J. Austral. Math. Soc. 25 (Series A) (1978), 215–229

ON GENERALIZATIONS OF C^*-EMBEDDING FOR WALLMAN RINGS

H. L. BENTLEY and B. J. TAYLOR

(Received 20 May 1976)
Communicated by E. Strzelecki

Abstract

Biles (1970) has called a subring A of the ring $C(X)$, of all real valued continuous functions on a topological space X, a Wallman ring on X whenever $Z(A)$, the zero sets of functions belonging to A, forms a normal base on X in the sense of Frink (1964). Previously, we have related algebraic properties of a Wallman ring A to topological properties of the Wallman compactification $w(Z(A))$ of X determined by the normal base $Z(A)$. Here we introduce two different generalizations of the concept of "a C^*-embedded subset" and study relationships between these and topological (respectively, algebraic) properties of $w(Z(A))$ (respectively, A).

Subject classification (Amer. Math. Soc. (MOS) 1970): 54 C 45, 54 C 40, 54 C 50 54 E 05.

1. Introduction

H. Wallman (1938) gave a method for associating a compact T_1 space $w(F)$ with a distributive lattice F; $w(F)$ is the space of all F-ultrafilters and the topology of $w(F)$ has as a base for closed sets a lattice F^* which is isomorphic to the lattice F.

O. Frink (1964) defined the concept of a normal base F on a Tychonoff space X and he applied Wallman's construction to obtain Hausdorff compactifications $w(F)$ of X. Throughout this paper X will denote a Tychonoff space (completely regular + Hausdorff).

1.1. Definition. A collection F of closed subsets of X is called a *lattice of closed subsets* of X provided that:

(1) $\emptyset, X \in F$; and

(2) if $A, B \in F$ then $A \cap B \in F$ and $A \cup B \in F$.

1.2. Definition. A base F for the closed subsets of X is called a *normal base* on X provided:

(1) F is a lattice of closed subsets of X.

(2) F is disjunctive (that is, if $A \in F$ and $x \in X - A$, then there exists $B \in F$ with $x \in B$ and $A \cap B = \emptyset$).

(3) F is normal (that is, if $A, B \in F$ with $A \cap B = \emptyset$, then there exist $C, D \in F$ with $A \cap D = \emptyset$, $B \cap C = \emptyset$ and $C \cup D = X$).

If F is a normal base on X, then $w(F)$ is the set of all F-ultrafilters which becomes a space as follows: If $A \in F$, let A^* be the set of all F-ultrafilters having A as a member. F^* then denotes the set of all A^* with $A \in F$. F^* is a base for the closed sets of a topology on $w(F)$. $w(F)$ with this topology is always a Hausdorff compactification of X. Here X is embedded into $w(F)$ by the map which sends each point $x \in X$ to the F-ultrafilter $\{A \in F \mid x \in A\}$.

Frink observed that the family $Z(X)$ of all zero sets of continuous real valued functions on X is a normal base on X which gives rise to a compactification $w(Z(X))$ equivalent to the Stone–Čech compactification βX of X. He also observed that if Y is any given compactification (all spaces are Hausdorff) of X, and if $E(X, Y)$ denotes the subset of $C(X)$ consisting of those real-valued continuous functions on X which are continuously extendible to all of Y, then $Z(E(X, Y))$, the zero sets of such functions, is a normal base on X. Biles (1970) later called a subring A of $C(X)$ a *Wallman ring* on X provided $Z(A)$, the zero sets of functions in A, is a normal base on X. Bentley and Taylor (1975) studied relationships between algebraic properties of a Wallman ring A and topological properties of the compactification $w(Z(A))$ of X.

We adopt our notation and terminology from our earlier paper (1975); these are mostly consistent with that of Gillman and Jerison (1960).

2. Generalizations of C^*-embedding

A well-known example of a Wallman ring is $C^*(X)$ whose Wallman compactification is equivalent to the Stone–Čech compactification. Since considerable work has been done with $C^*(X)$, it is natural to investigate which properties of $C^*(X)$ carry over to arbitrary Wallman rings.

In this paper we shall study two concepts which are related to a generalization of the question of when is a subset of a space C^*-embedded in the space.

Gillman and Jerison (1960), p. 89 proved that if S is a subspace of X, then S is C^*-embedded in X if and only if $\mathrm{Cl}_{\beta X} S \cong \beta S$. We begin with the investigation of a generalization of this property. If we replace Stone–Čech compactifications with Wallman compactifications induced by arbitrary Wallman rings on X and S, then $\mathrm{Cl}_{\beta X} S \cong \beta S$ leads us to the following definition.

2.1. Definition. If $S \subset X$, A is a Wallman ring on X, and B is a Wallman ring on S, then S is *(B, A)-embedded in X* if and only if $\mathrm{Cl}_{w(Z(A))} S \cong w(Z(B))$.

2.2. Theorem. *If $S \subset X$, then S is $(C^*(S), C^*(X))$-embedded in X if and only if S is C^*-embedded in X.*

The proof of this theorem, which shows (B, A)-embedding is in fact a generalization of C^*-embedding, is an immediate consequence of the aforementioned property from Gillman and Jerison (1960) and the property shown by Frink (1964) that for any Tychonoff space X, $w(Z(X)) \cong \beta X$.

Many of the results in Gillman and Jerison (1960) involving $C^*(X)$ and C^*-embedded subsets of X have generalizations involving (B, A)-embedding. We give a sample of these. The results from Gillman and Jerison (1960) which motivated the theorems are given as corollaries.

2.3. THEOREM. *If Y is a compactification of X and A is a Wallman ring on X, then $Y \cong w(Z(A))$ if and only if X is $(A, C^*(Y))$-embedded in Y.*

PROOF. $Y \cong w(Z(A))$ is equivalent to $\mathrm{Cl}_{w(Z(Y))} X \cong w(Z(A))$ which is equivalent to X is $(A, C^*(Y))$-embedded in Y.

