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CONGRUENCES ON REGULAR AND COMPLETELY REGULAR SEMIGROUPS

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Abstract

Congruences on regular semigroups have been characterized in terms of normal equivalences on sets of idempotents and kernels of congruences. A revised characterization is presented here with considerably simplified expressions for the least and greatest congruences associated with normal equivalences and with a new description of kernels. The results are then applied to characterize congruences on completely regular semigroups.

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A description of the congruences on an inverse semigroup in terms of normal equivalences of the subsemigroup of idempotents and kernels of homomorphisms has been given by H. E. Scheiblich (1974). In 1978 M. Petrich produced an elegant version of the description and used it to determine some properties associated with special congruences. In her doctoral dissertation, R. Feigenbaum (1975) extended the description to cover regular semigroups; she has published a detailed account (1976) for orthodox semigroups and a summary (1979) of the results for regular semigroups. In the regular case the description and proofs can be simplified, especially if use is made of sandwich sets as formulated by K. S. S. Nambooripad (1974). A characterization in terms of sandwich sets of normal equivalences on the set of idempotents and of the greatest congruence associated with a normal equivalence has been given by the author (1978).

The least and greatest congruences associated with normal equivalences on the set of idempotents of a regular semigroup play a crucial role in the description, via kernels, of congruences on the semigroup. As pointed out by A. H. Clifford

(1979) in a brief survey of the normal equivalence—kernel method, the known expressions for the least and greatest congruences are quite complicated and simpler expressions would be welcome.

Let π be a normal equivalence on the set of idempotents E of a regular semigroup S and let ρ be a congruence on S whose restriction to E is π. Preimages $R(\rho^\sharp)^{-1}$ and $L(\rho^\sharp)^{-1}$ of respective $\mathcal{R}$- and $\mathcal{L}$-classes R and L of S/ρ can be readily specified in terms of π (not ρ). Use is made of these classes and of sandwich sets in section 2 to obtain simplifications of Feigenbaum's results. New and less complex descriptions of the greatest and least congruences and kernels associated with π are obtained.

In Section 3 the results are applied to completely regular semigroups.

Congruences on regular semigroups

Throughout the paper S denotes a regular semigroup with a set of idempotents E. For $a \in S$, $V(a)$ denotes the set of inverses of a. For undefined terminology see Clifford and Preston (1961) or Howie (1976).

An equivalence relation π on E is called *normal* if and only if there is a congruence ρ on S so that $\pi = \rho \cap (E \times E)$. We then say π is *the normal equivalence associated with* ρ. For a congruence ρ on S define

$$\ker \rho = \{a \in S;\, a\rho = e\rho,\, e \in E\}$$

to be the *kernel* of ρ. The congruence ρ is uniquely determined by $\ker \rho$ and the associated normal equivalence on E (Feigenbaum (1979)).

For $e, f \in E$ the *sandwich set* of e, f is

$$S(e, f) = \{h \in E;\, he = h = fh,\, ehf = ef\}.$$

Nambooripad (1974) has shown that $\varnothing \neq S(e, f) \subseteq V(ef)$ and $S(a'a, bb') = S(a^*a, bb^*)$ for any $a, b \in S$, $a', a^* \in V(a)$ and $b', b^* \in V(b)$. Hence define $S(a, b) = S(a'a, bb')$. Furthermore $b'S(a, b)a' \subseteq V(ab)$.

THEOREM 2.1 (Trotter (1978)). *An equivalence relation π on E is normal if and only if for each $e, f \in E$, $c \in S$ and $c' \in V(c)$ then*

(i) $(e\pi)(f\pi) \cap E \subseteq g\pi$ *for all* $g \in S(ef, ef)$ *and*

(ii) $c(e\pi)c' \cap E \subseteq g\pi$ *for all* $g \in S(cec', cec')$.

Let π be a normal equivalence on E. Define relations on S by

$$\mathcal{R}_n = \{(a, b) \in S \times S;\, (aa'\pi)(bb'\pi) \cap bb'\pi \neq \varnothing,\, (bb')\pi(aa'\pi) \cap aa'\pi \neq \varnothing \text{ for some (any) } a' \in V(a), b' \in V(b)\}$$

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and

$$\mathcal{L}_\pi = \{(a, b) \in S \times S;\ (a'a\pi)(b'b\pi) \cap a'a\pi \neq \varnothing,\ (b'b\pi)(a'a\pi) \cap b'b\pi \neq \varnothing \text{ for some (any) } a' \in V(a), b' \in V(b)\}.$$

