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History of Moduli Problems

The moduli spaces of smooth or stable projective curves of genus $g \geq 2$ are, quite possibly, the most studied of all algebraic varieties.

The aim of this book is to generalize the moduli theory of curves to surfaces and to higher dimensional varieties. In this chapter, we aim to outline how this is done, and, more importantly, to explain why the answer for surfaces is much more complicated than for curves. On the positive side, once we get the moduli theory of surfaces right, the higher dimensional theory works the same.

Section 1.1 is a quick review of the history of moduli problems, culminating in an outline of the basic moduli theory of curves. A'Campo et al. (2016) is a very good overview. Reading some of the early works on moduli, including Riemann, Cayley, Klein, Hilbert, Siegel, Teichmüller, Weil, Grothendieck, and Mumford gives an understanding of how the modern theory relates to the earlier works. See Kollár (2021b) for an account that emphasizes the historical connections.

In Section 1.2, we outline how the theory should unfold for higher dimensional varieties. Details of going from curves to higher dimensions are given in the next two sections. Section 1.3 introduces canonical models, which are the basic objects of moduli theory in higher dimensions. Starting from stable curves, Section 1.4 leads up to the definition of stable varieties, their higher dimensional analogs. Then we show, by a series of examples, why flat families of stable varieties are *not* the correct higher dimensional analogs of flat families of stable curves. Finding the correct replacement has been one of the main difficulties of the whole theory.

While the moduli theory of curves serves as our guideline, it also has many good properties that do not generalize. Sections 1.5–1.8 are devoted to examples that show what can go wrong with moduli theory in general, or with stable varieties in particular.

First, in Section 1.5, we show that the simple combinatorial recipe of going from a nodal curve to a stable curve has no analog for surfaces. Next we give a collection of examples showing how easy it is to end up with rather horrible moduli problems. Hypersurfaces and other interesting examples are discussed in Section 1.6, as are alternative compactifications of the moduli of curves in Section 1.7. Section 1.8 illustrates the differences between fine and coarse moduli spaces.

Two major approaches to moduli – the geometric invariant theory of Mumford and the Hodge theory of Griffiths – are mostly absent from this book. Both of these are very powerful, and give a lot of information in the cases when they apply. They each deserve a full, updated treatment of their own. However, so far neither gave a full description of the moduli of surfaces, much less of higher dimensional varieties. It would be very interesting to develop a synthesis of the three methods and gain better understanding in the future.

1.1 Riemann, Cayley, Hilbert, and Mumford

Let $\mathbf{V}$ be a "reasonable" class of objects in algebraic geometry, for instance, $\mathbf{V}$ could be all subvarieties of $\mathbb{P}^n$, all coherent sheaves on $\mathbb{P}^n$, all smooth curves or all projective varieties. The aim of the theory of moduli is to understand all "reasonable" families of objects in $\mathbf{V}$, and to construct an algebraic variety (or scheme, or algebraic space) whose points are in "natural" one-to-one correspondence with the objects in $\mathbf{V}$. If such a variety exists, we call it the *moduli space* of $\mathbf{V}$, and denote it by $\mathrm{M}_{\mathbf{V}}$. The simplest, classical examples are given by the theory of linear systems and families of linear systems.

1.1 (Linear systems) Let X be a smooth, projective variety over an algebraically closed field k and L a line bundle on X. The corresponding linear system is

$$\mathcal{L}in\mathcal{S}ys(X, L) = \{\text{effective divisors } D \text{ such that } \mathscr{O}_X(D) \simeq L\}.$$

The objects in $\mathcal{L}in\mathcal{S}ys(X, L)$ are in natural one-to-one correspondence with the points of the projective space $\mathbb{P}(H^0(X, L)^\vee)$ which is traditionally denoted by $|L|$. (We follow the Grothendieck convention for $\mathbb{P}$ as in Hartshorne (1977, sec.II.7).) Thus, for every effective divisor D such that $\mathscr{O}_X(D) \simeq L$, there is a unique point $[D] \in |L|$.

Moreover, this correspondence between divisors and points is given by a universal family of divisors over $|L|$. That is, there is an effective Cartier divisor $\mathrm{Univ}_L \subset |L| \times X$ with projection $\pi\colon \mathrm{Univ}_L \to |L|$ such that $\pi^{-1}[D] = D$ for every effective divisor D linearly equivalent to L.

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The classical literature never differentiates between the linear system as a set and the linear system as a projective space. There are, indeed, few reasons to distinguish them as long as we work over a fixed base field k. If, however, we pass to a field extension $K \supset k$, the advantages of viewing $|L|$ as a k-variety appear. For any $K \supset k$, the set of effective divisors D defined over K such that $\mathscr{O}_X(D) \simeq L$ corresponds to the K-points of $|L|$. Thus the scheme-theoretic version automatically gives the right answer over every field.

1.2 (Jacobians of curves) Let C be a smooth projective curve (or Riemann surface) of genus g. As discovered by Abel and Jacobi, there is a variety $\mathrm{Jac}^\circ(C)$ of dimension g whose points are in natural one-to-one correspondence with degree 0 line bundles on C. As before, the correspondence is given by a universal line bundle $L^{\mathrm{univ}} \to C \times \mathrm{Jac}^\circ(C)$, called the Poincaré bundle. That is, for any point $p \in \mathrm{Jac}^\circ(C)$, the restriction of L^{univ} to $C \times \{p\}$ is the degree 0 line bundle corresponding to p.

Unlike in (1.1), the universal line bundle L^{univ} is not unique (and need not exist if the base field is not algebraically closed). This has to do with the fact that while an automorphism of the pair $D \subset X$ that is trivial on X is also trivial on D, any line bundle $L \to C$ has automorphisms that are trivial on C: we can multiply every fiber of L by the same nonzero constant.

1.3 (Cayley forms and Chow varieties) Cayley (1860, 1862) developed a method to associate a hypersurface in the Grassmannian $\mathrm{Gr}(\mathbb{P}^1, \mathbb{P}^3)$ to a curve in $\mathbb{P}^3$. The resulting moduli spaces have been used, but did not seem to have acquired a name. Chow understood how to deal with reducible and multiple varieties, and proved that one gets a projective moduli space; see Chow and van der Waerden (1937). The name *Chow variety* seems standard, we use Cayley–Chow for the correspondence that was discovered by Cayley. See Section 3.1 for an outline and Kollár (1996, secs.I.3–4) for a modern treatment.

Let k be an algebraically closed field and X a normal, projective k-variety. Fix a natural number m. An *m-cycle* on X is a finite, formal linear combination $\sum a_i Z_i$ where the Z_i are irreducible, reduced subvarieties of dimension m and $a_i \in \mathbb{Z}$. We usually assume tacitly that all the Z_i are distinct. An m-cycle is called *effective* if $a_i \geq 0$ for every i.

The points of the *Chow variety* $\mathrm{Chow}_m(X)$ are in "natural" one-to-one correspondence with the set of effective m-cycles on X. (Since we did not fix the degree of the cycles, $\mathrm{Chow}_m(X)$ is not actually a variety, but a countable disjoint union of projective, reduced k-schemes.) The point of $\mathrm{Chow}_m(X)$ corresponding to a cycle $Z = \sum a_i Z_i$ is also usually denoted by $[Z]$.

As for linear systems, it is best to describe the “natural correspondence” by a universal family. The situation is, however, more complicated than before.

There is a family (or rather an effective cycle) $\mathrm{Univ}_m(X)$ on $\mathrm{Chow}_m(X) \times X$ with projection $\pi\colon \mathrm{Univ}_m(X) \to \mathrm{Chow}_m(X)$ such that for every effective m-cycle $Z = \sum a_i Z_i$,

(1.3.1) the support of $\pi^{-1}[Z]$ is $\cup_i Z_i$, and

(1.3.2) the fundamental cycle (4.61.1) of $\pi^{-1}[Z]$ equals Z if $a_i = 1$ for every i.

If the characteristic of k is 0, then the only problem in (2) is a clash between the traditional cycle-theoretic definition of the Chow variety and the scheme-theoretic definition of the fiber, but in positive characteristic the situation is more problematic; see Kollár (1996, secs.I.3–4).

An example of a “perfect” moduli problem is the theory of *Hilbert schemes*, introduced in Grothendieck (1962, lect.IV). See Mumford (1966), (Kollár, 1996, I.1–2) or Sernesi (2006, sec.4.3) or Section 3.1 for a summary.

1.4 (Hilbert schemes) Let k be an algebraically closed field and X a projective k-scheme. Set

$$\mathcal{H}ilb(X) = \{\text{closed subschemes of } X\}.$$

Then there is a k-scheme $\mathrm{Hilb}(X)$, called the *Hilbert scheme* of X, whose points are in a “natural” one-to-one correspondence with closed subschemes of X. The point of $\mathrm{Hilb}(X)$ corresponding to a subscheme $Y \subset X$ is frequently denoted by $[Y]$. There is a universal family $\mathrm{Univ}(X) \subset \mathrm{Hilb}(X) \times X$ such that

(1.4.1) the first projection $\pi\colon\ \mathrm{Univ}(X) \to \mathrm{Hilb}(X)$ is flat, and

(1.4.2) $\pi^{-1}[Y] = Y$ for every closed subscheme $Y \subset X$.

The beauty of the Hilbert scheme is that it describes not just subschemes, but all flat families of subschemes as well. To see what this means, note that for any morphism $g\colon T \to \mathrm{Hilb}(X)$, by pull-back we obtain a flat family of subschemes $T \times_{\mathrm{Hilb}(X)} \mathrm{Univ}(X) \subset T \times X$. It turns out that every family is obtained this way:

(1.4.3) For every T and closed subscheme $Z \subset T \times X$ that is flat over T, there is a unique $g_Z\colon T \to \mathrm{Hilb}(X)$ such that $Z = T \times_{\mathrm{Hilb}(X)} \mathrm{Univ}(X)$.

This takes us to the functorial approach to moduli problems.

1.5 (Hilbert functor and Hilbert scheme) Let $X \to S$ be a morphism of schemes. Define the *Hilbert functor* of X/S as a functor that associates to a scheme $T \to S$ the set

$$\mathcal{H}ilb_{X/S}(T) = \{\text{subschemes } Z \subset T \times_S X \text{ that are flat and proper over } T\}.$$

The basic existence theorem of Hilbert schemes then says that, if $X \to S$ is quasi-projective, there is a scheme $\mathrm{Hilb}_{X/S}$ such that for any S scheme T,

$$\mathcal{H}ilb_{X/S}(T) = \mathrm{Mor}_S(T, \mathrm{Hilb}_{X/S}).$$

Moreover, there is a universal family $\pi\colon \mathrm{Univ}_{X/S} \to \mathrm{Hilb}_{X/S}$ such that the above isomorphism is given by pulling back the universal family.

We can summarize these results as follows:

Principle 1.6 $\pi\colon \mathrm{Univ}_{X/S} \to \mathrm{Hilb}_{X/S}$ *contains all the information about proper, flat families of subschemes of X/S, in the most succinct way.*

This example leads us to a general definition:

Definition 1.7 (Fine moduli spaces) Let $\mathbf{V}$ be a "reasonable" class of projective varieties (or schemes, or sheaves, or …). In practice "reasonable" may mean several restrictions, but for the definition we only need the following weak assumption:

(1.7.1) Let $K \supset k$ be a field extension. Then a k-variety X_k is in $\mathbf{V}$ iff $X_K := X_k \times_{\operatorname{Spec} k} \operatorname{Spec} K$ is in $\mathbf{V}$.