2.4. COROLLARY (Gillman and Jerison (1960), Theorem 2.5 (II)). *If Y is a compactification of X, then $Y \cong \beta X$ if and only if X is C^*-embedded in Y.*

2.5. THEOREM. *If $S \subset X$, A is a Wallman ring on X and B is a Wallman ring on S, then S is (B, A)-embedded in X if and only if S is $(B, C^*(w(Z(A))))$-embedded in $w(Z(A))$.*

PROOF. Since $w(Z(C^*(Y))) \cong Y$ for any compact space Y, we have that the following are equivalent:

(a) $\mathrm{Cl}_{w(Z(A))} S \cong w(Z(B))$;

(b) $\mathrm{Cl}_{w(Z(C^*(w(Z(A)))))} S \cong w(Z(B))$;

(c) S is $(B, C^*(w(Z(A))))$-embedded in $w(Z(A))$.

2.6. COROLLARY (Gillman and Jerison (1960), Theorem 6.9). *If $S \subset X$, then S is C^*-embedded in X if and only if it is C^*-embedded in βX.*

2.7. THEOREM. *If $S \subset X$, S is compact, A is a Wallman ring on X and B is a Wallman ring on S, then S is (B, A)-embedded in X.*

PROOF. Since S is compact and $S \subset w(Z[A])$, we have $w(Z(B)) \cong S$ and $\mathrm{Cl}_{w(Z(A))} S \cong S$. Therefore $\mathrm{Cl}_{w(Z(A))} S \cong w(Z(B))$.

2.8. COROLLARY (Gillman and Jerison (1960), Theorem 6.9b). *Every compact set S in X is C^*-embedded in X.*

2.9. THEOREM. *If $S \subset X$, A is a Wallman ring on X and B is a Wallman ring on S, then S is (B, A)-embedded in X if and only if S is $(B, C^*(\mathrm{Cl}_{w(Z(A))} S))$-embedded in $\mathrm{Cl}_{w(Z(A))} S$.*

PROOF. The following are equivalent:

(a) S is (B, A)-embedded in X;

(b) $\mathrm{Cl}_{w(Z(A))} S \cong w(Z(B))$;

(c) $\mathrm{Cl}_{\mathrm{Cl}_{w(Z(A))}S} S \cong w(Z(B))$;

(d) $\mathrm{Cl}_{w(Z(C^*(\mathrm{Cl}_{w(Z(A))}S)))} S \cong w(Z(B))$;

(e) S is $(B, C^*(\mathrm{Cl}_{w(Z(A))} S))$-embedded in $\mathrm{Cl}_{w(Z(A))} S$.

2.10. COROLLARY (Gillman and Jerison (1960), Theorem 6.9). *S is C^*-embedded in X if and only if S is C^*-embedded in* $\mathrm{Cl}_{\beta X} S$.

Proximities are a useful tool in the investigation of Wallman rings since they simplify the notation and help isolate critical ideas. So we now introduce a proximity induced by a Wallman ring. For basic information on proximities we refer the reader to Naimpally and Warrack (1970).

2.11. DEFINITION. If A is a Wallman ring on X then a *binary relation* δ_A on the power set of X is defined by: if $E, F \subset X$, then $E\,\delta_A\,F$ (E and F are A-near) if and only if whenever $f, g \in A$ with $E \subset Z(f)$ and $F \subset Z(g)$ then $Z(f) \cap Z(g) \neq \varnothing$. $E\,\delta^{\dagger}_A\,F$ means not ($E\,\delta_A\,F$) and is read as E and F are A-far.

The following theorem gives an equivalent definition for this proximity for inverse closed Wallman rings (Hager, 1960).

2.12. THEOREM. *If A is an inverse closed Wallman ring on X, then for H_1 and H_2 subsets of X, $H_1\,\delta^{\dagger}_A\,H_2$ if and only if there is a function $f \in A$ such that $f(H_1) = 0$ and $f(H_2) = 1$.*

PROOF. If $H_1\,\delta^{\dagger}_A\,H_2$, then there are functions f_1 and $f_2 \in A$ such that $H_1 \subset Z(f_1)$, $H_2 \subset Z(f_2)$ and $Z(f_1) \cap Z(f_2) = \varnothing$. Let $f = f_1^2/(f_1^2 + f_2^2)$. Then $f \in A$, since

$$Z(f_1^2 + f_2^2) = Z(f_1) \cap Z(f_2) = \varnothing;$$

also

$$f(H_1) = 0 \quad \text{and} \quad f(H_2) = 1.$$

If there is a function $f \in A$ such that $f(H_1) = 0$ and $f(H_2) = 1$, then $H_2 \subset Z(f-1)$ and $Z(f) \cap Z(f-1) = \varnothing$. $f - 1 \in A$ since the constant function 1 belongs to every inverse closed Wallman ring. Therefore $H_1\,\delta^{\dagger}_A\,H_2$.

2.13. DEFINITION. Let G and H be sets and let A be a collection of real valued functions, then G and H are *completely A-separated* if there are disjoint zero sets Z_1 and $Z_2 \in Z(A)$ such that $G \subset Z_1$ and $H \subset Z_2$.

2.14. DEFINITION. Let S be a set. Let A and B be two sets of real valued functions, whose domains may vary from function to function. Then we write:

(1) $A \leqslant_S B$ if and only if any two subsets of S which are completely A-separated are also completely B-separated.

(2) $A \cong_S B$ if and only if $A \leqslant_S B$ and $B \leqslant_S A$.

In terms of the notation of Bentley and Taylor (1975) we have the following.

2.15. THEOREM. *If X is a space and A and B are subsets of $C(X)$, then $A \leqslant_X B$ if and only if $A \leqslant B$, and $A \cong_X B$ if and only if $A \cong B$.*

In our study of subspaces which may be (B, A)-embedded in a space, we shall want to be able to consider the proximity δ_A restricted to the subspace and so we have the following theorem.

2.16. THEOREM. *If $S \subset X$, A is a Wallman ring on X, δ' is the proximity on S induced by δ_A and B is a Wallman ring on S, then $A \cong_S B$ if and only if $\delta' = \delta_B$.*

Given two proximities on the same space, as we had in the previous theorem, we have the following standard partial order relation.

