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LOCALLY HOMOGENEOUS S-STRUCTURES

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We prove a characterisation of locally s-regular manifolds using the theory of homogeneous structures.

1 INTRODUCTION

Pseudo-Riemannian locally s-regular manifolds are nice generalisations of locally symmetric spaces. Their geometry has remarkable properties [**1**], [**2**], [**3**]. In particular, they are all locally homogeneous spaces. So they all admit a locally homogeneous structure [**5**] which is of a special type (called a locally homogeneous S-structure).

Our purpose here is to show that, conversely, a pseudo-Riemannian manifold which admits such an S-structure and satisfies a further condition is then a locally s-regular manifold. This extra condition is satisfied, in particular, when S has finite order or the metric is positive definite. Also we prove that we can delete this extra condition when there exists a so-called naturally reductive S-structure.

2 PRELIMINARIES

Let (M,g) be a smooth connected n-dimensional manifold M with pseudo-Riemannian metric g, and denote by $\mathcal{T}_q^p$ the algebra of all smooth tensor fields on M with contravariant and covariant orders p and q respectively. In particular, write $\mathcal{T}_p^0 = \mathcal{T}_p$ and $\mathcal{T}_0^p = \mathcal{T}^p$. Let ∇ denote the Riemannian connection and R the curvature tensor field on M where we define the curvature operator R_{XY} by

$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$$

for all X, $Y \in \mathcal{T}^1$.

Any $S \in \mathcal{T}_1^1$ may be considered as a field of endomorphisms of tangent spaces and a tensor field $P \in \mathcal{T}_q^p$ is then called S*-invariant* if for all $\omega_1, \dots, \omega_p \in \mathcal{T}_1$ and $X_1, \dots, X_q \in \mathcal{T}^1$,

$$P(\omega_1 S, \dots, \omega_p S, X_1, \dots, X_q) = P(\omega_1, \dots, \omega_p, SX_1, \dots, SX_q)$$

where $(\omega S)X = \omega(SX)$ for $\omega \in \mathcal{T}_1$ and $X \in \mathcal{T}^1$.

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Next, S is called a *symmetry tensor field* if $I - S$ is non-singular and g is S-invariant. In particular, if ∇S and $\nabla^2 S$ are S-invariant, then we say S is *regular.* A *local symmetry* s_m is defined on each sufficiently small neighbourhood of m by

$$s_m = \exp_m \circ S_m \circ \exp_m^{-1}.$$

Clearly s_m is a local diffeomorphism and

$$d_{s_m} \text{ on } M_m = S_m.$$

We denote by s the map $m \longmapsto s_m$ so defined on M. Then (M, g) together with s is called a (pseudo-Riemannian) *locally s-regular manifold* if each s_m is a local isometry which preserves S. For a characterisation of such manifolds in terms of S we have

THEOREM 1. [**1**]. *Let S be a regular symmetry tensor field on any (M, g). Then (M, g) is a locally s-regular manifold with symmetry tensor field S if and only if R and ∇R are S-invariant.*

For a detailed study of s-manifolds and generalised symmetric spaces we refer also to [**3**].

3 Locally homogeneous S-structures

We recall from [**5**] that a *locally homogeneous structure* on any (M, g) is a tensor field $T \in \mathcal{T}_2^1$ such that the connection $\overline{\nabla} = \nabla - T$ satisfies

$$\overline{\nabla} g = \overline{\nabla} R = \overline{\nabla} T = 0.$$

If, in addition, a symmetry tensor field S is given on (M, g) satisfying

(i) $\overline{\nabla} S = 0$,

(ii) T is S-invariant,

then we call the pair (S, T) a *locally homogeneous S-structure* on (M, g). Such a stucture always exists on a locally s-regular manifold as shown by the following theorem from [**1**]. (The proofs given in [**1**] for this and for Theorem 1 apply also to the case when (M, g) is pseudo-Riemannian.)

THEOREM 2. *Let (M, g) be a locally s-regular manifold with symmetry tensor field S and define $T \in \mathcal{T}_2^1$ by*

$$T(X, Y) = (\nabla_{(I-S)^{-1}X} S) S^{-1} Y.$$

Then (S, T) is a locally homogeneous S-structure on (M, g).

Our purpose here is to consider the converse problem. First we remark that if (S, T) is given on (M, g) and if $p \in M$, then $S_p \in GL(M_p)$. We write G_p for the closure in $GL(M_p)$ of the subgroup generated by S_p. We now prove

THEOREM 3. *Let (S,T) be a locally homogeneous S-structure on a pseudo-Riemannian manifold (M,g) and suppose for some point $p \in M$ the subgroup G_p is compact. Then (M,g) is a locally s-regular manifold with symmetry tensor field S.*

PROOF: Since T is S-invariant and since for all $X, Y \in \mathcal{T}^1$,

$$(\nabla_X S)Y = (T_X S)Y$$

we obtain at once that ∇S is S-invariant. Next, let X be the tangent vector field to a smooth curve γ on M and let Y and Z be vector fields along γ which are parallel with respect to ∇. Then

$$\begin{aligned}(\nabla^2 S)(X,Y,Z) &= (\nabla_X(\nabla S))(Y,Z)\\ &= \nabla_X((\nabla_Y S)Z) - (\nabla_{\nabla_X Y} S)Z - (\nabla_Y S)\nabla_X Z\\ &= \nabla_X((T_Y S)Z) - (T_{\nabla_X Y} S)Z - (T_Y S)\nabla_X Z\\ &= T_X((T_Y S)Z) - (T_{T_X Y} S)Z - (T_Y S)T_X Z\end{aligned}$$

which shows that $\nabla^2 S$ is S-invariant. Thus S is regular.

