = (v_0, v_1, \ldots, v_s)$, $fy = y'$ and $fy' = y$. This implies $Y' = fY = fkX$. Hence X can be transformed into either of its successors by an automorphism of G. Similarly each predecessor of X can be transformed into X, and therefore X can be transformed into either of its predecessors.

Combining these results with (2.1) we see that all the $(s+1)$-paths of G are equivalent under $\mathbf{A}(G)$. But this contradicts the definition of s. The theorem follows.

We should perhaps remark here that in an s-regular cubic graph G the vertices of any s-path are all distinct. For any circuit of G has at least 3 vertices by our definitions and at least $2s - 2$ by (**3**, Theorem III).

3. 1-regular cubic graphs. Let G be any s-regular graph. Let S_0 be a fixed s-path of G. Any element x of $A(G)$ can be associated with a permutation x' of the set $\mathbf{S}$ of all s-paths of G defined as follows: $x'(hS_0) = hxS_0$ for all $h \in \mathbf{A}(G)$. If $x, y \in \mathbf{A}(G)$ the permutations x', y' satisfy the law $(xy)' = y'x'$. Moreover, the correspondence $x \leftrightarrow x'$ is $1-1$. For $x'(S_0) = xS_0$ for each $x \in \mathbf{A}(G)$, and there is only one element of $\mathbf{A}(G)$ mapping S_0 onto xS_0. It follows that the permutations x' of $\mathbf{S}$ are the elements of a permutation group H of the same order as $\mathbf{A}(G)$.

In (**3**) the symmetry of G is discussed in terms of H. There is a unique pair (r, l) of elements of H mapping S_0 into its two successors. It follows that rS and lS are the two successors of S for each $S \in \mathbf{S}$. Another important element of H, denoted by ξ, reverses the order of the vertices in each s-path S. Thus $\xi^2 = 1$.

It is clear that $\xi r \xi S$ and $\xi l \xi S$ are the two predecessors $r^{-1}S$ and $l^{-1}S$ of S, for each $S \in \mathbf{S}$. There are now two possibilities. In the first alternative $\xi r \xi = r^{-1}$ and $\xi l \xi = l^{-1}$; in the second $\xi r \xi = l^{-1}$ and $\xi l \xi = r^{-1}$. We say that G is of the *first* or *second class* according as the first or second alternative holds. These are the two classes mentioned in the Introduction.

(3.1) *Every* 1-*regular cubic graph is of the second class.*

Proof. Let G be any 1-regular cubic graph. Let a be any vertex of G. Let its incident edges join it to the three vertices b, c, and d. We may suppose $r(a, b) = (c, a)$ and $l(a, b) = (d, a)$. Then $l^{-1}rl^{-1}(d, a) = l^{-1}r(a, b) = l^{-1}(c, a)$. But $l^{-1}(c, a)$ is distinct from $r^{-1}(c, a) = (a, b)$, and is therefore $(a, d) = \xi(d, a)$. Hence $\xi = l^{-1}rl^{-1}$. Similarly $r^{-1}lr^{-1}(c, a) = \xi(c, a)$ and therefore $\xi = r^{-1}lr^{-1}$. Hence $\xi r \xi = r^{-1}lr^{-1}rl^{-1}rl^{-1} = l^{-1}$. Accordingly G is of the second class.

References

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