2.17. DEFINITION. If δ_1 and δ_2 are proximities on a space X, then $\delta_1 \leqslant \delta_2$ if and only if, for any two subsets G and H of X, $G \, \delta_1 \, H$ implies $G \, \delta_2 \, H$. Thus $\delta_1 = \delta_2$ if and only if $\delta_1 \leqslant \delta_2$ and $\delta_2 \leqslant \delta_1$.

2.18. THEOREM. *If A and B are Wallman rings on X, then $A \leqslant B$ if and only if $\delta_A \leqslant \delta_B$.*

A proximity on a space induces a compactification of that space called the Smirnov compactification. The following theorem, due to Njåstad (1966), states that the Wallman compactification induced by a Wallman ring and the Smirnov compactification induced by δ_A are equivalent.

2.19. THEOREM. *Let A be a Wallman ring on X, then the Smirnov compactification of the proximity space (X, δ_A) and the Wallman compactification $w(Z(A))$ are equivalent compactifications of X.*

The next theorem gives a convenient method for establishing which sets are far with respect to a proximity associated with a Wallman ring.

2.20. THEOREM. *If G and H are subsets of X, and A is a Wallman ring on X, then $G \, \delta_A^{\dagger} \, H$ if and only if*

$$\mathrm{Cl}_{w(Z(A))} G \cap \mathrm{Cl}_{w(Z(A))} H = \varnothing.$$

https://doi.org/10.1017/S1446788700038805 Published online by Cambridge University Press

PROOF. If $G\,\delta^{\dagger}_{A}\,H$, then there are disjoint zero sets F_1 and $F_2 \in Z(A)$ such that $G \subset F_1$ and $H \subset F_2$. Since

$$\mathrm{Cl}_{w(Z(A))} F_1 \cap \mathrm{Cl}_{w(Z(A))} F_2 = \varnothing, \quad \mathrm{Cl}_{w(Z(A))} G \cap \mathrm{Cl}_{w(Z(A))} H = \varnothing.$$

If $\mathrm{Cl}_{w(Z(A))} G \cap \mathrm{Cl}_{w(Z(A))} H = \varnothing$, then there are disjoint closed sets in $w(Z(A))$ of the form $\mathrm{Cl}_{w(Z(A))} F_1$ and $\mathrm{Cl}_{w(Z(A))} F_2$ where $F_1, F_2 \in Z(A)$ such that

$$\mathrm{Cl}_{w(Z(A))} G \subset \mathrm{Cl}_{w(Z(A))} F_1 \quad \text{and} \quad \mathrm{Cl}_{w(Z(A))} H \subset \mathrm{Cl}_{w(Z(A))} F_2.$$

Therefore $G \subset F_1$, $H \subset F_2$ and $F_1 \cap F_2 = \varnothing$.

In the work which follows, we will use a theorem due to Taĭmanov (1952), the proof of which is implicit in Engelking (1968, p. 127). Smirnov (1952) also proved the same result using proximity space theory.

2.21. THEOREM (Taĭmanov, 1952). *If Y_1 and Y_2 are compactifications of X, then a necessary and sufficient condition that $Y_2 \leqslant Y_1$ is the following: For any two closed subsets B_1 and B_2 of X, $\mathrm{Cl}_{Y_2} B_1 \cap \mathrm{Cl}_{Y_2} B_2 = \varnothing$ implies $\mathrm{Cl}_{Y_1} B_1 \cap \mathrm{Cl}_{Y_1} B_2 = \varnothing$.*

We are now in a position to prove a series of theorems which establish a relationship between the proximities we have just defined and (B, A)-embedding.

2.22. THEOREM. *If S is a subspace of X, A is a Wallman ring on X, B is a Wallman ring on S and δ' is the proximity on S induced by δ_A, then the following are equivalent:*
(1) $B \leqslant_S A$;
(2) $\delta_B \leqslant \delta'$;
(3) $w(Z(B)) \leqslant \mathrm{Cl}_{w(Z(A))} S$.

PROOF. That (1) and (2) are equivalent is clear.
(2)→(3): Let B_1 and B_2 be disjoint closed subsets of S such that

$$\mathrm{Cl}_{w(Z(B))} B_1 \cap \mathrm{Cl}_{w(Z(B))} B_2 = \varnothing.$$

Then $B_1\,\delta^{\dagger}_{B}\,B_2$ which implies by (2) that $B_1\,\delta'^{\dagger}\,B_2$. Thus $B_1\,\delta^{\dagger}_{A}\,B_2$ and so $\mathrm{Cl}_{w(Z(A))} B_1 \cap \mathrm{Cl}_{w(Z(A))} B_2 = \varnothing$ and $\mathrm{Cl}_{\mathrm{Cl}_{w(Z(A))} S} B_1 \cap \mathrm{Cl}_{\mathrm{Cl}_{w(Z(A))} S} B_2 = \varnothing$. Therefore by the Taĭmanov Theorem $w(Z[B]) \leqslant \mathrm{Cl}_{w(Z(A))} S$.

(3)→(2): Let A_1 and A_2 be subsets of S such that $A_1\,\delta^{\dagger}_{B}\,A_2$. Let $B_1 = \mathrm{Cl}_S A_1$, $B_2 = \mathrm{Cl}_S A_2$. Then $A_1\,\delta^{\dagger}_{B}\,A_2$ and so $B_1\,\delta^{\dagger}_{B}\,B_2$. Therefore

$$\mathrm{Cl}_{w(Z(B))} B_1 \cap \mathrm{Cl}_{w(Z(B))} B_2 = \varnothing$$

which is equivalent to $\mathrm{Cl}_{\mathrm{Cl}_{w(Z(A))} S} B_1 \cap \mathrm{Cl}_{\mathrm{Cl}_{w(Z(A))} S} B_2 = \varnothing$ and

$$\mathrm{Cl}_{w(Z(A))} B_1 \cap \mathrm{Cl}_{w(Z(A))} B_2 = \varnothing.$$

This means that $B_1\,\delta^{\dagger}_{A}\,B_2$. Therefore $B_1\,\delta'^{\dagger}\,B_2$ and $A_1\,\delta'^{\dagger}\,A_2$.