Let $\mathcal{H}_\pi = \mathcal{R}_\pi \cap \mathcal{L}_\pi$. Note that if ρ is a congruence on S associated with π then $a\mathcal{K}_\pi b$ if and only if $a\rho\mathcal{K}b\rho$ where $\mathcal{K}$ is $\mathcal{R}$, $\mathcal{L}$ or $\mathcal{H}$ (the author (1978)). $\mathcal{R}_\pi$ (and dually $\mathcal{L}_\pi$) can be defined in terms of $\mathcal{R}$ and π, or sandwich sets and π. For $aa'\mathcal{R}_\pi bb'$ if and only if there exists $e, f \in E$ so that $e\pi aa'$, $f\pi bb'$ and $e\mathcal{R}f$ (T. E. Hall (1972) Theorem 5) if and only if $S(aa', bb') \subseteq aa'\pi$ and $S(bb', aa') \subseteq bb'\pi$ (the author (1978) Lemma 1.3). Hence, given π, $\mathcal{H}_\pi$ can be readily calculated.

In the remainder of the paper π will denote a normal equivalence on E, while σ_π and μ_π will denote respectively the least and greatest congruences on S associated with π.

For $e \in E$ define $H_{e\pi} = \{a \in S;\ a\mathcal{H}_\pi e\}$. Let K be a subset of S and $K_e = K \cap H_{e\pi}$. Define K to be a *π-kernel* if and only if for each $e \in E$, $c \in S$ and $c' \in V(c)$ then

(1) $\ker \sigma_\pi \subseteq K \subseteq \ker \mu_\pi$

(2) K_e is a left unitary subsemigroup of $H_{e\pi}$, and

(3) whenever $cec' \in \ker \sigma_\pi$ then $cK_ec' \subseteq K$.

Note that if π is the identity equivalence then $\ker \sigma_\pi = E$; (2) becomes, K_e is a subgroup of the $\mathcal{H}$-class H_e. Define

$$\rho_K = \{(a, b) \in \mathcal{H}_\pi;\ \text{for some (any) } a' \in V(a), b' \in V(b) \text{ then } ab', a'b \in K\}.$$

THEOREM 2.2. *Let π be a normal equivalence and K be a π-kernel. Then ρ_K is a congruence on S with kernel K and is associated with π. Conversely if ρ is a congruence on S associated with π then* $\ker \rho$ *is a π-kernel and $\rho = \rho_{\ker\rho}$.*

PROOF. By Feigenbaum (Theorem 4.1 (1979)), K is the kernel of a congruence associated with π if and only if (i) K satisfies (1), (ii) $K_e = \{a \in H_{e\pi};\ xa \in K_e$ for some $x \in K_e\}$, (iii) for $a, b \in K$ and $a \in Sb$ or $a \in bS$ then ab or $ba \in K$ respectively, and (iv) for $a \in Sc$ or cS then cac' or $c'ac \in K$ respectively. Clearly K satisfies (i) and (ii). Suppose $a, b \in K$. We next check that K satisfies (iv). Say $a \in Sc$, then $a'a = a'ac'c$ for some $a' \in V(a)$, $c' \in V(c)$. Since $a \in \ker \mu_\pi$ then by Lemma 1.2 of the author (1978) $a\mu_\pi = p\mu_\pi$ for any $p \in S(a, a)$ and $a \in K_p$. We have $p = pa'a = pc'c$ so $cpc' \in E$. By (3), since $a \in K_p$, then $cac' \in K$. Similarly $a \in cS$ implies $c'ac \in K$. To see that K satisfies (iii) assume $a = xb$, $x \in S$. Since $b \in K \subseteq \ker \mu_\pi$ then $(b, b^2) \in \mu_\pi$ and $(xb, xb^2) \in \mu_\pi$ so $a\mathcal{H}_\pi ab$. Let $a' \in V(a)$, $b' \in V(b)$, $(ab)' \in V(ab)$ and select idempotents e, f so that $e\mathcal{H}_\pi a$ and $f\mathcal{H}_\pi b$. Then $(ea'e)\sigma_\pi^\# \in V(a\sigma_\pi^\#)$, $(e(ab)'e)\sigma_\pi^\# \in V(ab\sigma_\pi^\#)$, $(fb'f)\sigma_\pi^\# \in$

$V(b\sigma_\pi^\#)$, $ea'e\mathcal{H}_\pi a\mathcal{H}_\pi e(ab)'e$ and $fb'f\mathcal{H}_\pi b$. Since $a = xb$, then $a\sigma_\pi = (afb'fb)\sigma_\pi = (af)\sigma_\pi$. So $(aa')\sigma_\pi = (afa')\sigma_\pi$ and by (3), $aba' \in K$. By (3), since $aba'\mathcal{H}_\pi aa'$ and $e\sigma_\pi = (eaa'e)\sigma_\pi$ then $eaba'e \in K$. Since $eaba'e, a, e(ab)'eab \in K_\varepsilon$ then by (2) $eaba'eae(ab)'eab \in K_e$. But

$$(eaba'eae(ab)'e)\sigma_\pi = (eab(ea'ea)e(ab)'e)\sigma_\pi = (eab(e(ab)'e))\sigma_\pi = e\sigma_\pi$$

so $eaba'eae(ab)'e \in K_e$ by (1), and by (2) $ab \in K_e$. By a similar type of argument $a \in bS$ implies $ba \in K$. Hence K is the kernel of a congruence associated with π.