Following (1.5), define the corresponding moduli functor that associates to a scheme T the set

$$\mathcal{V}arieties_{\mathbf{V}}(T) := \left\{ \begin{array}{c} \text{Flat families } X \to T \text{ such that} \\ \text{every fiber is in } \mathbf{V}, \\ \text{modulo isomorphisms over } T. \end{array} \right\} \tag{1.7.2}$$

We say that a scheme $\mathrm{Moduli}_{\mathbf{V}}$ is a *fine moduli space* for the functor $\mathcal{V}arieties_{\mathbf{V}}$, if the following holds:

(1.7.3) For every scheme T, pulling back gives an equality

$$\mathcal{V}arieties_{\mathbf{V}}(T) = \mathrm{Mor}(T, \mathrm{Moduli}_{\mathbf{V}}).$$

Applying the definition to $T = \mathrm{Moduli}_{\mathbf{V}}$ gives a universal family $u\colon \mathrm{Univ}_{\mathbf{V}} \to \mathrm{Moduli}_{\mathbf{V}}$. Setting $T = \operatorname{Spec} K$, where K is a field, we see that the K-points of $\mathrm{Moduli}_{\mathbf{V}}$ correspond to the K-isomorphism classes of objects in $\mathbf{V}$.

We consider the existence of a fine moduli space as the ideal possibility. Unfortunately, it is rarely achieved.

Next we see what happens with the simplest case, for smooth curves.

1.8 (Moduli functor and moduli space of smooth curves) Following (1.7) we define the moduli functor of smooth curves of genus g as

$$\mathit{Curves}_g(T) := \left\{ \begin{array}{c} \text{Smooth, proper families } S \to T, \\ \text{every fiber is a curve of genus } g, \\ \text{modulo isomorphisms over } T. \end{array} \right\}$$

It turns out that there is no fine moduli space for curves of genus g. Every curve C with nontrivial automorphisms causes problems; there cannot be any point $[C]$ corresponding to it in a fine moduli space (see Section 1.8).

It was gradually understood that there is some kind of an object, denoted by M_g, and called the *coarse moduli space* (or simply *moduli space*) of curves of genus g, that comes close to being a fine moduli space.

For elliptic curves, we get $\mathrm{M}_1 \simeq \mathbb{A}^1$, and the moduli map is given by the j-invariant, as was known to Dedekind and Klein; see Klein and Fricke (1892). They also knew that there is no universal family over M_1. The theory of abelian integrals due to Abel, Jacobi, and Riemann does the same for all curves, though in this case a clear moduli-theoretic interpretation seems to have been done only later; see the historical sketch at the end of Shafarevich (1974), Siegel (1969, chap.4), or Griffiths and Harris (1978, chap.2) for modern treatments. For smooth plane curves, and more generally for smooth hypersurfaces in any dimension, the invariant theory of Hilbert produces coarse moduli spaces. Still, a precise definition and proof of existence of M_g appeared only in Teichmüller (1944) in the analytic case and in Mumford (1965) in the algebraic case. See A'Campo et al. (2016) or Kollár (2021b) for historical accounts.

1.9 (Coarse moduli spaces) Mumford (1965)

As in (1.7), let $\mathbf{V}$ be a "reasonable" class. When there is no fine moduli space, we still can ask for a scheme that best approximates its properties.

We look for schemes M for which there is a natural transformation

$$T_M \colon \mathcal{V}\mathit{arieties}_g(*) \longrightarrow \mathrm{Mor}(*, M).$$

Such schemes certainly exist: for instance, if we work over a field k, then we can take $M = \operatorname{Spec} k$. All schemes M for which T_M exists form an inverse system which is closed under fiber products. Thus, as long as we are not unlucky, there is a universal (or largest) scheme with this property. Though it is not usually done, it should be called the *categorical moduli space*.

This object can be rather useless in general. For instance, fix n, d and let $\mathbf{H}_{n,d}$ be the class of all hypersurfaces of degree d in $\mathbb{P}^{n+1}_k$, up to isomorphisms. We see in (1.56) that a categorical moduli space exists and it is $\operatorname{Spec} k$.

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To get something more like a fine moduli space, we require that it give a one-to-one parametrization, at least set theoretically. Thus we say that a scheme $\mathrm{Moduli}_{\mathbf{V}}$ is a *coarse moduli space* for $\mathbf{V}$ if the following hold:

(1.9.1) there is a natural transformation of functors

$$\mathrm{ModMap}\colon \mathcal{V}arieties_{\mathbf{V}}(*) \longrightarrow \mathrm{Mor}(*, \mathrm{Moduli}_{\mathbf{V}}),$$

(1.9.2) $\mathrm{Moduli}_{\mathbf{V}}$ is universal satisfying (1), and

(1.9.3) for any algebraically closed field $K \supset k$, we get a bijection

$$\mathrm{ModMap}\colon \mathcal{V}arieties_{\mathbf{V}}(\mathrm{Spec}\, K) \xrightarrow{\simeq} \mathrm{Mor}(\mathrm{Spec}\, K, \mathrm{Moduli}_{\mathbf{V}}) = \mathrm{Moduli}_{\mathbf{V}}(K).$$

1.10 (Moduli functors versus moduli spaces) While much of the early work on moduli, especially since Mumford (1965), put the emphasis on the construction of fine or coarse moduli spaces, recently the focus shifted toward the study of the families of varieties, that is, toward moduli functors and moduli stacks. The main task is to understand all "reasonable" families. Once this is done, the existence of a coarse moduli space should be nearly automatic. The coarse moduli space is not the fundamental object any longer, rather it is only a convenient way to keep track of certain information that is only latent in the moduli functor or stack.

1.11 (Compactifying M_g) While the basic theory of algebraic geometry is local, that is, it concerns affine varieties, most really interesting and important objects in algebraic geometry and its applications are global, that is, projective or at least proper.

The moduli spaces M_g are not compact, in fact the moduli functor of smooth curves discussed so far has a definitely local flavor. Most naturally occurring smooth families of curves live over affine schemes, and it is not obvious how to write down any family of smooth curves over a projective base. For many reasons it is useful to find geometrically meaningful compactifications of M_g. The answer to this situation is to allow not just smooth curves, but also certain singular curves in our families.

Concentrating on one-parameter families, we have the following:

Question 1.11.1 Let B be a smooth curve, $B^\circ \subset B$ an open subset, and $\pi^\circ\colon S^\circ \to B^\circ$ a smooth family of genus g curves. Is there a "natural" extension

$$\begin{tikzcd} S^\circ \arrow[r, hook] \arrow[d, "\pi^\circ"'] & S \arrow[d, "\pi"] \\ B^\circ \arrow[r, hook] & B, \end{tikzcd}$$

where $\pi\colon S \to B$ is a flat family of (possibly singular) curves?

There is no reason to think that there is a unique such extension. Deligne and Mumford (1969) construct one after a base change $B' \to B$, and by now it is hard to imagine a time when their choice was not the “obviously best” solution. We review their definition next. In Section 1.6 we see, by examples, why this concept has not been so obvious.

Definition 1.12 (Stable curve) A *stable curve* over an algebraically closed field k is a proper, geometrically connected k-curve C such that

(Local property) the only singularities of C are ordinary nodes, and

(Global property) the canonical class K_C is ample.

A *stable curve* over a scheme T is a flat, proper morphism $\pi\colon S \to T$ such that every geometric fiber of π is a stable curve. (The arithmetic genus of the fibers is a locally constant function on T, but we usually also tacitly assume that it is constant.) The moduli functor of stable curves of genus g is

$$\overline{Curves}_g(T) := \left\{ \begin{array}{c} \text{Stable curves of genus } g \text{ over } T, \\ \text{modulo isomorphisms over } T. \end{array} \right\}$$

Theorem 1.13 Deligne and Mumford (1969) *For every $g \geq 2$, the moduli functor of stable curves of genus g has a coarse moduli space $\overline{\mathrm{M}}_g$. Moreover, $\overline{M}_g$ is projective, normal, has only quotient singularities, and contains M_g as an open dense subset.*

$\overline{\mathrm{M}}_g$ has a rich and intriguing geometry, which is related to major questions in many branches of mathematics and physics; see Farkas and Morrison (2013) for a collection of surveys and Pandharipande (2018a,b) for overviews.

1.2 Moduli for Varieties of General Type

The aim of this book is to use the moduli of stable curves as a guideline, and develop a moduli theory for varieties of general type (1.30). (See (1.22) for some comments on the nongeneral type cases.)

Here we outline the main steps of the plan with some comments. Most of the rest of the book is then devoted to accomplishing these goals.

Step 1.14 (Higher dimensional analogs of smooth curves) It has been understood since the beginnings of the theory of surfaces that, for surfaces of Kodaira dimension ≥ 0 (p.xiv), the correct moduli theory should be birational, not biregular. That is, the points of the moduli space should correspond not to *isomorphism* classes of surfaces, but to *birational* equivalence classes of surfaces. There are two ways to deal with this problem.

First, one can work with smooth families, but consider two families $V \to B$ and $W \to B$ equivalent if there is a *fiber-wise birational* map between them; that is, a rational map $V \dashrightarrow W$ that induces a birational equivalence of the fibers $V_b \dashrightarrow W_b$ for every $b \in B$. This seems rather complicated technically.

The second, much more useful method relies on the observation that every birational equivalence class of surfaces of Kodaira dimension ≥ 0 contains a unique *minimal model*, that is, a smooth projective surface S^{m} whose canonical class is nef (p.xv). Therefore, one can work with families of minimal models, modulo isomorphisms. With the works of Mumford (1965) and Artin (1974) it became clear that, for surfaces of general type, it is even better to work with the *canonical model*, which is a mildly singular projective surface S^{c} whose canonical class is ample. The resulting class of singularities has since been established in all dimensions; they are called *canonical singularities* (1.33).

Principle 1.14.1 In moduli theory, the main objects of study are projective varieties with ample canonical class and with canonical singularities.

Implicit in this claim is that every smooth family of varieties of general type produces a flat family of canonical models, we discuss this in (1.36).

See Section 1.3 for more details on this step.

Step 1.15 (Higher dimensional analogs of stable curves) The correct definition of the higher dimensional analogs of stable curves was much less clear. An approach through geometric invariant theory (GIT) was investigated by Mumford (1977), but never fully developed. In essence, the GIT approach starts with a particular method of construction of moduli spaces, and then tries to see for which class of varieties it works. The examples of Wang and Xu (2014) suggest that GIT is unlikely to give a good compactification for the moduli of surfaces.

A different framework was proposed in Kollár and Shepherd-Barron (1988). Instead of building on geometric invariant theory, it focuses on one-parameter families, and uses Mori's program as its basic tool.

Before we give the definition, recall a key step of the proof of (1.13) that establishes separatedness and properness of $\overline{\mathrm{M}}_g$. (The traditional name is stable "reduction," but "extension" is more descriptive.)