2.23. THEOREM. *If S is a subspace of X, A is a Wallman ring on X, B is a Wallman ring on S and δ' is the proximity on S induced by δ_A, then the following are equivalent:*

(1) $A \leqslant_S B$;

(2) $\delta' \leqslant \delta_B$;

(3) $\mathrm{Cl}_{w(Z(A))} S \leqslant w(Z(B))$.

PROOF. That (1) and (2) are equivalent is clear.

(2)→(3): Let B_1 and B_2 be disjoint closed subsets of S such that

$$\mathrm{Cl}_{\mathrm{Cl}_{w(Z(A))}S} B_1 \cap \mathrm{Cl}_{\mathrm{Cl}_{w(Z(A))}S} B_2 = \varnothing.$$

Then $\mathrm{Cl}_{w(Z(A))} B_1 \cap \mathrm{Cl}_{w(Z(A))} B_2 = \varnothing$ which means that $B_1 \,\delta_A^\dagger\, B_2$. Thus $B_1 \,\delta'^\dagger\, B_2$ and by (2) $B_1 \,\delta_B^\dagger\, B_2$ which implies that $\mathrm{Cl}_{w(Z(B))} B_1 \cap \mathrm{Cl}_{w(Z(B))} B_2 = \varnothing$. Therefore by the Taĭmanov Theorem we have $\mathrm{Cl}_{w(Z(A))} S \leqslant w(Z(B))$.

(3)→(2): Let A_1 and A_2 be subsets of S such that $A_1 \,\delta'^\dagger\, A_2$. Let $B_1 = \mathrm{Cl}_S A_1$ and $B_2 = \mathrm{Cl}_S A_2$. Then we have $B_1 \,\delta'^\dagger B_2$ and so $B_1 \,\delta_A^\dagger\, B_2$.

This means $\mathrm{Cl}_{w(Z(A))} B_1 \cap \mathrm{Cl}_{w(Z(A))} B_2 = \varnothing$ which is equivalent to

$$\mathrm{Cl}_{\mathrm{Cl}_{w(Z(A))}S} B_1 \cap \mathrm{Cl}_{\mathrm{Cl}_{w(Z(A))}S} B_2 = \varnothing \quad \text{and} \quad \mathrm{Cl}_{w(Z(B))} B_1 \cap \mathrm{Cl}_{w(Z(B))} B_2 = \varnothing.$$

This means $B_1 \,\delta_B^\dagger\, B_2$ and so $A_1 \,\delta_B^\dagger\, A_2$.

As a corollary to these two theorems we have the following theorem.

2.24. THEOREM. *If $S \subset X$, A is a Wallman ring on X, B is a Wallman ring on S and δ' is the proximity on S induced by δ_A, then the following are equivalent:*

(1) $A \cong_S B$;

(2) $\delta' = \delta_B$;

(3) *S is (B, A)-embedded in X.*

In the special case when $A = C^*(X)$, and $B = C^*(S)$, Theorem 2.24 yields the following.

2.25. THEOREM. *If S is a subspace of X, then the following are equivalent:*

(1) *S is C^*-embedded in X;*

(2) $C^*(S) \cong_S C^*(X)$;

(3) $C^*(S) \leqslant_S C^*(X)$;

(4) $Z(S) \leqslant_X Z(X)$;

(5) *any two completely separated sets in S are completely separated in X.*

PROOF. By Theorem 2.2, S is C^*-embedded in X if and only if S is $(C^*(S), C^*(X))$-embedded in X. Therefore (1) and (2) are equivalent. The restriction to S of any function in $C^*(X)$ is a function in $C^*(S)$, so (3), (4), and (5) are equivalent to (2).

The next corollary to Theorem 2.24 gives a sufficient condition for (B, A)-embedding.

https://doi.org/10.1017/S1446788700038805 Published online by Cambridge University Press

2.26. COROLLARY. *If S is a subspace of X, A is a Wallman ring on X, B is a Wallman ring on S, $Z(B) \subset Z(A)$, and for each $F \in Z[A]$, $F \cap S \in Z(B)$, then S is (B, A)-embedded in X.*

PROOF. $Z(B) \subset Z(A)$ implies $B \leqslant_S A$ and $F \cap S \in Z(B)$ for each $F \in Z(A)$ implies $A \leqslant_S B$.

Problem 1F (2) from Gillman and Jerison (1960) is a corollary to Corollary 2.26.

2.27. COROLLARY. *If S is a subspace of X and $Z(S) \subset Z(X)$, then S is C^*-embedded in X.*

PROOF. By hypothesis $Z(C^*(S)) \subset Z(C^*(X))$. If $F \in Z(C^*(X))$, there is a function $f \in C^*(X)$ such that $F = Z(f)$. $F \cap S = Z(f|S) \in Z(C^*(S))$. Therefore S is $(C^*(S), C^*(X))$-embedded in X. By Theorem 2.2 S is C^*-embedded in X.

A second concept which is related to C^*-embedding will be introduced in the next theorem and then generalized to arbitrary normal bases.

2.28. THEOREM. *If S is C^*-embedded in X, then the following condition is satisfied: For each pair of sets $F_1, F_2 \in Z(X)$ such that $F_1 \cap F_2 \cap S = \emptyset$ there are sets $E_1, E_2 \in Z(X)$ such that $F_1 \cap S = E_1 \cap S$, $F_2 \cap S = E_2 \cap S$ and $E_1 \cap E_2 = \emptyset$.*

PROOF. Let $F_1 = Z(f_1)$ and $F_2 = Z(f_2)$ for $f_1, f_2 \in C^*(X)$. Since $F_1 \cap F_2 \cap S = \emptyset$, $Z(f_1|S) \cap Z(f_2|S) = \emptyset$. By Theorem 2.25 there are functions $f'_1, f'_2 \in C^*(X)$ such that $Z(f_1|S) \subset Z(f'_1)$, $Z(f_2|S) \subset Z(f'_2)$ and $Z(f'_1) \cap Z(f'_2) = \emptyset$. Let $E_i = Z(f_i) \cap Z(f'_i)$ for $i = 1, 2$.