By Feigenbaum (Theorem 4.1 (1979)) ρ_K is a congruence associated with π having kernel K. The word "any" can be added to her description since for $(a, b) \in \rho_K$ then $(ab', bb') \in \rho_K$ and $(a'b, a'a) \in \rho_K$ for any $a' \in V(a)$ and $b' \in V(b)$. It is easy to see that $\ker\rho$ is a π-kernel. The remainder of the theorem is from Feigenbaum's result.

In order to complete the description of the normal equivalence-kernel method, a description of $\ker\sigma_\pi$ and $\ker\mu_\pi$ is required. The following lemma is used in obtaining a characterization of these kernels.

LEMMA 2.3. *Let ρ be a congruence on S and $e, f \in E$.*
(i) $S(f\rho^\#, e\rho^\#) \subseteq \{(S(f, g))\rho^\#;\ g \in E\}$.
(ii) $(ef)\rho = e\rho$ *if and only if there exists* $s \in e\rho \cap E$ *so that* $sf = s = es$.
(iii) $e\rho^\# \leqslant f\rho^\#$ *if and only if there exists* $t \in e\rho \cap E$ *so that* $t \leqslant f$.

PROOF. There exists $p \in E$ so that $p\rho^\# \in S(f\rho^\#, e\rho^\#)$. Then $(pf)\rho = p\rho = (ep)\rho$ and by the author (Lemma 1.2 (1978)), $S(pf, pf) \subseteq p\rho$. Let $q \in S(pf, pf)$ and $h \in S(p, f)$, then $h \in V(pf)$, $hp = h = fh$ and $S(pf, pf) = S(hpf, pfh) = S(hf, ph)$ so $qhf = q$; in particular $qf = q$. Likewise for $r \in S(ep, ep)$ then $r\rho = p\rho$ and $er = r$. For $s \in S(q, r)$, since $q\rho = r\rho$ we have $s \in p\rho$ by the author (Lemma 1.3 (1978)). Then $sq = s = rs$ and since $qf = q$, $er = r$ then $sf = s = es$. Now put $g = se$. Then $s \in S(f, g)$ and $s\rho = p\rho$, so (i) is proved.

Suppose $(ef)\rho^\# = e\rho^\#$ then $e\rho^\# \in S(f\rho^\#, e\rho^\#)$. By the proof of (i) with $p = e$, we get s as required in (ii). If furthermore $e\rho^\# \leqslant f\rho^\#$ let $t = fs$. Since $(fs)\rho = (fe)\rho = e\rho$ and $t \leqslant f$ then t is as required in (iii). The converses of (ii) and (iii) are immediate.

THEOREM 2.4. $\mu_\pi = \{(a, b) \in \mathcal{H}_\pi$; *for some (any)* $a' \in V(a)$, $b' \in V(b)$ *and each idempotent* $t \leqslant aa'$ *then* $S(a'ab'tb, a'ab'tb) \subseteq (a'ta)\pi\} = \{(a, b) \in \mathcal{H}_\pi$; *for some (any)* $a' \in V(a)$, $b' \in V(b)$ *and each idempotent* $t \leqslant aa'$ *then* $a'ta\mathcal{H}_\pi a'ab'tb\}$.

PROOF. By Theorem 2.3 of the author (1978) and its proof, $a\mu_\pi b$ if and only if $a\mathcal{H}_\pi b$ and for some (any) $a' \in V(a)$, $b' \in V(b)$ and for each idempotent e,