1.15.1 (Stable extension for curves) Let B be a smooth curve, $B^\circ \subset B$ a dense, open subset, and $\pi^\circ\colon S^\circ \to B^\circ$ a flat family of smooth, projective curves of genus ≥ 2. Then there is a finite surjection $p\colon A \to B$ and a diagram

$$\begin{tikzcd} S^\circ \times_B A \arrow[r, hook] \arrow[d] & S_A^{\mathrm{ss}} \arrow[r, "\tau"] \arrow[d, "\pi_A^{\mathrm{ss}}"] & S_A^{\mathrm{stab}} \arrow[d, "\pi_A^{\mathrm{stab}}"] \\ B^\circ \times_B A \arrow[r, hook] & A \arrow[r, equal] & A, \end{tikzcd}$$

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where

(a) $\pi_A^{\mathrm{ss}}: S_A^{\mathrm{ss}} \to A$ is a flat family of reduced, nodal curves,
(b) $\tau: S_A^{\mathrm{ss}} \to S_A^{\mathrm{stab}}$ is the relative canonical model (11.26), and
(c) $\pi_A^{\mathrm{stab}}: S_A^{\mathrm{stab}} \to A$ is a flat family of stable curves.

A detailed proof is given in (2.51): for now we build on this to state the main theses of Kollár and Shepherd-Barron (1988) about moduli problems.

Principle 1.15.2 We should follow the proof of the Stable extension theorem (1.15.1). The resulting fibers give the right class of stable varieties.

Principle 1.15.3 As in (1.12), a connected k-scheme X is stable iff it satisfies two conditions, whose precise definitions are not important for now:
(Local property) Semi-log-canonical singularities, see (1.41).
(Global property) The canonical class K_X is ample, see (1.23).

1.15.4 (Warning about positive characteristic) The examples of Kollár (2022) suggest that, in positive characteristic, (1.15.2) gives the right families, but not quite the right objects in dimensions ≥ 3; see Section 8.8 for details.

Step 1.16 (Higher dimensional analogs of families of stable curves I) The definition (1.7) is very natural within our usual framework of algebraic geometry, but it hides a very strong supposition:

1.16.1 (Unwarranted assumption) If **V** is a “reasonable” class of varieties, then any flat family whose fibers are in **V** is a “reasonable” family.

In Grothendieck’s foundations of algebraic geometry, flatness is one of the cornerstones, and there are many “reasonable” classes for which flat families are indeed the “reasonable” families. Nonetheless, even when the base of the family is a smooth curve, (1.16.1) needs arguing, but the assumption is especially surprising when applied to families over nonreduced schemes T. Consider, for instance, the case when T is the spectrum of an Artinian k-algebra. Then T has only one closed point $t \in T$. A flat family $p: X \to T$ has only one fiber X_t, and our only restriction is that X_t be in our class **V**. Thus (1.16.1) declares that we care only about X_t. Once X_t is in **V**, every flat deformation of X_t over T is automatically “reasonable.”

A crucial conceptual point in the moduli theory of higher dimensional varieties is the realization that, starting with families of surfaces, flatness of the map $X \to T$ is not enough: allowing all flat families whose fibers are stable varieties leads to the wrong moduli problem.

The simple fact is that basic numerical invariants, like the self intersection of the canonical class, or even the Kodaira dimension, fail to be locally constant in

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flat families of stable varieties, even when the singularities are quite mild and the base is a smooth curve. We give a series of such examples in (1.42–1.47).

The difficulty of working out the correct concept has been one of the main stumbling blocks of the general theory.

Principle 1.16.2 Flat families of stable varieties $X \to T$ are **not** the correct higher dimensional analogs of flat families of stable curves (1.12).

For families over smooth curves, the Stable extension theorem (1.15.1) is again our guide to the correct definition.

1.16.3 (Stable morphisms) Let $p\colon Y \to B$ be a proper morphism from a normal variety to a smooth curve. Then p is *stable* iff, for every $b \in B$,

(a) Y_b has semi-log-canonical singularities,

(b) $K_{Y_b} = K_Y|_{Y_b}$ is ample, and

(c) mK_Y is Cartier for some $m > 0$, that is, K_Y is $\mathbb{Q}$-Cartier (p.xv).

This is a direct generalization of the notion of stable family of curves (1.12), except that here we have to add condition (c) for K_Y. If the K_{Y_b} are Cartier, then so is K_Y (2.6), this is why (c) was not necessary for curves. See (2.3) for other versions and (2.4) for comments on the positive characteristic cases.

Note that the K_{Y_b} are $\mathbb{Q}$-Cartier by (1.15.3), but this does not imply that K_Y is $\mathbb{Q}$-Cartier; this is a quite subtle issue with restrictions of non-Cartier divisors. We discuss this in detail in Section 2.4.

Step 1.17 (Higher dimensional analogs of families of stable curves II) Extending the definition (1.16.3) to general base schemes turned out to be very difficult. There were two main proposals in Kollár and Shepherd-Barron (1988) and Viehweg (1995). They are equivalent over reduced base schemes; we explain this in Section 3.4. However, the two versions differ for families of surfaces with quotient singularities over Spec $\mathbb{C}[\varepsilon]$ by Altmann and Kollár (2019). We treat these topics in Sections 6.2–6.3 and 6.6.

The problem becomes even harder when we treat not just stable varieties, but stable pairs. Finding the correct definition turned out to be the longest-standing open question of the theory. An answer was developed in Kollár (2019) and we devote Chapter 7 to explaining it.

Step 1.18 (Representability of moduli functors) The question is the following. Let $p\colon X \to S$ be an arbitrary projective morphism. Can we understand all morphisms $q\colon T \to S$ such that $X \times_S T \to T$ is a family in our moduli theory?

A moduli theory $\mathbf{M}$ is *representable* if, for every projective morphism $p\colon X \to S$, there is a morphism $j\colon S^{\mathbf{M}} \to S$ with the following property:

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Given any $q: T \to S$, the pulled-back family $X \times_S T \to T$ is in $\mathbf{M}$ iff q factors uniquely as $q: T \to S^{\mathbf{M}} \to S$.

That is, $X \times_S S^{\mathbf{M}} \to S^{\mathbf{M}}$ is in $\mathbf{M}$ and $S^{\mathbf{M}}$ is universal with this property.

Representability is rarely mentioned for the moduli of curves, since it easily follows from general principles. The Flattening decomposition theorem (3.19) says that flatness is representable, and for proper, flat morphisms, being a family of stable curves is represented by an open subscheme.

Both of these become quite complicated in higher dimensions. Since flatness is only part of our assumptions, we need a different way of pulling back families. The theory of hulls and husks in Kollár (2008a) was developed for this reason, leading to the notion of generically Cartier pull-back, defined in Section 4.1. With these, representability is proved in Sections 3.5, 4.6, and 7.6 in increasing generality.

Representability also implies that being a stable family can be tested on 0-dimensional subschemes of T, that is, on spectra of Artinian rings. This is the reason why formal deformation theory is such a powerful tool: see Illusie (1971); Artin (1976); Sernesi (2006).

The previous steps form the basis of a good moduli theory. Once we have them, it is quite straightforward to construct the corresponding moduli space.

Step 1.19 (Two moduli spaces) Let C be a stable curve of genus $g \geq 2$. Then rK_C is very ample for $r \geq 3$, and any basis of its global sections gives an embedding $C \hookrightarrow \mathbb{P}^{r(2g-2)-g}$. Thus all stable curves of genus g appear in the Chow variety or Hilbert scheme of $\mathbb{P}^{r(2g-2)-g}$. Representability (1.18) then implies that we get a moduli space of all r-canonically embedded stable curves

$$\mathrm{EmbStab}_g \subset \mathrm{Hilb}(\mathbb{P}^{r(2g-2)-g}). \tag{1.19.1}$$

For a fixed C, the embedding $C \hookrightarrow \mathbb{P}^{r(2g-2)-g}$ gives an orbit of $\mathbf{Aut}(\mathbb{P}^{r(2g-2)-g})$, thus we should get the moduli space as

$$\overline{\mathrm{M}}_g = \mathrm{EmbStab}_g \,/\, \mathbf{Aut}(\mathbb{P}^{r(2g-2)-g}). \tag{1.19.2}$$

Starting with Mumford (1965) and Matsusaka (1964), much effort was devoted to understanding quotients like (1.19.2). Already for curves the method of Mumford (1965) is quite subtle; generalizations to surfaces in Gieseker (1977) and to higher dimensions in Viehweg (1995) are quite hard. For surfaces and in higher dimensions, these approaches handle only the interior of the moduli space (where we have only canonical singularities). When GIT works, it automatically gives a quasi-projective moduli space, but Wang and Xu (2014) suggest that GIT methods do not work for the whole moduli space.

It turns out to be much easier to obtain quotients that are algebraic spaces. The general quotient theorems of Kollár (1997) and Keel and Mori (1997) take care of this question completely; see Section 8.6 for details.

The same approach works in all dimensions. We fix $r > 0$ such that rK_X is very ample, and the rest of the proof works without changes.

For curves any $r \geq 3$ works, but, starting with surfaces, a uniform choice of r is no longer possible. The strongest results say that if we fix the dimension n and the volume v (10.31), then there is an $r = r(n, v)$ such that rK_X is very ample. We discuss this in (1.21).

Once we have our moduli spaces, we start to investigate their properties. We should not expect to get moduli spaces that are as nice as those for curves. For instance, even for smooth surfaces with ample canonical class, the moduli spaces can have arbitrarily complicated singularities and scheme structures (Vakil, 2006). Nonetheless, we have two types of basic positive results.

Step 1.20 (Separatedness and properness) The valuative criteria of separatedness and properness translate to functors as follows.

We start with a smooth curve B, an open subset $B^\circ \subset B$, and a stable family $\pi^\circ : X^\circ \to B^\circ$.

1.20.1 (Separatedness) There is at most one stable extension to

$$\begin{tikzcd} X^\circ \arrow[r, hook] \arrow[d, "\pi^\circ"] & X \arrow[d, "\pi"] \\ B^\circ \arrow[r, hook] & B. \end{tikzcd}$$

We obtain a similar translation of properness, but here we have to pay attention to the difference between coarse and fine moduli spaces.

1.20.2 (Valuative-properness) There is a finite surjection $p: A \to B$ such that there is a unique stable extension

$$\begin{tikzcd} X^\circ \times_B A \arrow[r, hook] \arrow[d, "\pi_A^\circ"] & X_A \arrow[d, "\pi_A"] \\ B^\circ \times_B A \arrow[r, hook] & A. \end{tikzcd}$$

Thus the valuative criterion of properness is exactly the general version of the Stable extension theorem (1.15.1).

Step 1.21 (Discrete invariants, boundedness, and projectivity) The most important discrete invariant of a smooth projective curve C is its genus. The

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genus is unchanged under smooth deformations, and all smooth curves with the same genus form a single family M_g. Thus, in effect, the genus is the only discrete invariant of a smooth projective curve; it completely determines the other ones, like the Euler characteristic $\chi(C, \mathscr{O}_C) = 1 - g$, or the Hilbert polynomial $\chi(C, \mathscr{O}_C(mK_C)) = (2g - 2)m + (1 - g)$.

In a similar manner, we would like to find discrete invariants of (locally) stable varieties that are unchanged by (locally) stable deformations.

The basic such invariant is the Hilbert "polynomial" of K_X. We have to keep in mind that K_X need not be Cartier. Therefore, $m \mapsto \chi(X, \mathscr{O}_X(mK_X))$ is not a polynomial, rather a polynomial with periodic coefficients.

For stable varieties the most important invariant is $\mathrm{vol}(X) := (K_X^n)$ (where $n = \dim X$), called the *volume* (10.31) of X. This is also the leading coefficient of the Hilbert polynomial (times $n!$). The volume is positive, but it is frequently a rational number since K_X is only $\mathbb{Q}$-Cartier; it can be quite small: see Alexeev and Liu (2019a); Esser et al. (2021).