2.29. DEFINITION. If F is a normal base on X and S is a subspace of X, then S is *F-embedded in X* if and only if the following condition is satisfied: For each pair of sets $F_1, F_2 \in F$ such that $F_1 \cap F_2 \cap S = \emptyset$, there are sets $E_1, E_2 \in F$ such that $F_1 \cap S = E_1 \cap S$, $F_2 \cap S = E_2 \cap S$ and $E_1 \cap E_2 = \phi$.

Later it will be shown that if X is a metric space, then S is $Z(X)$-embedded in X if and only if S is C^*-embedded in X; but, in general $Z(X)$-embedding does not imply C^*-embedding.

The next theorem gives a necessary and sufficient condition for F-embedding, one which is frequently easier to exhibit than the condition in the definition.

https://doi.org/10.1017/S1446788700038805 Published online by Cambridge University Press

2.30. THEOREM. *If F is a normal base on X and $S \subset X$, then S is F-embedded in X if and only if the following condition is satisfied: For each pair of sets $F_1, F_2 \in F$ such that $F_1 \cap F_2 \cap S = \emptyset$, there are sets $E_1, E_2 \in F$ such that $F_1 \cap S \subset E_1$, $F_2 \cap S \subset E_2$,* and *$E_1 \cap E_2 = \emptyset$.*

PROOF. The sufficiency of the condition follows from the fact that $F_1 \cap S \subset E_1$, $F_2 \cap S \subset E_2$ and $E_1 \cap E_2 = \emptyset$ implies $F_1 \cap S = (E_1 \cap F_1) \cap S$, $F_2 \cap S = (E_2 \cap F_2) \cap S$ and $(E_1 \cap F_1) \cap (E_2 \cap F_2) = \emptyset$. The necessity is obvious.

If we had defined F-embedding by considering any finite number of sets $F_1, \ldots, F_n$ instead of two sets we would have had an equivalent definition.

2.31. THEOREM. *If S is a subspace of X and F is a normal base on X, then S is F-embedded in X if and only if the following condition is satisfied: For each finite collection of sets $F_1, \ldots, F_n \in F$ such that*

$$S \cap \left(\bigcap_{i=1}^{n} \right) F_i = \emptyset$$

there are sets $E_1, \ldots, E_n \in F$ such that $F_i \cap S = E_i \cap S$ and $\bigcap_{i=1}^{n} E_i = \emptyset$.

PROOF. Obviously this condition implies F-embedding. To prove the other direction let S be F-embedded in X and let $F_1, \ldots, F_n \in F$ such that

$$S \cap \left(\bigcap_{i=1}^{n} \right) F_i = \emptyset.$$

Then there exist E_1 and $P \in F$ such that

$$S \cap F_1 = S \cap E_1, \quad S \cap \left(\bigcap_{i=2}^{n} F_i \right) = S \cap P \quad \text{and} \quad E_1 \cap P = \emptyset.$$

So for $m = 1$ we have shown the following property: There exist $E_1, \ldots, E_m \in F$ (where $m < n$) such that $S \cap E_i = S \cap F_i$, $i = 1, \ldots, m$; and there exists $P \in F$ such that

$$\left(\bigcap_{i=1}^{m} E_i \right) \cap P = \emptyset \quad \text{and} \quad S \cap P = \left(\bigcap_{i=m+1}^{n} F_i \right) \cap S.$$

Now assume this property holds for some $m \in \{1, 2, \ldots, n-2\}$ and show it holds for $m+1$. Since by assumption the property holds for m, we have $E_1, \ldots, E_m$ and $P \in F$ such that

$$\left(\bigcap_{i=1}^{m} \right) E_i \cap P = \emptyset.$$

So by the normality of F, there are $J, L \in F$ such that

$$\left(\bigcap_{i=1}^{m} E_i \right) \cap J = \emptyset, \quad P \cap L = \emptyset \quad \text{and} \quad J \cup L = X.$$

https://doi.org/10.1017/S1446788700038805 Published online by Cambridge University Press

Also by assumption

$$S\cap P = S\cap\left(\bigcap_{i=m+1}^{n} F_i\right) \;\; = S\cap F_{m+1}\cap\left(\bigcap_{i=m+1}^{n} F_i\right).$$

So $S\cap(F_{m+1})\cap(\bigcap_{i=m+2}^{n} F_i)\cap L = S\cap P\cap L = \varnothing$ since $P\cap L=\varnothing$.

We now have two elements of F, F_{m+1} and $(\bigcap_{i=m+2}^{n} F_i)\cap L$ disjoint on S. So by the hypothesis that S is F-embedded in X, we obtain T and $R\in F$ such that

$$S\cap F_{m+1} = S\cap T,\quad S\cap\left(\bigcap_{i=m+2}^{n} F_i\right)\cap L = S\cap R \quad\text{and}\quad R\cap T=\varnothing.$$