$e\sigma_\pi^\# \leqslant (aa')\sigma_\pi^\#$ then $(S(a'ea, a'ea))\pi = (S(a'ab'aa'eb, a'ab'aa'eb))\pi$; since $a\mathcal{H}_\pi b$, $(a'ea)\sigma_\pi^\#$ and $(a'ab'aa'eb)\sigma_\pi^\#$ are idempotents. It can be easily seen, for an idempotent $x\sigma_\pi^\#$, that $(S(x, x))\sigma_\pi^\# \subseteq S(x\sigma_\pi^\#, x\sigma_\pi^\#) = \{x\sigma_\pi^\#\}$. Thus, if $aa' \geqslant t \in e\pi$, $(S(a'ea, a'ea))\pi = (a'ea)\sigma_\pi \cap E = (a'ta)\sigma_\pi \cap E = (S(a'ta, a'ta))\pi = (a'ta)\pi$. Likewise $(S(a'ab'aa'eb, a'ab'aa'eb))\pi = (S(a'ab'tb, a'ab'tb))\pi$. The first equality of the theorem now follows by Lemma 2.3(iii). The second equality follows since as an idempotent $(a'ta)\sigma_\pi^\# = (a'ab'tb)\sigma_\pi^\#$.

It should be noted that this theorem allows simplifications of the descriptions of μ_π by Reilly and Scheiblich (1967) and Feigenbaum (1976) for inverse and orthodox semigroups respectively. In the inverse case, using Theorem 2.1 and our notation, these give

$$\mu_\pi = \{(a, b) \in S \times S;\ a^{-1}ea\pi b^{-1}eb \text{ for each } e \in E\}.$$

We now have *for S an inverse semigroup*

$$\mu_\pi = \{(a, b) \in S \times S;\ aa^{-1}\pi bb^{-1},\ a^{-1}a\pi b^{-1}b \textit{ and } a^{-1}ta\pi b^{-1}tb \textit{ for each idempotent } t \leqslant aa^{-1}\}.$$

We get $a^{-1}ta\pi b^{-1}tb$ since $(a^{-1}ab^{-1}tb)\sigma_\pi = (b^{-1}tb)\sigma_\pi$ so the idempotent $b^{-1}tb$ is $\mathcal{H}_\pi$-related to $a^{-1}ab^{-1}tb$. The simplification in the orthodox case is more extensive.

COROLLARY 2.5. $\ker \mu_\pi = \{b \in S$; *for some (any) $b' \in V(b)$ and idempotent $f\mathcal{H}_\pi b$ and for each idempotent $t \leqslant f$ then $S(fb'tb, fb'tb) \subseteq t\pi$ (equivalently $t\mathcal{H}_\pi fb'tb$)*$\}$.

The following definitions are required for our description of σ_π. If U is a subset of S let $\langle U \rangle$ denote the subsemigroup of S generated by U. For $e \in E$ define

$$N_{e\pi} = \langle \cup \{x(g\pi)y;\ x, y \in S^1, g, xgy \in E \text{ and } xgys = s \text{ for some } s \in e\pi\}\rangle$$

and

$$E_{e\pi} = \langle \cup \{f\pi;\ f \in E \text{ and } sf = s \text{ for some } s \in e\pi\}\rangle.$$

THEOREM 2.6. $\sigma_\pi = \{(a, b) \in \mathcal{H}_\pi$; *for some (any) $a' \in V(a)$ there exists $u \in E_{aa'\pi}$ and $v \in N_{aa'\pi}$ so that $ba'u = v$*$\}$.

PROOF. We will need to select elements of $V(xgzz')$ for $g \in E$, $x, z \in S^1$, $x' \in V(x)$ and $z' \in V(z)$ ($x' = 1$ if $x = 1$ and likewise for z). This can be done as follows. Let $h \in S(g, zz') \subseteq V(gzz')$, so $hg = h = zz'h$ and $ghzz' = gzz'$. Let

$k \in S(x, gzz') = S(x'x, gzz'h) = S(x'x, gh)$, then $kx'x = k = ghk$, $xkgz = x(x'xkgh)zz'z = xx'xghzz'z = xgz$ and $hkx' \in V(xgzz')$. Note that if $xgz(xgz)' \in e\pi$ for some $(xgz)' \in V(xgz)$ then $x(g\pi)hkx' \in N_{e\pi}$; this follows since $xghkx' = xkx' \in E$ and $xkx'xgz(xgz)' = xkgz(xgz)' = xgz(xgz)'$.

Now suppose $(a, b) \in \sigma_\pi$ and $a' \in V(a)$. Clearly $(a, b) \in \mathcal{H}_\pi$. Let $e = aa'$, then $(e, ba') \in \sigma_\pi$ and (by Clifford and Preston (1967) Lemma 10.3) there exists $x_i, y_i \in S^1$ and $(e_i, f_i) \in \pi$, $1 \leq i \leq n$, so that

$$e = x_1 e_1 y_1, \qquad x_j f_j y_j = x_{j+1} e_{j+1} y_{j+1}, \qquad x_n f_n y_n = ba' \quad \text{for } 1 \leq j < n.$$