For $m = 0$ we get the Euler characteristic $\chi(X, \mathscr{O}_X)$, but it turns out that the individual groups $h^i(X, \mathscr{O}_X)$ are also deformation invariants by Kollár and Kovács (2010); see Section 2.5.

Next we would like to show that all stable varieties with fixed volume can be "parametrized" by a scheme of finite type; this is called *boundedness*. To state it, let $\mathcal{SV}^{\mathrm{set}}(n, v)$ denote the set of all stable varieties of dimension n and volume v. There are three, roughly equivalent versions.

- There is an $m = m(n, v)$ such that mK_X is very ample for $X \in \mathcal{SV}^{\mathrm{set}}(n, v)$.
- There is a $D = D(n, v)$ such that every $X \in \mathcal{SV}^{\mathrm{set}}(n, v)$ is isomorphic to a subvariety of $\mathbb{P}^D$ of degree $\leq D$.
- There is a morphism $\pi\colon U \to S$ of schemes of finite type such that every $X \in \mathcal{SV}^{\mathrm{set}}(n, v)$ is isomorphic to a fiber of π.

Proving these three turned out to be extremely difficult. For smooth varieties this was solved by Matsusaka (1972), for stable surfaces by Alexeev (1993), and the general stable case is settled in Hacon et al. (2018).

Our moduli spaces satisfy the valuative criterion of properness. Together with boundedness this implies that our moduli spaces are proper.

Once we have a proper moduli space, one would like to prove that it is projective. For surfaces this was done in Kollár (1990), and extended to higher dimensions in Fujino (2018) and Kovács and Patakfalvi (2017).

These last two topics each deserve a detailed treatment of their own; we make only a few more comments in (6.5).

1.22 (Moduli for varieties of nongeneral type) The moduli theory of varieties of nongeneral type is quite complicated.

A general problem, illustrated by abelian, elliptic, and K3 surfaces, is that a typical deformation of such an algebraic surface over $\mathbb{C}$ is a nonalgebraic complex analytic surface. Thus any algebraic theory captures only a small part of the full analytic deformation theory.

The moduli question for analytic surfaces has been studied, especially for complex tori and K3 surfaces. In both cases it seems that one needs to add some extra structure (for instance, fixing a basis in some topological homology group) in order to get a sensible moduli space. (As an example of what could happen, note that the three-dimensional space of Kummer surfaces is dense in the 20-dimensional space of all K3 surfaces.)

Even if one restricts to the algebraic case, compactifying the moduli space seems rather difficult. Detailed studies of abelian varieties and K3 surfaces show that there are many different compactifications depending on additional choices: see Kempf et al. (1973) and Ash et al. (1975).

It is only with the works of Alexeev (2002) that a geometrically meaningful compactification of the moduli of principally polarized abelian varieties became available. This relies on the observation that a pair (A, Θ) consisting of a principally polarized abelian variety A and its theta divisor Θ behaves as if it were a variety of general type.

A moduli theory for K-stable Fano varieties was developed quite recently; see Xu (2020) for an overview.

Definition 1.23 (Canonical class, bundle, and sheaf I) Let X be a smooth variety over a field k. As in Shafarevich (1974, III.6.3) or Hartshorne (1977, p.180), the *canonical line bundle* of X is $\omega_X := \wedge^{\dim X}\Omega_{X/k}$. Any divisor D such that $\mathscr{O}_X(D) \simeq \omega_X$ is called a *canonical divisor*. Their linear equivalence class is called the *canonical class*, denoted by K_X. (Both books tacitly assume that k is algebraically closed. The definition, however, works over any field k, as long as X is smooth over k.)

Let X be a normal variety over a perfect field k. Let $j\colon X^{\mathrm{sm}} \hookrightarrow X$ be the inclusion of the locus of smooth points. Then $X \setminus X^{\mathrm{sm}}$ has codimension ≥ 2, therefore, restriction from X to X^{sm} is a bijection on Weil divisors and on linear equivalence classes of Weil divisors. Thus there is a unique linear equivalence class K_X of Weil divisors on X such that $K_X|_{X^{\mathrm{sm}}} = K_{X^{\mathrm{sm}}}$. It is called the *canonical class* of X. The divisors in K_X need not be Cartier.

The push-forward $\omega_X := j_*\omega_{X^{\mathrm{sm}}}$ is a rank 1 coherent sheaf on X, called the *canonical sheaf* of X. The canonical sheaf ω_X agrees with the *dualizing sheaf* ω_X° as defined in Hartshorne (1977, p.241). (Note that Hartshorne (1977) defines the dualizing sheaf only if X is proper. In general, take a normal compactification $\bar{X} \supset X$ and use $\omega_{\bar{X}}^\circ|_X$ instead. For more details, see Kollár and Mori

(1998, sec.5.5), Hartshorne (1966), or Conrad (2000).) Note that ω_X satisfies Serre's condition S_2 (10.3.2), but frequently it is not locally free.

More generally, as long as X itself is normal or S_2, and ω_X is locally free outside a codimension ≥ 2 subset of X, we can work with ω_X and K_X as in the normal case. Then

$$\mathscr{O}_X(mK_X) \simeq \omega_X^{[m]} := (\omega_X^{\otimes m})^{**}. \tag{1.23.1}$$

We use this mostly when X has at worst nodes at codimension 1 points (11.35).

1.3 From Smooth Curves to Canonical Models

Here we discuss the considerations that led to Principle 1.14.1.

In the theory of curves, the basic objects are smooth projective curves. We frequently study any other curve by relating it to smooth projective curves. As a close analog, in higher dimensions, the moduli functor of smooth varieties is

$$\mathcal{S}mooth(S) := \left\{ \begin{array}{c} \text{Smooth, proper families } X \to S, \\ \text{modulo isomorphisms over } S. \end{array} \right\}$$

This, however, gives a rather badly behaved and mostly useless moduli functor already for surfaces. First of all, it is very nonseparated.

1.24 (Nonseparatedness of the moduli of smooth surfaces of general type) We construct two smooth families of projective surfaces $f_i\colon X^i \to B$ over a pointed smooth curve $b \in B$ such that

(1.24.1) all the fibers are smooth, projective surfaces of general type,

(1.24.2) $X^1 \to B$ and $X^2 \to B$ are isomorphic over $B \setminus \{b\}$,

(1.24.3) the fibers X^1_b and X^2_b are *not* isomorphic.

As the construction shows, this type of behavior happens every time we look at deformations of a surface that contains at least three (-1)-curves.

Let $f\colon X \to B$ be a smooth family of projective surfaces over a smooth (affine) pointed curve $b \in B$. Let $C_1, C_2, C_3 \subset X$ be three sections of f, all passing through a point $x_b \in X_b$ with independent tangent directions and disjoint elsewhere.

Set $X^1 := B_{C_1}B_{C_2}B_{C_3}X$, where we first blow up $C_3 \subset X$, then the birational transform of C_2 in $B_{C_3}X$, and finally the birational transform of C_1 in $B_{C_2}B_{C_3}X$. Similarly, set $X^2 := B_{C_1}B_{C_3}B_{C_2}X$. Since the C_i are sections, all these blow-ups give smooth families of projective surfaces over B.

Over $B \setminus \{b\}$ the curves C_i are disjoint, thus X^1 and X^2 are both isomorphic to $B_{C_1+C_2+C_3}X$, the blow-up of $C_1 + C_2 + C_3 \subset X$.

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We claim that, by contrast, the fibers of X_b^1 and X_b^2 are not isomorphic to each other for a general choice of the C_i.

To see this, choose local analytic coordinates t at $b \in B$ and (x, y, t) at $x_b \in X$. The curves C_i are defined by equation

$$C_i = (x - a_i t - (\text{higher terms}) = y - b_i t - (\text{higher terms}) = 0).$$

The blow-up $B_{C_i} X$ is given by

$$B_{C_i} X = \big(u_i(x - a_i t - (\text{higher terms})) = v_i(y - b_i t - (\text{higher terms}))\big) \subset X \times \mathbb{P}^1_{u_i v_i}.$$

On the fiber over b, these give the same blow-up

$$B_{x_b}(X_b) = (ux = vy) \subset X_b \times \mathbb{P}^1_{uv}.$$

Thus we see that the birational transform of C_j intersects the central fiber $(B_{C_i} X)_b = B_{x_b}(X_b)$ at the point

$$\frac{v}{u} = \frac{a_j - a_i}{b_j - b_i} \in \{x_b\} \times \mathbb{P}^1_{uv}.$$

The fibers $(B_{C_2} B_{C_3} X)_b$ and $(B_{C_3} B_{C_2} X)_b$ are isomorphic to each other since they are obtained from $B_{x_b}(X_b)$ by blowing up the same point

$$\frac{v}{u} = \frac{a_2 - a_3}{b_2 - b_3} \quad \text{resp.} \quad \frac{v}{u} = \frac{a_3 - a_2}{b_3 - b_2}.$$

When we next blow up the birational transform of C_1 on $(B_{C_2} B_{C_3} X)_b$ (resp. on $(B_{C_3} B_{C_2} X)_b$), this gives the blow-up of the point

$$\frac{a_1 - a_3}{b_1 - b_3} \quad \text{resp.} \quad \frac{a_1 - a_2}{b_1 - b_2}, \tag{1.24.4}$$

and these are different, unless $C_1 + C_2 + C_3$ is locally planar at x_b.

So far we have seen that the identity $X_b = X_b$ does not extend to an isomorphism between the fibers X_b^1 and X_b^2. If X_b is of general type, then $\operatorname{Aut} X_b$ is finite, hence, to ensure that X_b^1 and X_b^2 are not isomorphic, we need to avoid finitely many other possible coincidences in (1.24.4).

The main reason, however, why we do not study the moduli functor of smooth varieties up to isomorphism is that, in dimension two, smooth projective surfaces do not form the *smallest* basic class. Given any smooth projective surface S, one can blow up any set of points $Z \subset S$ to get another smooth projective surface $B_Z S$, which is very similar to S. Therefore, the basic object is not a single smooth, projective surface, but a whole *birational equivalence class* of smooth, projective surfaces. Thus it would be better to work with

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smooth, proper families $X \to S$ modulo birational equivalence over S. That is, with the moduli functor

$$\mathcal{G}en\mathcal{T}ype_{bir}(S) := \left\{ \begin{array}{c} \text{Smooth, proper families } X \to S, \\ \text{every fiber is of general type,} \\ \text{modulo birational equivalences over } S. \end{array} \right\} \qquad (1.24.5)$$

In essence this is what we end up doing – see (1.36) – but it is very cumbersome to deal with birational equivalence over a base scheme. Nonetheless, working with birational equivalence classes leads to a separated moduli functor.