Next we note from the relation

$$\nabla_X R = T_X R$$

and the S-invariance of T that if R is S-invariant, then so is ∇R. Thus, by Theorem 1, it remains only to prove the S-invariance of R.

Since $\overline{\nabla} T = 0$ we have

$$\overline{R}_{XY} = R_{XY} + B_{XY}$$

where $\overline{R}$ is the curvature tensor related to $\overline{\nabla}$ and

$$B_{XY} = [T_X, T_Y] + T_Z, \qquad Z = T_Y X - T_X Y.$$

Define $A \in \mathcal{T}_4$ by

$$A_{XYZW} = \overline{R}_{SXSYSZSW} - \overline{R}_{XYZW} \tag{1}$$

where

$$\overline{R}_{XYZW} = g(\overline{R}_{XY} Z, W).$$

Clearly, B is S-invariant, so

$$A_{XYZW} = R_{SXSYSZSW} - R_{XYZW}$$

from which

$$(2) \qquad A_{XYZW} = A_{ZWXY}.$$

Also, since $\overline{\nabla} S = 0$ and g is S-invariant, we have

$$(3) \qquad \overline{R}_{XYSZSW} = \overline{R}_{XYZW}.$$

Hence

$$(4) \qquad \begin{aligned} A_{XYSZSW} &= \overline{R}_{SXSYS^2ZS^2W} - \overline{R}_{XYSZSW} \\ &= \overline{R}_{SXSYSZSW} - \overline{R}_{XYZW} \\ &= A_{XYZW}. \end{aligned}$$

Consequently, from (2) and (4),

$$A_{SXSYSZSW} = A_{SXSYZW} = A_{ZWSXSY} = A_{ZWXY} = A_{XYZW}.$$

Thus A is S-invariant. Now consider M_p and any vectors $X, Y, Z, W \in M_p$. It follows using (1) and the S-invariance of A that, for each positive integer m,

$$(5) \qquad mA_{XYZW} = \overline{R}_{S^{m+1}XS^{m+1}YS^{m+1}ZS^{m+1}W} - \overline{R}_{XYZW}.$$

However, since G_p is compact, the right hand side of (5) is a bounded sequence in m. Hence $A_{XYZW} = 0$. Thus $A = 0$ at p. Finally, we note that if $q \in M$ and γ is any smooth curve from p to q, then parallel transport with respect to $\overline{\nabla}$ of M_p along γ induces an isomorphism of $GL(M_p)$ onto $GL(M_q)$ and of G_p onto G_q since $\nabla S = 0$. Hence each G_p is compact and so $A = 0$ on M. Thus R is S-invariant on M and the proof is complete. ∎

Remark. The compactness condition on G_p in Theorem 3 clearly holds if S has finite order or if g is positive definite since then S is orthogonal.

4 Hermitian-homogeneous manifolds

Suppose (M,g) has a locally homogeneous structure T and an almost Hermitian structure J such that

(a) $\overline{\nabla} J = 0$,
(b) $T(JX,Y) = T(X,JY) = -JT(X,Y)$

for all X, $Y \in \mathcal{T}^1$. Then (M,g,J) is said to be *locally Hermitian-homogeneous* (see [4]). In particular, it is well-known that any pseudo-Riemannian locally 3-symmetric space is Hermitian-homogeneous with respect to its canonical almost complex structure

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$J = \frac{1}{\sqrt{3}}(2S + I)$. Conversely, suppose (M, g, J) is almost Hermitian and locally Hermitian-homogeneous. Define

$$S = -\frac{1}{2}I + \frac{\sqrt{3}}{2}J.$$

Then $I - S$ is non-singular and g is S-invariant. Also, (a) and (b) imply that $\nabla S = 0$ and that T is S-invariant. Thus (S, T) is a locally homogeneous S-structure and since S has order 3, Theorem 3 can be applied. Thus we have an alternative proof of the following:

THEOREM 4. [**4**]. *Any pseudo-Riemannian locally Hermitian-homogeneous almost Hermitian manifold (M, g, J) is a locally 3-symmetric space with J as canonical almost complex structure.*

For the study of 3-symmetric spaces we refer also to [**2**].

5 Naturally reductive structures

An important class of locally 3-symmetric spaces is that of nearly Kähler 3-symmetric spaces. These are characterised by the additional property

$$(\nabla_X J)X = 0$$

for all $X \in \mathcal{T}^1$. It is proved in [**5**] that this is equivalent to the existence of a Hermitian-homogeneous structure which is naturally reductive or, equivalently,

$$T_X X = 0 \tag{6}$$

for all $X \in \mathcal{T}^1$ (see also [**4**]).

We now prove that the conclusion of Theorem 3 remains true when this condition replaces that on G_p.

THEOREM 5. *Let (S, T) be a locally homogeneous naturally reductive S-structure on a pseudo-Riemannian manifold (M, g). Then (M, g) is a locally s-regular manifold with symmetry tensor field S.*

PROOF: The first part of the proof of Theorem 3 still applies, so we need only to prove that R is S-invariant. Since T is naturally reductive we have explicitly, using (6),

$$\overline{R}_{XYZW} = R_{XYZW} + g(T_X Z, T_Y W) - g(T_Y Z, T_X W) - 2g(T_X Y, T_Z W).$$

Further, this implies that $\overline{R}$ has the same symmetries as a Riemannian curvature tensor.

Now we consider again the tensor A defined by (1) and note that (3) is still valid. So

$$\overline{R}_{SXSYSZSW} = \overline{R}_{SXSYZW} = \overline{R}_{ZWSXSY} = \overline{R}_{ZWXY} = \overline{R}_{XYZW}.$$

From this we obtain again $A = 0$ and hence the required result. ∎

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