We proceed by induction on n to find the required u and v. Say $n = 1$ and put $x = x_1$, $z = y_1 e$ and $g = e_1$, so $e = xgz$, $xf_1 z = ba'e$. Select h and k as in the last paragraph, so $hkx' \in V(ez')$. Note that $ez' \mathcal{L} zz'$ since $xgzz' = ez'$ and $zez' = zz'$, so $zz' = zz'(hkx')ez' = hkx'ez'$. Since $ez'z = xgzz'z = e$ then $z = hkx'ez'z = hkx'e$. We now have $(xf_1 hkx')e = xf_1 z = ba'e$. Put $u = e$ and $v = (xf_1 hkx')e$. Clearly $u \in E_{e\pi}$ and since $g \in f_1\pi$ then by the note of the last paragraph $xf_1 hkx' \in N_{e\pi}$, so $v \in N_{e\pi}$. Thus $ba'u = v$ as required.

Continuing by induction assume $x_{n-1} f_{n-1} y_{n-1} r = w$ for some $r = er \in E_{e\pi}$ and $w = we \in N_{e\pi}$. Then we have

$$ww' = x_n e_n (y_n rw'), \qquad x_n f_n (y_n rw') = ba'rw' \quad \text{for } w' \in V(w).$$

This is of the form for $n = 1$ with the relabelling $x = x_n$, $z = y_n rw'$, $e_n = g$ and and e and ba' replaced by ww' and $ba'rw'$ respectively. So for h and k as in the first paragraph we get as for $n = 1$ that $xf_n hkx'ww' = ba'rw'$ where $xf_n hkx' \in N_{ww'\pi}$. Hence $ba'u = v$ where $u = rw'w$ and $v = xf_n hkx'w$. Since $w = we \in N_{e\pi}$ then from the definition of $N_{e\pi}$ we easily see that $(w, e) \in \sigma_\pi$. Thus $(ww'e)\sigma_\pi = e\sigma_\pi = (ew'w)\sigma_\pi$. By the note of the first paragraph $xghkx' = xkx' \in E$ where $(g, f_n) \in \pi$ and $xkx'ww' = ww'$. So $(xkx'e)\sigma_\pi = (xkx'ww'e)\sigma_\pi = (ww'e)\sigma_\pi = e\sigma_\pi$. So by the dual of Lemma 2.3(ii), $xf_n hkx' \in N_{e\pi}$. Thus $v \in N_{e\pi}$. Since $r = er \in E_{e\pi}$ then $(r, e) \in \sigma_\pi$ and $(rw'w)\sigma_\pi = (ew'w)\sigma_\pi = e\sigma_\pi$ so by Lemma 2.3(ii) $w'w$ and $u \in E_{e\pi}$. Therefore by induction, if $(a, b) \in \sigma_\pi$ then there exists u and v as required.

Conversely suppose a and b are as described in the statement of the theorem. Then by the definitions of $E_{e\pi}$ and $N_{e\pi}$ we readily see that $(aa'u)\sigma_\pi = (aa')\sigma_\pi = (vaa')\sigma_\pi$. Since $a \mathcal{H}_\pi b$ then $(b'ba'a)\sigma_\pi = (b'b)\sigma_\pi$ so $b\sigma_\pi = (ba'a)\sigma_\pi = (ba'aa'ua)\sigma_\pi = (ba'ua)\sigma_\pi = (va)\sigma_\pi = (vaa'a)\sigma_\pi = a\sigma_\pi$.

COROLLARY 2.7. Ker $\sigma_\pi = \{b \in S$; *for some (any) idempotent* $e \mathcal{H}_\pi b$ *there exists* $u \in E_{e\pi}$ *and* $v \in N_{e\pi}$ *so that* $beu = v\}$.

Theorem 2.6 is a considerable refinement of the previous characterization of σ_π. Let U be the subsemigroup of S generated by conjugates of products of idempotents and for $e \in E$ let $W_e = e\sigma_\pi \cap U$. The characterization of σ_π by Feigenbaum ((1979) Theorem 3.3) makes use of the sets W_e. These sets are hard to determine; the suggested method is via chains of elementary $(\pi \cup \iota)$-transitions. In Theorem 2.6 we make use of the explicitly described semigroups $N_{e\pi}$, $E_{e\pi} \subseteq W_e$. It should be noted that the previous characterization also included duals of conditions of the type used in Theorem 2.6. By the Theorem, or directly, these dual conditions are superfluous. The same statement about dual conditions applies to the description of σ_π for orthodox semigroups (Feigenbaum (1976), Theorem 4.1).

The major complication in our description of the normal equivalence-kernel method for regular semigroups, from a computational point of view, now seems to be in the determination of π.