Proposition 1.25 *Let $f_i\colon X^i \to B$ be two smooth families of projective varieties over a smooth curve B. Assume that the generic fibers $X^1_{k(B)}$ and $X^2_{k(B)}$ are birational, and the pluricanonical system $|mK_{X^1_{k(B)}}|$ is nonempty for some $m > 0$. Then, for every $b \in B$, the fibers X^1_b and X^2_b are birational.*

Proof Pick a birational map $\phi\colon X^1_{k(B)} \dashrightarrow X^2_{k(B)}$, and let $\Gamma \subset X^1 \times_B X^2$ be the closure of the graph of ϕ. Let $Y \to \Gamma$ be the normalization with projections $p_i\colon Y \to X^i$. Note that both of the p_i are open embeddings on $Y \setminus (\mathrm{Ex}(p_1) \cup \mathrm{Ex}(p_2))$. Thus if we prove that neither $p_1(\mathrm{Ex}(p_1) \cup \mathrm{Ex}(p_2))$ nor $p_2(\mathrm{Ex}(p_1) \cup \mathrm{Ex}(p_2))$ contains a fiber of f_1 or f_2, then $p_2 \circ p_1^{-1}\colon X^1 \dashrightarrow X^2$ restricts to a birational map $X^1_b \dashrightarrow X^2_b$ for every $b \in B$. (Thus the fiber Y_b contains an irreducible component that is the graph of the birational map $X^1_b \dashrightarrow X^2_b$, but it may have other components too; see (1.27.1) for such an example.)

We use the canonical class to compare $\mathrm{Ex}(p_1)$ and $\mathrm{Ex}(p_2)$. Since the X^i are smooth,

$$K_Y \sim p_i^* K_{X^i} + E_i, \quad \text{where } E_i \geq 0 \text{ and } \operatorname{Supp} E_i = \mathrm{Ex}(p_i). \qquad (1.25.1)$$

We may assume that B is affine and let $\mathrm{Bs}\,|mK_{X^i}|$ denote the set-theoretic base locus. By assumption, $|mK_{X^i}|$ is not empty. Since B is affine, $\mathrm{Bs}\,|mK_{X^i}|$ does not contain any of the fibers of f_i.

Every section of $\mathscr{O}_Y(mK_Y)$ pulls back from X^i, thus

$$\mathrm{Bs}\,|mK_Y| = p_i^{-1}\big(\mathrm{Bs}\,|mK_{X^i}|\big) + \operatorname{Supp} E_i.$$

Comparing these for $i = 1, 2$, we conclude that

$$p_1^{-1}\big(\mathrm{Bs}\,|mK_{X^1}|\big) + \operatorname{Supp} E_1 = p_2^{-1}\big(\mathrm{Bs}\,|mK_{X^2}|\big) + \operatorname{Supp} E_2.$$

Therefore, $p_1(\operatorname{Supp} E_2) \subset p_1(\operatorname{Supp} E_1) + \mathrm{Bs}\,|mK_{X^1}|$.

Since E_1 is p_1-exceptional, $p_1(E_1)$ has codimension ≥ 2 in X^1, hence it does not contain any of the fibers of f_1. We saw that $\mathrm{Bs}\,|mK_{X^1}|$ does not contain any of the fibers either. Thus $p_1(\mathrm{Ex}(p_1) \cup \mathrm{Ex}(p_2))$ does not contain any of the fibers, and similarly for $p_2(\mathrm{Ex}(p_1) \cup \mathrm{Ex}(p_2))$. □

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Remark 1.26 By Matsusaka and Mumford (1964) and Kontsevich and Tschinkel (2019), the conclusion holds even if the pluricanonical systems are empty.

The proof focuses on the role of the canonical class. It is worthwhile to go back and check that the proof works if the X^i are normal, as long as (1.25.1) holds; the latter is essentially the definition of terminal singularities. It is precisely the property (1.25.1) and its closely related variants that lead us to the correct class of singular varieties for moduli purposes.

Since it is much harder to work with a whole equivalence class, it would be desirable to find a particularly nice surface in every birational equivalence class. This is achieved by the theory of minimal models of algebraic surfaces. By a result of Enriques (Barth et al., 1984, III.4.5), every birational equivalence class of surfaces **S** contains a unique smooth projective surface whose canonical class is nef, except when **S** contains a ruled surface $C \times \mathbb{P}^1$ for some curve C. This unique surface is called the *minimal model* of **S**.

It would seem at first sight that (1.25) implies that the moduli functor of minimal models is separated. There are, however, quite subtle problems.

1.27 (Families of minimal models) Let Y be a projective 3-fold whose only singularities are ordinary nodes. Take a general pencil and blow up its base locus to get $f\colon X \to \mathbb{P}^1$. The general fiber is a smooth surface. At the nodes, in local coordinates we can write it as

$$\begin{tikzcd} (xy + z^2 - t^2 = (\text{higher terms})) \arrow[r, hook] \arrow[d, "f"'] & \mathbb{A}^4_{xyzt} \arrow[d] \\ \mathbb{A}^1_t \arrow[r, equal] & \mathbb{A}^1_t . \end{tikzcd}$$

By the Morse lemma, with a suitable analytic coordinate change we can eliminate the higher terms (10.43). Then we can blow up either of the the 2-planes $(x = z \pm t = 0)$ to get $\pi^\pm\colon X^\pm \to X$.

By explicit computation as in (10.45), we get smooth morphisms $f^\pm\colon X^\pm \to \mathbb{A}^1$, and the fiber over the origin $X_0^\pm$ is the blow-up of X_0 at the origin. However, the composite map $X^+ \to X \dashrightarrow X^-$ is not an isomorphism. Also, the exceptional set of $\pi^\pm$ is a smooth rational curve $C^\pm \subset X^\pm$.

To get a concrete example, start with a general sextic hypersurface $Y \subset \mathbb{P}^4$ that contains a 2-plane P. Let $P + Q$ be a general hyperplane section containing P. Blow up the birational transforms of P and Q in X to get $X^\pm \to X$.

1.27.1 (Nonseparatedness in the moduli of minimal models) We get two projective morphisms $f^\pm\colon X^\pm \to \mathbb{P}^1$ and a finite set $B \subset \mathbb{P}^1$ such that

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(a) general fibers are smooth, canonical models,
(b) the $X^{\pm}$ are isomorphic over $\mathbb{P}^1 \setminus B$,
(c) the fibers X_b^+ and X_b^- are isomorphic minimal models for $b \in B$, but
(d) $X^+ \to \mathbb{P}^1$ and $X^- \to \mathbb{P}^1$ are *not* isomorphic to each other. □

Starting with a general sextic hypersurface $Y \subset \mathbb{CP}^4$ that has a single node, and using that every divisor on Y is Cartier by Cheltsov (2010), gives the next example.

1.27.2 (Nonprojective families of projective surfaces) We get two smooth, compact, complex manifolds $X^{\pm}$ and morphisms $f^{\pm}: X^{\pm} \to \mathbb{P}^1$ such that every fiber is a projective minimal model, yet the $X^{\pm}$ are not projective.

Proof If $X^{\pm}$ is projective, let $S^{\pm}$ be an ample divisor. We claim that $S := f^{\pm}(S^{\pm})$ is not Cartier at the node. Indeed, since $f^{\pm}$ has no exceptional divisors, we must have $S^{\pm} = (f^{\pm})^*(S)$. This is impossible since $(S^{\pm} \cdot C^{\pm}) > 0$, but $(S^{\pm} \cdot (f^{\pm})^*(S)) = 0$. Thus, if every divisor on Y is Cartier, then the $X^{\pm}$ cannot be projective. □

All such problems go away when the canonical class is ample.

Proposition 1.28 *Let $f_i: X^i \to B$ be two smooth families of projective varieties over a smooth curve B. Assume that the canonical classes K_{X^i} are f_i-ample. Let $\phi: X^1_{k(B)} \simeq X^2_{k(B)}$ be an isomorphism of the generic fibers.*

Then ϕ extends to an isomorphism $\Phi: X^1 \simeq X^2$.

Proof Let $\Gamma \subset X^1 \times_B X^2$ be the closure of the graph of ϕ. Let $Y \to \Gamma$ be the normalization, with projections $p_i: Y \to X^i$ and $f: Y \to B$. As in (1.25), we use the canonical class to compare the X^i. Since the X^i are smooth,

$$K_Y \sim p_i^* K_{X^i} + E_i \quad \text{where } E_i \text{ is effective and } p_i\text{-exceptional.} \tag{1.28.1}$$

Since $(p_i)_* \mathscr{O}_Y(mE_i) = \mathscr{O}_{X^i}$ for every $m \geq 0$, we get that

$$\begin{aligned}(f_i)_* \mathscr{O}_{X^i}(mK_{X^i}) &= (f_i)_*(p_i)_* \mathscr{O}_Y(mp_i^* K_{X^i}) \\ &= (f_i)_*(p_i)_* \mathscr{O}_Y(mp_i^* K_{X^i} + mE_i) = (f_i)_*(p_i)_* \mathscr{O}_Y(mK_Y) = f_* \mathscr{O}_Y(mK_Y).\end{aligned}$$

Since the K_{X^i} are f_i-ample, $X^i = \operatorname{Proj}_B \oplus_{m \geq 0} (f_i)_* \mathscr{O}_{X^i}(mK_{X^i})$. Putting these together, we get the isomorphism

$$\begin{aligned}\Phi: X^1 &\simeq \operatorname{Proj}_B \oplus_{m \geq 0} (f_1)_* \mathscr{O}_{X^1}(mK_{X^1}) \simeq \operatorname{Proj}_B \oplus_{m \geq 0} f_* \mathscr{O}_Y(mK_Y) \\ &\simeq \operatorname{Proj}_B \oplus_{m \geq 0} (f_2)_* \mathscr{O}_{X^2}(mK_{X^2}) \simeq X^2. \quad \square\end{aligned}$$

Remark 1.29 As in (1.26), it is again worthwhile to investigate the precise assumptions behind the proof. The smoothness of the X^i is used only through the pull-back formula (1.28.1), which is weaker than (1.25.1).

If (1.28.1) holds, then, even if the K_{X^i} are not f_i-ample, we obtain an isomorphism

$$\operatorname{Proj}_B \oplus_{m\geq 0} (f_1)_* \mathscr{O}_{X^1}(mK_{X^1}) \simeq \operatorname{Proj}_B \oplus_{m\geq 0} (f_2)_* \mathscr{O}_{X^2}(mK_{X^2}). \tag{1.29.1}$$

Thus it is of interest to study objects as in (1.29.1) in general.

Let us start with the absolute case, when X is a smooth projective variety over a field k. Its *canonical ring* is the graded ring

$$R(X, K_X) := \oplus_{m\geq 0} H^0(X, \mathscr{O}_X(mK_X)). \tag{1.29.2}$$

In some cases the canonical ring tells us very little about X. For instance, if X is rational or Fano then $R(X, K_X)$ is the base field k; if X is Calabi–Yau then $R(X, K_X)$ is isomorphic to the polynomial ring $k[t]$. One should thus focus on the cases when the canonical ring is large. The following notion is due to Iitaka (1971). See (Lazarsfeld, 2004, sec.2.1.C) for a detailed treatment.

Definition 1.30 Let X be a smooth proper variety. Its *Kodaira dimension*, denoted by $\kappa(X)$, is the dimension of the image of $|mK_X|: X \dashrightarrow \mathbb{P}^{\dim |mK_X|}$ for m sufficiently large and divisible. One can also define $\kappa(X)$ by the property: the limsup of $h^0(X, \mathscr{O}_X(mK_X))/m^{\kappa(X)}$ is positive and finite. We set $\kappa(X) = -\infty$ if $|mK_X|$ is empty for all $m > 0$.

If $\kappa(X) = \dim X$, we say that X is of *general type*. In this case $|mK_X|$ defines a birational map for all $m \gg 1$, and the limit of $h^0(X, \mathscr{O}_X(mK_X))/m^{\dim(X)}$ is positive and finite. See (3.34) for more on $h^0(X, \mathscr{O}_X(mK_X))$.