Now let $E_{m+1}=T$ and $Q=(J\cup R)\cap(\bigcap_{i=m+2}^{n} F_i)$. So E_{m+1} and $Q\in F$ are such that:

$$S\cap E_{m+1} = S\cap T = S\cap F_{m+1};$$

$$\begin{aligned}
&\left(\bigcap_{i=1}^{m+1} E_i\right)\cap Q\\
&\quad=\left(\bigcap_{i=1}^{m+1} E_i\right)\cap(J\cup R)\cap\left(\bigcap_{i=m+2}^{n} F_i\right)\\
&\quad=\left[\left(\bigcup_{i=1}^{m+1} E_i\right)\cap J\cap\left(\bigcap_{i=m+2}^{n} F_i\right)\right]\cup\left[\left(\bigcap_{i=1}^{m+1} E_i\right)\cap R\cap\left(\bigcap_{i=m+2}^{n} F_i\right)\right]\\
&\quad=\left[\left(\bigcap_{i=1}^{m} E_i\right)\cap J\cap E_{m+1}\cap\left(\bigcap_{i=m+2}^{n} F_i\right)\right]\cup\left[\left(\bigcap_{i=1}^{m} E_i\right)\cap T\cap R\cap\left(\bigcap_{i=m+2}^{n} F_i\right)\right]\\
&\quad=\varnothing
\end{aligned}$$

since $(\bigcap_{i=1}^{m} E_i)\cap J=\varnothing$ and $R\cap T=\varnothing$, and

$$\begin{aligned}
S\cap Q &= S\cap\left[(J\cup R)\cap\left(\bigcap_{i=m+2}^{n} F_i\right)\right]\\
&=\left[S\cap J\cap\left(\bigcap_{i=m+2}^{n} F_i\right)\right]\cup\left[\left(S\cap R\cap\left(\bigcap_{i=m+2}^{n} F_i\right)\right]\\
&=\left[S\cap J\cap\left(\bigcap_{i=m+2}^{n} F_i\right)\right]\cup\left[\left(S\cap\left(\bigcap_{i=m+2}^{n} F_i\right)\cap L\right)\cap\left(\bigcap_{i=m+2}^{n} F_i\right)\right]\\
&= S\cap\left(\bigcap_{i=m+2}^{n} F_i\right)\cap(J\cup L)\\
&= S\cap\left(\bigcap_{i=m+2}^{n} F_i\right)\cap X\\
&= S\cap\left(\bigcap_{i=m+2}^{n} F_i\right).
\end{aligned}$$

So we have shown that the property stated above holds for $m+1$. We know there exists sets $E_1, \ldots, E_{n-1}$ from F such that $S \cap E_i = S \cap F_i$, $i = 1, \ldots, n-1$ and for these sets there is a set $P \in F$ such that $(\bigcap_{i=1}^{n-1} E_i) \cap P = \emptyset$ and $P \cap S = F_n \cap S$. We set $E_n = P$ and our proof is complete.

We now look at a couple of examples of types of sets which are F-embedded. Every member of a normal base is F-embedded.

2.32. THEOREM. *If F is a normal base on X and $S \in F$, then S is F-embedded in X.*

PROOF. If $F_1, F_2 \in F$ such that $F_1 \cap F_2 \cap S = \emptyset$, then $F_1 \cap S$ and $F_2 \cap S \in F$, and $(F_1 \cap S) \cap (F_2 \cap S) = \emptyset$.

In terms of Wallman rings, this theorem tells us that the zero sets of a Wallman ring A are $Z[A]$-embedded.

Hamburger (1971) gave the following definition.

2.33. DEFINITION. Let F be a closed base in X. We say that S is *F-dense* in X if $\mathrm{Cl}_X(S \cap A) = A$ for each $A \in F$.

Being F-dense is a sufficient but not necessary condition for a subset to be F-embedded.

2.34. THEOREM. *If S is a subspace of X, F is a normal base on X and S is F-dense in X, then S is F-embedded in X.*

PROOF. Let $F_1, F_2 \in F$ such that $F_1 \cap F_2 \cap S = \emptyset$, then $\mathrm{Cl}_X(F_1 \cap F_2 \cap S) = \emptyset$. So if S is F-dense in X, $F_1 \cap F_2 = \emptyset$.

An example of a set which is F-embedded but not F-dense is $S = N$ which is $Z(X)$-embedded in $X = \beta N$, since it is C^*-embedded in X. S is dense in X; but it is not $Z(X)$-dense in X since $\mathrm{Cl}_X(S \cap Z(g)) \neq Z(g)$ where g is the continuous extension to βN of the function f: $N \to R$ defined by $f(n) = 1/n$. Clearly $S \cap Z(g) = \emptyset$ but $Z(g) \neq \emptyset$.

The principal use of the concept of F-embedding is indicated in the following theorem.

2.35. THEOREM. *If F is a normal base on X and S is F-embedded in X, then $\{E \cap S$: $E \in F\}$ is a normal base on X.*

PROOF. Let $G = \{E \cap S$: $E \in F\}$. If it can be shown that G is normal on S, then it will be clear that G is a normal base on S.

Let $G_1, G_2 \in G$ such that $G_1 \cap G_2 = \emptyset$. By definition of G, there are sets $F_1, F_2 \in F$ such that $G_1 = F_1 \cap S$ and $G_2 = F_2 \cap S$. Then $S \cap F_1 \cap F_2 = G_1 \cap G_2 = \emptyset$. So there are sets $E_1, E_2 \in F$ such that $S \cap F_1 = S \cap E_1$, $S \cap F_2 = S \cap E_2$ and $E_1 \cap E_2 = \emptyset$. Since F is a normal base on X, there are sets $P_1, P_2 \in F$ such that $E_1 \cap P_1 = \emptyset$, $E_2 \cap P_2 = \emptyset$ and $P_1 \cup P_2 = X$.

Let $Q_1 = P_1 \cap S$, $Q_2 = P_2 \cap S$. Then $G_i \cap Q_i = F_i \cap S = E_i \cap P_i \cap S = \emptyset$, for $i = 1, 2$, and $Q_1 \cup Q_2 = S$. Therefore G is normal on S.

This theorem shows us that the restriction of a Wallman ring A to a $Z(A)$-embedded subspace is again a Wallman ring.

2.36. COROLLARY. *If A is a Wallman ring on X and S is $Z(A)$-embedded in X, then $\{f|S: f \in A\}$ is a Wallman ring on S.*

The next theorem gives a necessary and sufficient condition for F-embedding involving the restrictions of F-sets.