Completely regular semigroups

In this section the normal equivalence-kernel method is applied to completely regular semigroups. Some special cases are considered for which the method simplifies. We determine as applications some special congruences on completely regular semigroups.

A semigroup S is *completely regular* if and only if the $\mathcal{H}$-classes of S are all groups.

Throughout this section S will denote a completely regular semigroup. For $a \in S$ let a^* denote the identity of the $\mathcal{H}$-class containing a, and $a^{-1} \in V(a)$ denote the unique inverse of a so that $aa^{-1} = a^* = a^{-1}a$. It is well known (Clifford and Preston (1961) Theorem 4.6) that $S = \bigcup\{S_\alpha;\ \alpha \in J\}$ where S_α is a completely simple $\mathcal{J}$-class of S, J is a semilattice and $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for all $\alpha, \beta \in J$.

For $a \in S$, then $S(a, a) = S(a^*, a^*) = a^*$ so we get from Theorem 2.1,

THEOREM 3.1. *An equivalence relation π on E is normal if and only if for each $e, f \in E$, $c \in S$ and $c' \in V(c)$ then*

(i) $e\pi f\pi \cap E \subseteq (ef)^*\pi$ *and*

(ii) $c(e\pi)c' \cap E \subseteq (cec')^*\pi$.

It can be easily checked that a homomorphic image of S is completely regular. In fact if S is also orthodox or is a band of groups then its homomorphic images are respectively orthodox or bands of groups. Hence, for a normal equivalence π and $a \in S$, $H_{a\pi} \cap E = a^*\pi$, where $H_{a\pi}$ is the $\mathcal{H}_\pi$-class containing a. Clearly $H_{a\pi} = \bigcup\{H_g;\ g \in a^*\pi\}$, where H_g is the $\mathcal{H}$-class containing g. We have $a\mathcal{H}_\pi b$ if and only if $a^*\pi = b^*\pi$ if and only if $a^{-1}\mathcal{H}_\pi b^{-1}$. Note that $H_{a\pi}$ is itself a completely regular semigroup.

COROLLARY 3.2. *Conditions* (i) *and* (ii) *of Theorem* 3.1 *are equivalent to the conditions*

$$e\pi f\pi = (ef)^*\pi \text{ and } ce\pi c' = (cec')^*\pi \text{ if } S \text{ is orthodox},$$
$$e\pi f\pi \subseteq \bigcup \{H_g;\, g \in (ef)^*\pi\} \text{ if } S \text{ is a band of groups},$$
$$e\pi f\pi = (ef)^*\pi \text{ if } S \text{ is an orthodox band of groups}.$$

PROOF. The result is immediate in the orthodox case. Suppose S is a band of groups. From the preceding discussion (i) and (ii) imply the condition. Conversely, suppose $e\pi f\pi \subseteq \bigcup\{H_g;\ g \in (ef)^*\pi\}$. Condition (i) follows. Since $(c^*e)^*\mathcal{H}(ce)^*$ then

$$c^*e\pi(c')^* \subseteq \bigcup \{H_g;\, g \in (ce)^*\pi\}(c')^*$$
$$= \bigcup \{H_h;\, h \in (ce)^*\pi(c')^*\} \subseteq \bigcup \{H_k;\, k \in (cec')^*\pi\}.$$

Since $cec'\mathcal{H}c^*e(c')^*$ then condition (ii) follows. Hence the result for bands of groups. The result for orthodox bands of groups is a consequence of the other two results.

With a π-kernel K as defined in Section 2, we may redefine ρ_K in Theorem 2.2 by

$$\rho_K = \{(a, b) \in S \times S;\, a^*\pi = b^*\pi \text{ and } ab^{-1}, a^{-1}b \in K\}.$$

THEOREM 3.3. $\mu_\pi = \{(a, b) \in S \times S;\ a^*\pi = b^*\pi$ *and* $(a^{-1}ea)\pi = (b^{-1}eb)\pi$ *for all* $e \in E$ *so that* $a^*e = e = eb^*\}$.

PROOF. Suppose $t \in E$ and $t \leq a^*$. Since $a^*\pi = b^*\pi$ then $t\sigma_\pi^\# \leq b^*\sigma_\pi^\#$, so by Lemma 2.3(iii) there exists $s \in t\pi$ so that $s \leq b^*$. Choose $e \in S(s, t)$ then $e\pi = t\pi$ (the author (1978) Lemma 1.3) and $a^*e = a^*te = te = e = es = esb^* = eb^*$. We now have $a^{-1}ea$, $b^{-1}eb \in E$, and since $(a^*b^{-1})\sigma_\pi = b^{-1}\sigma_\pi$ then $a^*b^{-1}eb\mathcal{H}_\pi b^{-1}eb$. Proceeding as in the proof of Theorem 2.4 we get the result.