Definition 1.31 (Canonical models) Let X be a smooth projective variety of general type over a field k such that its canonical ring $R(X, K_X)$ (1.29.2) is finitely generated. We define its *canonical model* as

$$X^c := \operatorname{Proj}_k R(X, K_X).$$

If Y is a smooth projective variety birational to X, then Y^c is isomorphic to X^c. Thus X^c is also the canonical model of the whole birational equivalence class containing X. (Taking Proj of a nonfinitely generated ring may result in a quite complicated scheme. It does not seem profitable to contemplate what would happen in our case.)

The canonical ring $R(X, K_X)$ is always finitely generated in characteristic 0 (11.28), thus X^c is an irreducible, projective variety. On the other hand, X^c can be singular. Originally this was viewed as a major obstacle, but now it seems only a technical problem.

We can now give an abstract characterization of canonical models.

Theorem 1.32 *A normal proper variety Y is a canonical model iff*

(1.32.1) *K_Y is $\mathbb{Q}$-Cartier (p.xv) and ample, and*

(1.32.2) *there is a resolution $f\colon X \to Y$ (p.xv) and an effective, f-exceptional $\mathbb{Q}$-divisor E such that $K_X \sim_{\mathbb{Q}} f^*K_Y + E$.*

Proof For now we prove only the "if" part. For the converse, see Reid (1980) or (Kollár, 2013b, 1.15) or (11.62.2).

Choose m_0 such that m_0K_X is Cartier, then so is m_0E. Note that for any $r > 0$, $f_*\mathscr{O}_X(rm_0E) = \mathscr{O}_Y$ since E is effective and f-exceptional. Thus

$$\begin{aligned} H^0(X, \mathscr{O}_X(rm_0K_X)) &= H^0(Y, f_*\mathscr{O}_X(rm_0K_X)) \\ &= H^0(Y, \mathscr{O}_Y(rm_0K_Y) \otimes f_*\mathscr{O}_X(rm_0E)) = H^0(Y, \mathscr{O}_Y(rm_0K_Y)). \end{aligned}$$

Therefore,

$$\begin{aligned} \operatorname{Proj} \oplus_m H^0(X, \mathscr{O}_X(mK_X)) &= \operatorname{Proj} \oplus_r H^0(X, \mathscr{O}_X(rm_0K_X)) \\ &= \operatorname{Proj} \oplus_r H^0(Y, \mathscr{O}_Y(rm_0K_Y)) = Y. \qquad \square \end{aligned}$$

This makes it possible to give a local definition of the singularities that occur on canonical models, using $\mathbb{Q}$-linear equivalence $\sim_{\mathbb{Q}}$ as in (p.xv).

Definition 1.33 A normal variety Y has *canonical singularities* if

(1.33.1) K_Y is $\mathbb{Q}$-Cartier, and

(1.33.2) there is a resolution $f\colon X \to Y$ and an effective, f-exceptional $\mathbb{Q}$-divisor E such that $K_X \sim_{\mathbb{Q}} f^*K_Y + E$.

It is easy to show that this is independent of the resolution $f\colon X \to Y$; see (Kollár, 2013b, sec.2.12). (One can define canonical singularities without resolutions, see (Kollár, 2013b, sec.2.1) or Luo (1987).)

Equivalently, Y has canonical singularities iff every point $y \in Y$ has an étale neighborhood which is an open subset on some canonical model.

A complete list of canonical singularities is known in dimension 2 and almost known in dimension 3; see Reid (1980). The following examples are useful to keep in mind:

(1.33.3) $(x_1x_2 + f(x_3, \dots, x_n) = 0)$ is canonical iff f is not identically 0.

(1.33.4) The quotient singularity $\mathbb{A}^n/\frac{1}{m}(1, m-1, a_3, \dots, a_n)$ (1.40.2) is canonical for $n \geq 3$ if $\gcd(m, a_3, \dots, a_n) = 1$. Its canonical class is Cartier iff $m \mid a_3 + \cdots + a_n$.

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(1.33.5) The cone $C_d(\mathbb{P}^n)$ over the d-uple Veronese embedding has a canonical singularity iff $d \le n+1$. Its canonical class is Cartier iff $d|n+1$. (See (2.35) or (Kollár, 2013b, 3.1) for the case of general cones.)

Warning 1.34 ($\mathbb{Q}$-Cartier condition) While (1.33.1) may seem like a small technical condition, in many cases it turns out to be extremely important.

First of all, one cannot pull back arbitrary divisors, so (1.33.2) does not even make sense if K_Y is not $\mathbb{Q}$-Cartier. This is a substantial problem starting with dimension 3; cf. (11.57) and (11.58).

The issue becomes more serious for families of varieties. Unexpected jumps of the Kodaira dimension happen precisely when the canonical class of the total space is not $\mathbb{Q}$-Cartier; see (1.43–1.46).

The most difficult aspects appear for nonnormal varieties. The gluing theory of (Kollár, 2013b, chap.5) is almost entirely devoted to proving that in some cases the canonical divisor is $\mathbb{Q}$-Cartier; see (11.38) for a key consequence.

Definition 1.35 The *moduli functor of canonical models* is

$$\mathcal{C}an\mathcal{M}od(S) := \left\{ \begin{array}{c} \text{Flat, proper families } X \to S, \\ \text{every fiber is a canonical model,} \\ \text{modulo isomorphisms over } S. \end{array} \right\} \tag{1.35.1}$$

This is an improved version of $GenType_{bir}$ defined in (1.24.5).

Warning. In retrospect, it seems only by luck that this definition gives the correct functor. See (1.16.2), the examples in (1.42–1.47), and (2.8).

1.36 (From $\mathcal{G}en\mathcal{T}ype$ to $\mathcal{C}an\mathcal{M}od$) Let $p\colon Y \to S$ be a smooth, projective morphism of varieties over a field of characteristic 0. Assume that S is reduced and the fibers Y_s are of general type. By (1.37), we get the flat family of canonical models $p^c\colon Y^c \to S$. This gives a natural transformation T_{CanMod} which, for any reduced scheme S gives a map of sets

$$T_{\mathrm{CanMod}}(S)\colon \mathcal{G}en\mathcal{T}ype_{bir}(S) \to \mathcal{C}an\mathcal{M}od(S). \tag{1.36.1}$$

By definition, if $X_i \to S$ are two smooth, proper families of varieties of general type such that $T_{\mathrm{CanMod}}(S)(X_1) = T_{\mathrm{CanMod}}(S)(X_2)$, then X_1 and X_2 are birational, thus $T_{\mathrm{CanMod}}(S)$ is injective. It is not surjective, but we have the following partial surjectivity statement.

Claim 1.36.2 Let $Y \to S$ be a flat family of canonical models. Then there is a dense open subset $S^\circ \subset S$ and a smooth, proper family of varieties of general type $Y^\circ \to S^\circ$ such that $T_{\mathrm{CanMod}}(S^\circ)(Y^\circ) = [Y|_{S^\circ}]$. □

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Theorem 1.37 *Let $p : Y \to S$ be a flat, projective morphism, whose fibers are of general type and have canonical singularities. Assume that S is reduced. Then the canonical models of the fibers form a flat, projective morphism $p^{\mathrm{stab}} : Y^{\mathrm{stab}} \to S$, and the natural map $Y \dashrightarrow Y^{\mathrm{stab}}$ is fiber-wise birational.*

For surfaces, this goes back to Kodaira and Spencer (1958); the 3-fold case is proved in Kollár and Mori (1992, 12.5.1). See (2.48) for a proof using MMP. The complex analytic case is in Kollár (2021a).

The theorem implies the deformation invariance of plurigenera as in (5.1.3). Conversely, the deformation invariance of plurigenera, due to Siu (1998) and (Nakayama, 2004, chap.VI), shows that, if the Y_s have canonical models, then they form a flat family $p^{\mathrm{stab}} : Y^{\mathrm{stab}} \to S$.

The case when S is nonreduced is open.

1.4 From Stable Curves to Stable Varieties

Next we discuss the reasoning behind Step 1.15.

Let C be a nodal curve with normalized irreducible components C_i. We frequently view C as an object assembled from the pieces C_i. Note that the pull-back of K_C to C_i is not K_{C_i}, rather $K_{C_i} + P_i$, where $P_i \subset C_i$ are the preimages of the nodes of C.

Similarly, if X is a scheme with simple normal crossing singularities (p.xvi) and normalized irreducible components X_i, then the pull-back of K_X to X_i is not K_{X_i}, rather $K_{X_i} + D_i$, where $D_i \subset X_i$ is the divisorial part of the preimage of $\operatorname{Sing} X$ on X_i.

This suggests that we should develop a theory of "canonical models" where the role of the canonical class is played by a divisor of the form $K_X + D$, where D is a simple normal crossing divisor (p.xvi).

Definition 1.38 (Canonical models of pairs) Let (X, D) be a projective snc pair (p.xvi). We define the *canonical ring*[1] of the pair (X, D) as

$$R(X, K_X + D) := \oplus_{m\geq 0} H^0(X, \mathscr{O}_X(mK_X + mD)).$$

It is conjectured (but known only for $\dim X \leq 4$ in characteristic 0) that the canonical ring of a pair (X, D) is finitely generated. If this holds, then $X^{\mathrm{c}} := \operatorname{Proj}_k R(X, K_X + D)$ is a normal projective variety. We say that (X, D) is of *general type* if the natural map $\pi\colon X \dashrightarrow X^{\mathrm{c}}$ is birational, and then $(X^{\mathrm{c}}, D^{\mathrm{c}} := \pi_* D)$ is called the *canonical model* of (X, D).

The proof of the "if" part of the following goes exactly as in (1.32).

[1] *Log* canonical ring and *log* general type is also frequently used; see (1.39.3).

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Theorem 1.39 *A pair (Y, B), consisting of a proper normal variety Y and an effective, reduced, Weil divisor B, is the canonical model of a simple normal crossing pair iff*

(1.39.1) *$K_Y + B$ is $\mathbb{Q}$-Cartier, ample, and*

(1.39.2) *there is a resolution $f\colon X \to Y$, an effective, reduced, simple normal crossing divisor $D \subset X$ such that $f(D) = B$, and an effective, f-exceptional $\mathbb{Q}$-divisor E such that $K_X + D \sim_{\mathbb{Q}} f^*(K_Y + B) + E$.*

Warning 1.39.3 If $B = 0$, it can happen that $(X, 0)$ is the canonical model of a pair, but X is not a canonical model (1.32). To see this, choose a resolution $f\colon X \to Y$ and let $E_i \subset X$ be the f-exceptional divisors. Although $B = 0$, in (1.39.2) we can still take $D = \sum E_i$. Thus (1.39.2) can be rewritten as

$$K_X \sim f^* K_Y + E - \textstyle\sum E_i.$$

This looks like (1.32.2), but $E - \sum E_i$ need not be effective; it can contain divisors with coefficients ≥ -1.

This is the source of some terminological problems. Originally $R(X, K_X + D)$ was called the "log canonical ring" and $\operatorname{Proj}_k R(X, K_X + D)$ the "log canonical model." Since the canonical ring is just the $D = 0$ special case of the "log canonical ring," it seems more convenient to drop the prefix "log." However, log canonical singularities are quite different from canonical singularities, so "log" cannot be omitted there. (See also p.xvi for other inconsistencies in the usage of "canonical model.")