2.37. THEOREM. *If F is a normal base on X and S is a subspace of X, then S is F-embedded in X if and only if*

$$\{E \cap S: E \in F\} \leqslant_S F.$$

PROOF. If $\{E \cap S: E \in F\} \leqslant_S F$, then for each pair of sets $F_1, F_2 \in F$ such that $F_1 \cap F_2 \cap S = \emptyset$, there exist $E_1, E_2 \in F$ such that $F_1 \cap S \subset E_1$, $F_2 \cap S \subset E_2$ and $E_1 \cap E_2 = \emptyset$. Therefore, by Theorem 2.30, S is F-embedded in X.

Suppose S is F-embedded in X. Then if $F_1, F_2 \in F$ such that $F_1 \cap F_2 \cap S = \emptyset$, there are sets $E_1, E_2 \in F$ such that $F_1 \cap S \subset E_1$, $F_2 \cap S \subset E_2$ and $E_1 \cap E_2 = \emptyset$. Therefore

$$\{E \cap S: E \in F\} \leqslant_S F.$$

2.38. COROLLARY. *If A is a Wallman ring on X and $S \subset X$, then the following are equivalent:*

(1) *S is $Z(A)$-embedded in X;*
(2) *$\{f|S: f \in A\} \leqslant_S A$;*
(3) *$\{f|S: f \in A\} \cong_S A$.*

2.39. COROLLARY. *If A is an inverse closed Wallman ring on X and $S \subset X$, then S is $Z(A)$-embedded in X if and only if for each pair F_1 and $F_2 \in Z(A)$ such that $F_1 \cap F_2 \cap S = \emptyset$, there exists $g \in A$ such that $g[S \cap F_1] = 0$ and $g[S \cap F_2] = 1$.*

PROOF. By Theorem 2.12 two subsets G and H of X are completely A-separated if and only if there is a function $f \in A$ such that $g[G] = 0$ and $g[H] = 1$. Therefore, for $F_1, F_2 \in Z(A)$ such that $F_1 \cap F_2 \cap S = \emptyset$, there are sets $E_1, E_2 \in Z(A)$ such that $F_1 \cap S \subset E_1$, $F_2 \cap S \subset E_2$ and $E_1 \cap E_2 = \emptyset$ if and only if there is a function $g \in A$ such that $g[S \cap F_1] = 0$ and $g[S \cap F_2] = 1$.

Continuing in our investigation of conditions under which a subspace is $Z(A)$-embedded for a Wallman ring A we present the following theorem.

2.40. THEOREM. *If $S \subset X$, A is an inverse closed Wallman ring on X and $B = \{f \mid S : f \in A\}$ is an inverse closed Wallman ring on S, then S is $Z(A)$-embedded in X.*

PROOF. Let $f_1, f_2 \in A$ such that $Z(f_1) \cap Z(f_2) \cap S = \emptyset$. Let $g_1 = f_1 \mid S$, $g_2 = f_2 \mid S$ and $g = g_1^2/(g_1^2 + g_2^2)$. Then $Z(g_1^2 + g_2^2) = \emptyset$ so $g \in B$. Also $g[Z(f_1) \cap S] = 0$ and $g[Z(f_2) \cap S] = 1$. Since there is a function $f \in A$ such that $f \mid S = g$ there is a function in A which is zero on $Z(f_1) \cap S$ and one on $Z(f_2) \cap S$. Therefore by the previous theorem S is $Z(A)$-embedded in X.

We naturally wonder how $Z(A)$-embedding is related to (B, A)-embedding, for some appropriate B. The next theorem tells us for which Wallman rings B, a $Z[A]$-embedded subset is (B, A)-embedded.

2.41. THEOREM. *If S is a subspace of X, A is a Wallman ring on X, B is a Wallman ring on S, and S is $Z(A)$-embedded in X, then S is (B, A)-embedded in X if and only if $B \cong \{f \mid S : f \in A\}$.*

PROOF. By Corollary 2.36, $\{f \mid S : f \in A\}$ is a Wallman ring on S. Clearly $A \cong_S \{f \mid S : f \in A\}$. Theorem 2.24 established that S is (B, A)-embedded if and only if $A \cong_S B$. Therefore S is (B, A)-embedded in X if and only if $B \cong_S \{f \mid S : f \in A\}$. So, since B and $\{f \mid S : f \in A\} \subset C(S)$, S is (B, A)-embedded in X if and only if $B \cong \{f \mid S : f \in A\}$.

2.42. COROLLARY. *If S is a subspace of X, A is an inverse closed Wallman ring on X and $B = \{f \mid S : f \in A\}$ is an inverse closed Wallman ring on S, then S is (B, A)-embedded in X.*

The next theorem is the converse to part of Theorem 2.41.

2.43. THEOREM. *If A is a Wallman ring on X, $S \subset X$, B is a Wallman ring on S, $B \cong \{f \mid S : f \in A\}$ and S is (B, A)-embedded in X, then S is $Z(A)$-embedded in X.*

PROOF. By Theorem 2.24 S is (B, A)-embedded in X if and only if $A \cong_S B$. Therefore S is (B, A)-embedded in X implies $A \leqslant_S \{f \mid S : f \in A\}$. Then by Corollary 2.38 S is $Z(A)$-embedded in X.

For an arbitrary Wallman ring B on S, (B, A)-embedding may not imply $Z(A)$-embedding, as is illustrated by this example.

2.44. EXAMPLE. Let $X = [0, 1]$, $S = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$, $A = C(X)$ and

$$B = \{f | S : f \in C(X) \text{ and is constant on a neighborhood of } \tfrac{1}{2}\}.$$

Then by Bentley and Taylor (1975, Corollary 4.10), B is a Wallman ring and $w(Z(B)) \cong X \cong \mathrm{Cl}_{w(Z(A))} S$ so S is (B, A)-embedded in X. Now let $F_1 = [0, \frac{1}{2}]$, $F_2 = [\frac{1}{2}, 1]$. These are zero sets of A since every closed subset of a metric space is a zero set.

$F_1 \cap F_2 \cap S = \varnothing$, but the zero sets of A are the closed sets of $[0, 1]$, so there are no disjoint zero sets of A which separate $[0, \frac{1}{2})$ and $(\frac{1}{2}, 1]$. Therefore S is not $Z(A)$-embedded in X.