If S is orthodox the restriction $a^*e = e = eb^*$ of Theorem 3.3 *may be replaced by* $e \leq a^*$. This is an immediate consequence of Theorem 2.4 since $b^{-1}eb$ is an idempotent $\mathcal{H}_\pi$-related to $a^*b^{-1}eb$.

If $a \in \ker \mu_\pi$ then $a\mu_\pi a^*$ hence

COROLLARY 3.4. $\ker \mu_\pi = \{a \in S;\ a^{-1}ea \in e\pi$ *for all idempotents* $e \leq a^*\}$.

EXAMPLE 3.5. Let $S = \mathfrak{M}(G, I, \Lambda, P)$ be a regular Rees matrix semigroup over the group G, with $\Lambda \times I$ sandwich matrix $P = (p_{\lambda i})$ and elements $(a, i, \lambda) \in G \times I \times \Lambda$. There are various descriptions of the congruences on S in the literature. The following is easily derived from (Howie (1976) III.4). Let π be an equivalence relation on the set of idempotents of S and let I_π be the equivalence relation on I induced by π, so

$$I_\pi = \left\{(i, j) \in I \times I;\ \left(p_{\lambda i}^{-1}, i, \lambda\right)\pi\left(p_{\mu j}^{-1}, j, \mu\right) \text{ for some } \lambda, \mu \in \Lambda\right\}.$$

Likewise let Λ_π be the equivalence relation on Λ induced by π. Then π is a normal equivalence if and only if

$$\pi = \left\{\left(\left(p_{\lambda i}^{-1}, i, \lambda\right), \left(p_{\mu j}^{-1}, j, \mu\right)\right);\ (i, j) \in I_\pi, (\lambda, \mu) \in \Lambda_\pi\right\}.$$

Let M be the least normal subgroup of G so that $p_{\lambda i} p_{\mu i}^{-1} p_{\mu j} p_{\lambda j}^{-1} \in M$ for each $i, j \in I$ and $\lambda, \mu \in \Lambda$ so that $(i, j) \in I_\pi$ or $(\lambda, \mu) \in \Lambda_\pi$. A subset K of S is a π-kernel if and only if $K = \{(p_{\lambda i}^{-1} n, i, \lambda);\ (i, \lambda) \in I \times \Lambda$ and $n \in N\}$ where $M \subseteq N$ and N is a normal subgroup of G. Since the congruence ρ_K with kernel K, associated with π, is uniquely determined by π and N (see Theorem 2.2) we write $\rho(\pi, N) = \rho_K$. It follows that $\mu_\pi = \rho(\pi, G)$ and $\sigma_\pi = \rho(\pi, M)$.

Recall that $S = \bigcup\{S_\alpha;\ \alpha \in J\}$ where S_α is a completely-simple subsemigroup for all α in the semilattice J. Since π denotes a normal equivalence on E then clearly $\pi_\alpha = \pi \cap (S_\alpha \times S_\alpha)$ is a normal equivalence on the idempotents of S_α.

For $\alpha \in J$ suppose $S_\alpha = \mathfrak{M}(G_\alpha, I_\alpha, \Lambda_\alpha, P_\alpha)$, a regular Rees matrix semigroup. For any $e \in E \cap S_\beta$ where $\beta \geqslant \alpha$ define $e_{ij\lambda\mu} \in G_\alpha$ by $(e_{ij\lambda\mu}, i, \mu) = (p_{\lambda i}^{-1}, i, \lambda)e(p_{\mu j}^{-1}, j, \mu)$. Now define N_α to be the least normal subgroup of G_α that contains $\{e_{ij\lambda\mu} f_{ij\lambda\mu}^{-1};\ (e, f) \in \pi \cap (S_\beta \times S_\gamma)$ where $\beta\gamma \geqslant \alpha$ and $i, j \in I_\alpha$, $\lambda, \mu \in \Lambda_\alpha\}$. It can be readily checked that N_α contains M_α (as defined in Example 3.5).

THEOREM 3.6. *$\sigma_\pi = \{(a, b) \in S \times S;\ a^*\pi = b^*\pi$ and there exists an idempotent $f \in a^*\pi$ so that $fbf\ \rho(\pi_\alpha, N_\alpha)\ faf$ where $f \in S_\alpha\}$.*

PROOF. Suppose $(e, f) \in \pi \cap (S_\beta \times S_\gamma)$ where $\beta\gamma \geqslant \alpha$. Then for any $i, j \in I_\alpha$, $\lambda, \mu \in \Lambda_\alpha$ we have $(e_{ij\lambda\mu}, i, \mu)\sigma_\pi(f_{ij\lambda\mu}, i, \mu)$. Hence by Howie ((1976) Lemma III, 4.20), or by a direct calculation using Theorem 2.2 and Example 3.5, we see that $\rho(\pi_\alpha, N_\alpha) \subseteq \sigma_\pi$. With a, b, f and α as in the statement of the Theorem, then $b\sigma_\pi = (fbf)\sigma_\pi = (faf)\sigma_\pi = a\sigma_\pi$.