As in (1.33), this can be reformulated as a definition. (For now we assume that every irreducible component of B appears in B with coefficient 1; later we also consider cases when the coefficients are rational or real.)

Definition 1.40 Let (Y, B) be a pair consisting of a normal variety Y and a reduced Weil divisor B. Then (Y, B) is *log canonical*, or has *log canonical singularities,* iff the conditions (1.39.1–2) are satisfied. We say that Y is log canonical if $(Y, \emptyset)$ is.

If (Y, B) is log canonical and B is $\mathbb{Q}$-Cartier then Y is also log canonical (11.5.1). However, if B is not $\mathbb{Q}$-Cartier, then K_Y is also not $\mathbb{Q}$-Cartier, so Y is not log canonical.

A complete list of log canonical singularities is known in dimension 2, see Section 2.2 or Kollár (2013b, sec.2.2). The following examples of log canonical singularities are useful to keep in mind:

1.40.1 (Simple normal crossing) $(\mathbb{A}^n, (x_1 \cdots x_r = 0))$ for any $r \leq n$.

1.40.2 (Quotient singularities) $\mathbb{A}^n / \frac{1}{m}(a_1, \ldots, a_n)$ denotes the quotient of $\mathbb{A}^n_{\mathbf{x}}$ by the action $x_i \mapsto \varepsilon^{a_i} x_i$ where ε is a primitive mth root of unity. The canonical class is Cartier iff $m \mid \sum a_i$. These are even log terminal.

1.40.3 (Cones) A cone $C(X)$ over a Calabi–Yau variety; see (2.35).

We are now ready to define the higher dimensional analogs of stable curves.

Definition 1.41 (Stable varieties) Let k be a field and Y a reduced, proper, pure dimensional scheme over k. Let $Y_i \to Y$ be the irreducible components of the normalization of Y, and $D_i \subset Y_i$ the divisorial part of the preimage of the nonnormal locus of Y. Then Y is *semi-log-canonical* – usually abbreviated as *slc* – or *locally stable* iff

(1.41.1) at codimension 1 points, Y is either smooth or has a node (11.35),

(1.41.2) each (Y_i, D_i) is log canonical, and

(1.41.3) K_Y is $\mathbb{Q}$-Cartier.

Y is a *stable variety* iff, in addition,

(1.41.4) Y is projective and K_Y is ample.

As we noted in (1.34), the $\mathbb{Q}$-Cartier condition for K_Y is quite hard to interpret in terms of the (Y_i, D_i). See (11.38) or the more detailed Kollár (2013b, chap.5). For now we only deal with examples where K_Y is obviously Cartier or $\mathbb{Q}$-Cartier.

1.41.5 (Note on terminology) This usage of “stable” has very little to do with the GIT notion of “stable” in Mumford (1965). They agree for curves, and originally there was hope of a close relationship in all dimensions. The two versions aimed to answer the same question, but from different viewpoints. They ended up quite different.

Jump of K^2 and of the Kodaira Dimension

We give examples of flat families of projective surfaces $\{S_t : t \in \mathbb{C}\}$ such that S_0 has quotient singularities and the S_t are smooth for general $t \neq 0$, but the self intersection of the canonical class $(K_{S_t}^2)$ jumps at $t = 0$. We also give examples where K_{S_t} is ample for $t = 0$, but not even big for $t \neq 0$. Among log canonical singularities, quotient singularities (1.40.2) are the mildest.

As we already noted in (1.34), such jumps happen when the canonical class of the total space is not $\mathbb{Q}$-Cartier.

Example 1.42 (Degree 4 surfaces in $\mathbb{P}^5$) There are two families of nondegenerate degree 4 smooth surfaces in $\mathbb{P}^5$. These were classified by Del Pezzo; see Eisenbud and Harris (1987) for a modern treatment.

One family consists of Veronese surfaces $\mathbb{P}^2 \subset \mathbb{P}^5$ embedded by $\mathcal{O}(2)$. The general member of the other family is $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^5$ embedded by $\mathcal{O}(2, 1)$, special members are embeddings of the ruled surface $\mathbb{F}_2$. The two families are

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distinct since $(K^2_{\mathbb{P}^2}) = 9$ and $(K^2_{\mathbb{P}^1\times\mathbb{P}^1}) = 8$. For both of these surfaces, a smooth hyperplane section is a degree 4 rational normal curve in $\mathbb{P}^4$.

Let $T_0 \subset \mathbb{P}^5$ denote the cone over the degree 4 rational normal curve in $\mathbb{P}^4$. The minimal resolution of T_0 is the ruled surface $p\colon \mathbb{F}_4 \to T_0$. Let $E, F \subset \mathbb{F}_4$ be the exceptional curve and the fiber of the ruling. Then $K_{\mathbb{F}_4} = -2E - 6F$ and $p^*(2K_{T_0}) = -3E - 12F$. Thus $2(K_{\mathbb{F}_4} + E) = p^*(2K_{T_0}) + E$ shows that T_0 has log canonical singularities. We also get that $(K^2_{T_0}) = 9$.

A key feature is that one can write T_0 as a limit of smooth surfaces in two distinct ways, corresponding to the two ways of writing the degree 4 rational normal curve in $\mathbb{P}^4$ as a hyperplane section of a surface; see (2.36).

From the first family, we get T_0 as the special fiber of a flat family whose general fiber is $\mathbb{P}^2$. This family is denoted by $\{T_t\colon t \in \mathbb{C}\}$. From the second family, we get T_0 as the special fiber of a flat family whose general fiber is $\mathbb{P}^1 \times \mathbb{P}^1$. This family is denoted by $\{T'_t\colon t \in \mathbb{C}\}$. Note that (K^2) is constant in the family $\{T_t\colon t \in \mathbb{C}\}$, but jumps at $t = 0$ in the family $\{T'_t\colon t \in \mathbb{C}\}$. (In general, one needs to worry about the possibility of getting embedded points at the vertex. However, by (2.36), in both cases the special fiber is indeed T_0.)

Alternatively, the degree 4 rational normal curve $(s{:}t) \mapsto (s^4{:}s^3t{:}s^2t^2{:}st^3{:}t^4)$ can be given by determinantal equations in 2 ways, giving the families

$$\begin{aligned} T'_t &= \left(\operatorname{rank}\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 + tx_5 & x_3 & x_4 \end{pmatrix} \le 1\right), \quad \text{and} \\ T_t &= \left(\operatorname{rank}\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 + tx_5 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix} \le 1\right). \end{aligned}$$

These are, however, families of rational surfaces with negative canonical class, but we are interested in stable varieties.

Next we take a suitable cyclic cover (11.24) of the two families to get similar examples with ample canonical class.

Example 1.43 (Jump of Kodaira dimension I) We give two flat families of projective surfaces S_t and S'_t such that

(1.43.1) $S_0 \simeq S'_0$ has log canonical singularities and ample canonical class,

(1.43.2) S_t is a smooth surface with ample canonical class for $t \neq 0$, and

(1.43.3) S'_t is a smooth, elliptic surface with $(K^2_{S'_t}) = 0$ for $t \neq 0$.

With T_0 as in (1.42), let $\pi_0\colon S_0 \to T_0$ be a double cover, ramified along the intersection of T_0 with a general quartic hypersurface. Note that $K_{T_0} \sim_{\mathbb{Q}} -\frac{3}{2}H$, where H is the hyperplane class. Thus, by the Hurwitz formula,

$$K_{S_0} \sim_{\mathbb{Q}} \pi_0^*(K_{T_0} + 2H) \sim_{\mathbb{Q}} \tfrac{1}{2}\pi_0^*H.$$

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So S_0 has ample canonical class and $(K_{S_0}^2) = 2$. Since π_0 is étale over the vertex of T_0, S_0 has two singular points, locally (in the analytic or étale topology) isomorphic to the singularity on T_0. Thus S_0 is a stable surface.

Both of the smoothings in (1.42) lift to smoothings of S_0.

From T_t we get a smoothing S_t where $\pi_t\colon S_t \to \mathbb{P}^2$ is a double cover, ramified along a smooth octic. Thus S_t is smooth, $K_{S_t} \sim_{\mathbb{Q}} \pi_t^*\mathscr{O}_{\mathbb{P}^2}(1)$ is ample and $(K_{S_t}^2) = 2$.

From T_t' we get a smoothing S_t' where $\pi_t'\colon S_t' \to \mathbb{P}^1 \times \mathbb{P}^1$ is a double cover, ramified along a smooth curve of bidegree $(8, 4)$. One of the families of lines on $\mathbb{P}^1 \times \mathbb{P}^1$ pulls back to an elliptic pencil on S_t' and $(K_{S_t'}^2) = 0$. Thus S_t' is not of general type for $t \neq 0$.

Example 1.44 (Jump of Kodaira dimension II) A similar pair of examples is obtained by working with triple covers ramified along a cubic hypersurface section of the surface families in (1.42). The family over T_t has ample canonical class and $(K^2) = 3$. As before, the family over T_t' is elliptic and has $(K^2) = 0$.

Example 1.45 (Jump of Kodaira dimension III) We construct a flat family of surfaces whose central fiber is the quotient of the square of the Fermat cubic curve by $\mathbb{Z}/3$:

$$S_F^* \simeq (u_1^3 = v_1^3 + w_1^3) \times (u_2^3 = v_2^3 + w_2^3)/\tfrac{1}{3}(1, 0, 0; 1, 0, 0), \tag{1.45.1}$$

thus it has Kodaira dimension 0. The general fiber is $\mathbb{P}^2$ blown up at 12 points.

In $\mathbb{P}^3$, consider two lines $L_1 = (x_0 = x_1 = 0)$ and $L_2 = (x_2 = x_3 = 0)$. The linear system $|\mathscr{O}_{\mathbb{P}^2}(2)(-L_1 - L_2)|$ is spanned by the four reducible quadrics $x_i x_j$ for $i \in \{0, 1\}$ and $j \in \{2, 3\}$. They satisfy a relation $(x_0x_2)(x_1x_3) = (x_0x_3)(x_1x_2)$. Thus we get a morphism $\pi\colon B_{L_1+L_2}\mathbb{P}^3 \to \mathbb{P}^1 \times \mathbb{P}^1$, which is a $\mathbb{P}^1$-bundle whose fibers are the birational transforms of lines that intersect both of the L_i.

Let $S \subset \mathbb{P}^3$ be a cubic surface such that $\mathbf{p} := S \cap (L_1 + L_2)$ is six distinct points. Then we get $\pi_S : B_{\mathbf{p}}S \to \mathbb{P}^1 \times \mathbb{P}^1$.

In general, none of the lines connecting two points of $\mathbf{p}$ is contained in S; in this case π_S is a finite triple cover.

At the other extreme, we have the Fermat-type surface

$$S_F := (x_0^3 + x_1^3 = x_2^3 + x_3^3) \subset \mathbb{P}^3.$$

We can factor both sides and write its equation as $m_1m_2m_3 = n_1n_2n_3$. The nine lines $L_{ij} := (m_i = n_j = 0)$ are all contained in S_F. Let $L_{ij}' \subset B_{\mathbf{p}}S_F$ denote their birational transforms. Then the self-intersections $(L_{ij}' \cdot L_{ij}')$ equal -3 and π_{S_F} contracts these nine curves L_{ij}'. Thus the Stein factorization of π_{S_F} gives a triple

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cover $S^*_F \to \mathbb{P}^1 \times \mathbb{P}^1$. Here S^*_F has nine singular points of type $\mathbb{A}^2/\frac{1}{3}(1,1)$. We see furthermore that $-3K_{S_F} \sim \sum_{ij} L_{ij}$ and $-3K_{B_{\mathbf{P}}S_F} \sim \sum_{ij} L'_{ij}$. Thus $-3K_{S^*_F} \sim 0$.