In the case of a metric space X with subspace S and Wallman rings $A = C^*(X)$ and $B = C^*(S)$, $Z(A)$-embedding and (B, A)-embedding are equivalent.

2.45. THEOREM. *If X is a metric space and $S \subset X$, then the following are equivalent*:

(1) *S is C^*-embedded in X*;

(2) *S is $(C^*(S), C^*(X))$-embedded in X*;

(3) *S is $Z(X)$-embedded in X.*

PROOF. That (1) and (2) are equivalent was shown in 2.2 for all spaces. That (1) implies (3) was shown in 2.28 for all spaces.

(3)$\to$(1). Since X is a metric space, the zero sets of X are precisely the closed subsets of X and the zero sets of S are the intersections with S of closed subsets of X. Since, by the $Z(A)$-embedded hypothesis, disjoint sets of this type are completely $C^*(X)$-separated, we have $Z(S) \leqslant Z(X)$. Therefore by Theorem 2.25, S is C^*-embedded in X.

The only part of the previous theorem which depended on X being a metric space was the proof that $Z(X)$-embedding implies C^*-embedding. If X were not a metric space, this conclusion would not necessarily be valid as is illustrated by the following example of a space which is not a metric space. (The example of the non-metric space is from Gillman and Jerison (1960), Problem 3K, p. 50.)

2.46. EXAMPLE. Let X denote the subset $\{(x, y) : y \geqslant 0\}$ of $R \times R$ provided with the following enlargement of the product topology: for $r > 0$, the sets

$$V_r(x, 0) = \{(x, 0)\} \cup \{(u, v) \in X : (u - x)^2 + (v - r)^2 < r^2\}$$

are also neighborhoods of the point $(x, 0)$.

$S = \{(x, 0) : x \in R\}$ is $Z(X)$-embedded in X since it is a zero set of X; however, it is not C^*-embedded in X.

REFERENCES

J. M. Aarts (1968), "Every metric compactification is a Wallman type compactification", *Proceedings of the International Symposium on Topology and its Applications*, 29–34.

R. A. Alo and H. L. Shapiro (1968), "Normal bases and compactifications", *Math. Annalen* **175**, 337–340.

B. Banaschewski (1963), "On Wallman's method of compactification", *Math. Nachr.* **27**, 105–114.

H. L. Bentley (1972), "Normal bases and compactifications", *Proceedings of the Topology Conference, Univ. of Oklahoma*, 23–37.

H. L. Bentley (1972), "Some Wallman compactifications determined by retracts", *Proc. Amer. Math. Soc.* **33**, 587–593.

H. L. Bentley (1972), "Some Wallman compactifications of locally compact spaces", *Fund. Math.* **75**, 13–24.

H. L. Bentley and B. J. Taylor (1975), "Wallman rings", *Pacific J. Math.* **58**, 15–35.

C. M. Biles (1970), "Gelfand and Wallman-type compactifications", *Pacific J. Math.* **35**, 267–278.

C. M. Biles (1970), "Wallman-type compactifications", *Proc. Amer. Math. Soc.* **25**, 363–368.

R. M. Brooks (1967), "On Wallman compactifications", *Fund. Math.* **60**, 157–173.

R. Engelking (1968), *Outline of General Topology* (Panstwowe Wydawnictwo Naukowe, Warsaw).

O. Frink (1964), "Compactifications and semi-normal spaces", *Amer. J. Math.* **86**, 602–607.

L. Gillman and M. Jerison (1960), *Rings of Continuous Functions* (Van Nostrand, Princeton).

A. W. Hager (1969), "On inverse-closed subalgebras of $C(X)$", *Proc. London Math. Soc.* (3) **19**, 233–257.

A. W. Hager and D. G. Johnson (1968), "A note on certain subalgebras of $C(X)$", *Canad. J. Math.* **20**, 389–393.

P. Hamburger (1971), "On Wallman-type, regular Wallman-type and Z-compactifications", *Periodica Mathematica Hungarica* (1) **4**, 303–309.

M. Henricksen and J. Isbell (1962), "Lattice ordered rings and function rings", *Pacific J. Math*, **12**, 533–565.

J. R. Isbell (1958), "Algebras of uniformly continuous functions", *Ann. Math.* **68**, 96–125.

S. G. Mrówka (1964), "Some approximation theorems for rings of unbounded functions". *Amer. Math. Soc. Not.* **11**, 666.

S. A. Naimpally and D. B. Warrack (1970), *Proximity Spaces* (Cambridge University Press, London).

O. Njåstad (1966), "On Wallman-type compactifications", *Math. Zeitschr.* **91**, 267–276.

N. A. Šanin (1943), "On special extensions of topological spaces", *Dokl. Akad. Nauk S.S.S.R.* **38**, 6–9.

G. F. Simmons (1963), *Introduction to Topology and Modern Analysis* (McGraw-Hill, New York).

Y. M. Smirnov (1952), "Mappings of systems of open sets", *Mat. Sbornik* **31** (73), 152–166.

E. F. Steiner (1968), "Wallman spaces and compactifications", *Fund. Math.* **61**, 295–304.

A. K. Steiner and E. F. Steiner (1968), "Products of compact metric spaces are regular Wallman", *Indag. Math.* **30**, 428–430.

A. K. Steiner and E. E. Steiner (1968), "Wallman and Z-compactifications", *Duke Math. J.* **35**, 269–276.

M. H. Stone (1962), "A generalized Weierstrass approximation theorem", *Studies in Modern Analysis, M.A.A. Studies in Mathematics*, Vol. I (Prentice Hall, Englewood Cliffs, N.J.).

A. Taĭmanov (1952), "On extension of continuous mappings of topological spaces", *Mat. Sbornik* **31**, 459–463.

H. Wallman (1938), "Lattices and topological spaces", *Ann. of Math.* **39**, 112–126.

The University of Toledo
Toledo
Ohio 43606, USA

IBM Corporation
Sylvania
Ohio 43560, USA

https://doi.org/10.1017/S1446788700038805 Published online by Cambridge University Press