Conversely suppose $a, b \in \sigma_\pi$. As in the proof of Theorem 2.6 there exists x_i, $y_i \in S^1$ and $(e_i, f_i) \in \pi$, $1 \leqslant i \leqslant n$, so that

$$a = x_1 e_1 y_1, \qquad x_j f_j y_j = x_{j+1} e_{j+1} y_{j+1} \quad \text{and} \quad x_n f_n y_n = b, \qquad 1 \leqslant j < n.$$

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The terms of these equations all lie in one σ_π-class so $a^*\pi = b^*\pi$. Let $\alpha \in J$ be least so that a or $x_p f_p y_p \in S_\alpha$ for some p, $1 \leqslant p \leqslant n$. Then there exists $f \in a^*\pi \cap S_\alpha$ (the $\mathcal{H}_\pi$-class for a meets S_α). Put $u_i = fx_i$, $v_i = y_i f$, $h_i = (fx_i)^* e_i (y_i f)^*$ and $k_i = (fx_i)^* f_i (y_i f)^*$ for each i. Since $(e_i, f_i) \in \pi$ we get (for example using Howie ((1976) Lemma III, 4.19)) that $(h_i, k_i) \in \rho(\pi_\alpha, N_\alpha)$. Furthermore, since

$$faf = u_1 h_1 v_1, \qquad u_j k_j v_j = u_{j+1} h_{j+1} v_{j+1} \quad \text{and} \quad u_n k_n v_n = fbf, \qquad 1 \leqslant j < n,$$

in S_α then $faf\ \rho(\pi_\alpha, N_\alpha)\ fbf$.

Notice that if S is an orthodox union of groups then $|N_\alpha| = 1$. So *if S is an orthodox union of groups then the condition $fbf\, \rho(\pi_\alpha, N_\alpha)\, faf$ can be replaced by fbf σ_{π_α} faf* ($\ker \sigma_{\pi_\alpha} = E \cap S_\alpha$).

COROLLARY 3.7. $\ker \sigma_\pi = \{b \in S$; *there exists an idempotent* $f \in b^*\pi$ *so that* $fbf \in \ker \rho(\pi_\alpha, N_\alpha)$ *where* $f \in S_\alpha\}$.

EXAMPLES 3.8. The finest group congruence on S is σ_π where $\pi = E \times E$. The finest semilattice of groups congruence on S is σ_π where $\pi = \{(e, f) \in (E \cap S_\alpha) \times (E \cap S_\alpha);\ \alpha \in J\}$ (for an alternative description of these congruences see T. L. Pirnot (1973)).

The finest orthodox union of groups congruence on a completely regular semigroup has a complex description by these methods. Restricting ourselves to bands of groups, we can obtain a neat expression. Let the semigroup S be a band of groups and as above suppose $S_\alpha = \mathfrak{M}(G_\alpha, I_\alpha, \Lambda_\alpha, P_\alpha)$. It can be readily checked that the least orthodox congruence on S_α is $\rho(\iota, N_\alpha)$ where ι is the identity normal equivalence and N_α is the least normal subgroup of G_α that contains $\{p_{\lambda i}^{-1} p_{\lambda j} p_{\mu j}^{-1} p_{\mu i};\ i, j \in I_\alpha,\ \lambda, \mu \in \Lambda_\alpha\}$. Let ρ be the least orthodox congruence on S. Then ρ is the least congruence on S so that $(ef, (ef)^*) \in \rho$ for any $e, f \in E$. Notice that for $p \in S(e, f)$ and $ef \in S_\alpha$ then $ef \mathcal{R} ep \mathcal{L} p \mathcal{R} pf$; $ep, pf \in E$ and $(ep)(pf) = ef$; so ef is a product of idempotents in S_α. Using this and the fact that $\rho \subseteq \mathcal{H}$ it can be checked by a proof similar to that of Theorem 3.6 that

$$\rho = \{(a, b) \in S \times S;\ a^* = b^* \text{ and } a\, \rho(\iota, N_\alpha)\, b \text{ where } a \in S_\alpha\}.$$

Hence $\rho = \bigcup \{\rho(\iota, N_\alpha);\ \alpha \in J\}$.

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