To see that the two surfaces denoted by S^*_F are isomorphic, use the map of the surface (1.45.1) to $\mathbb{P}^1 \times \mathbb{P}^1$ given by

$$(u_1{:}v_1{:}w_1) \times (u_2{:}v_2{:}w_2) \mapsto (v_1{:}w_1) \times (v_2{:}w_2),$$

and the rational map to the cubic surface is given by

$$(u_1{:}v_1{:}w_1) \times (u_2{:}v_2{:}w_2) \mapsto (u_1v_2{:}u_1w_2{:}v_1u_2{:}w_1u_2).$$

Example 1.46 (Jump of Kodaira dimension IV) The previous examples are quite typical in some sense. If S_0 is any projective rational surface with quotient singularities, then there is a flat family of surfaces $\{S_t\}$ such that S_t is a smooth rational surface for $t \neq 0$.

To see this, take a minimal resolution $S'_0 \to S_0$. Since S'_0 is a smooth rational surface, it can be obtained from a minimal smooth rational surface by blowing up points. We can deform S'_0 by moving these points into general position (and also deforming the minimal smooth rational surface if necessary). Thus we see that if S_0 is singular, then a general deformation S'_t of S'_0 is obtained by blowing up points in $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ in general position. In the second case, if we blow up at least one point, it is also a blow-up of $\mathbb{P}^2$. There are no curves with negative self-intersection on $\mathbb{P}^1 \times \mathbb{P}^1$, and on a blow-up of $\mathbb{P}^2$ at general points, every smooth rational curve with negative self-intersection is a (-1)-curve by (de Fernex, 2005, 2.4). In particular, none of the exceptional curves of $S'_0 \to S_0$ lift to S'_t.

Let H'_0 be the pull-back of a very ample Cartier divisor from S_0 whose higher cohomologies vanish. Since S'_0 is a smooth rational surface, $\mathrm{Pic}(S'_0) = H^2(S'_0, \mathbb{Z})$, so H'_0 lifts to a family of semiample Cartier divisors H'_t. As we discussed, none of the exceptional curves of $S'_0 \to S_0$ lift to S'_t for general t, so H'_t is ample for general t. As before, we get a flat deformation $\{S_t\}$ such that $S_t \simeq S'_t$ for $t \neq 0$.

Many recent constructions of surfaces of general type start with a particular rational surface S_0 with quotient singularities, and show that it has a flat deformation to a smooth surface with ample canonical class; see Lee and Park (2007); Park et al. (2009a,b). Thus such an S_0 has flat deformations of general type and also flat deformations that are rational.

Even more surprisingly, a surface with ample canonical class can have nonalgebraic deformations.

Example 1.47 (Nonalgebraic deformations) (Kollár, 2021a) We construct a projective surface X_0 with a quotient singularity, ample canonical class and two

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deformations. An algebraic one $g^{\mathrm{alg}} : X^{\mathrm{alg}} \to D$, where g^{alg} is flat, projective, and a complex analytic one $g^{\mathrm{an}} : X^{\mathrm{an}} \to \mathbb{D}$ over the *unit disc* $\mathbb{D} \subset \mathbb{C}$, where g^{an} is flat, proper such that

(1.47.3) X^{alg}_s is a smooth, algebraic, K3 surface blown up at three points for $s \neq 0$,

(1.47.4) X^{an}_s is a smooth, nonalgebraic, K3 surface blown up at three points for very general $s \in \mathbb{D}$.

Let us start with a K3 surface $Y \subset \mathbb{P}^3$ with a hyperplane section $C \subset Y$ and three points $p_i \in C$. Blow up these points to get $\pi : Z \to Y$ with exceptional curves $E = E_1 + E_2 + E_3$. Let $C_Z \subset Z$ be the birational transform of C and $H = \pi^* C - \frac{2}{3} E$.

If the p_i are smooth points on C, then $\pi^* C = C_Z + E$, hence $H = C_Z + \frac{1}{3} E$. Since $(H \cdot C_Z) = 2$, $(H \cdot E_i) = \frac{2}{3}$ and $Z \setminus (C_Z + E) \simeq Y \setminus C$ is affine, we see that H is ample by the Nakai–Moishezon criterion.

If the p_i are double points on C, then $\pi^* C = C_Z + 2E$, hence $H = C_Z + \frac{4}{3} E$. Then $(C_Z \cdot E_i) = 2$, $(H \cdot C_Z) = 0$ and $(H \cdot E_i) = \frac{2}{3}$. So $3H$ is semiample and it contracts C_Z. Let the resulting surface be X_0 and $F_i \subset X_0$ the images of the E_i.

Note that in this case C_Z is a smooth, rational curve and $(C_Z^2) = -8$. Thus X_0 has a single quotient singularity of type $\mathbb{C}^2 / \frac{1}{8}(1, 1)$. We also get that $(F_i^2) = -\frac{1}{2}$ and $(F_i \cdot F_j) = \frac{1}{2}$ for $i \neq j$. Furthermore, $K_{X_0} \sim F_1 + F_2 + F_3$ is ample.

In order to construct the algebraic family, start with $C \subset Y$ where C is a rational curve with three nodes. The deformation is obtained by moving the points into general position. Blowing up the points we get H that is ample on the general fibers and contracts the birational transform of C in the special fiber. Thus we get $g^{\mathrm{alg}} : X^{\mathrm{alg}} \to D$.

For the analytic case, we choose a deformation $Y \to \mathbb{D}$ of Y_0 whose very general fibers are nonalgebraic K3 surfaces. Take three sections $B_i \subset Y$ that pass through the three nodes of C. Blow them up and then contract the birational transform of C. The contraction extends to the total space by Kollár and Mori (1992, 11.4). We get $g^{\mathrm{an}} : X^{\mathrm{an}} \to \mathbb{D}$ whose central fiber is X_0. The other fibers are nonalgebraic, K3 surfaces blown up at three points.

Example 1.48 (More rational surfaces with ample canonical class) (Kollár, 2008b, sec.5) Given natural numbers a_1, a_2, a_3, a_4, consider the surface

$$S = S(a_1, a_2, a_3, a_4) := (x_1^{a_1} x_2 + x_2^{a_2} x_3 + x_3^{a_3} x_4 + x_4^{a_4} x_1 = 0) \subset \mathbb{P}(w_1, w_2, w_3, w_4),$$

where $w_i' = a_{i+1} a_{i+2} a_{i+3} - a_{i+2} a_{i+3} + a_{i+3} - 1$ (with indices modulo 4) and $w_i = w_i' / \gcd(w_1', w_2', w_3', w_4')$. It is easy to see that S has only quotient singularities (at the four coordinate vertices). It is proved in (Kollár, 2008b, thm.39) that S

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is rational if $\gcd(w_1', w_2', w_3', w_4') = 1$. (By Kollár, 2008b, 38, this happens with probability ≥ 0.75.)

$\mathbb{P}(w_1, w_2, w_3, w_4)$ has isolated singularities iff the $\{w_i\}$ are pairwise relatively prime. (It is easy to see that for $1 \leq a_i \leq N$, this happens for at least $c \cdot N^{4-\varepsilon}$ of the 4-tuples.) In this case $K_S = \mathscr{O}_{\mathbb{P}}(\prod a_i - 1 - \sum w_i)|_S$. From this it is easy to see that if $a_1, a_2, a_3, a_4 \geq 4$ then K_S is ample and (K_S^2) converges to 1 as $a_1, a_2, a_3, a_4 \to \infty$. See Urzúa and Yáñez (2018) for more on these surfaces.

1.5 From Nodal Curves to Stable Curves and Surfaces

We discussed stable extension for families of curves $C \to B$ over a smooth curve B in (1.15.1). Similarly, working over a higher dimensional reduced base $C \to S$ involves two main steps.

- First, we transform a given proper family of curves $C \to S$ into a proper, flat family $C_1 \to S_1$, whose fibers are reduced, nodal curves. This needs a base change $S_1 \to S$ that involves choices, and then a sequence of blow-ups that again involves choices. We can choose S_1 to be smooth.
- Once we have a proper, flat family $C_1 \to S_1$ whose fibers are reduced, nodal curves, and whose base is smooth, we take the relative canonical model (11.28) to get the stable family $C_1^{\mathrm{stab}} \to S_1$. For MMP to work, we need S_1 to have at worst log canonical singularities.

Nonetheless, we show that one can go from flat families of nodal curves to flat families of stable curves in a functorial way over an arbitrary base.

Theorem 1.49 *For every $g \geq 2$ there is a natural transformation $C \mapsto C^{stab}$*

$$\left\{ \begin{array}{c} \textit{proper, flat families of} \\ \textit{reduced, nodal, genus } g \textit{ curves} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \textit{stable families of} \\ \textit{genus } g \textit{ curves} \end{array} \right\},$$

such that that if C is a smooth, projective curve, then $C^{stab} = C$. (We assume that the curves are geometrically connected. By the genus of a proper nodal curve C we mean $h^1(C, \mathscr{O}_C)$.)

Proof We outline the main steps, leaving some details to the reader. We use C' to denote any irreducible component of the curve that we work with.

First, let C be a proper, reduced, nodal curve over an algebraically closed field. We start with two recipes to construct C^{stab}. With both approaches, we first obtain the largest semistable subcurve $C^{\mathrm{ss}} \subset C$.

Step 1.a (Using MMP) Find $C' \subset C$ on which K_C has negative degree. Equivalently, $C' \simeq \mathbb{P}^1$ and it meets the rest of C in one point only. Contract (or discard) this component. Repeat if possible.
Step 1.b (Using K_C) $C^{\rm ss}$ is the support of the global sections of $\mathscr{O}_C(K_C)$.

Once we have $C^{\rm ss}$, we continue to get $C^{\rm stab}$ as follows.
Step 2.a Find $C' \subset C^{\rm ss}$ on which $K_{C^{\rm ss}}$ has degree 0. Equivalently, $C' \simeq \mathbb{P}^1$ and it meets the rest of $C^{\rm ss}$ in two points only; call them p, q. Contract this component. Equivalently, discard C' and identify the points p, q. Repeat if possible.
Step 2.b (Using the canonical ring) $C^{\rm stab} = \operatorname{Proj} \oplus_m H^0(C^{\rm ss}, \mathscr{O}_{C^{\rm ss}}(mK_{C^{\rm ss}}))$.

Once we know $C^{\rm stab}$, we can also recover it in one step as follows.
Step 3 Let $\{C^i \subset C : i \in I\}$ be the irreducible components that are kept in the above process; call them *stable.* Pick nonnodal points $p^i \in C^i$ and set $L := \mathscr{O}_C(\sum p^i)$. Then, for $m \gg 1$, $H^1(C, L^m) = 0$, L^m is globally generated and maps C onto $C^{\rm stab}$.
Step 4 Over an arbitrary field k with algebraic closure $\bar{k}$, we show that if C is defined over k, then $(C_{\bar{k}})^{\rm stab}$ is also defined over k, giving us $C^{\rm stab}$.

Now to the general case. Let $g : C_S \to S$ be a proper, flat family of reduced, nodal curves over an arbitrary base. We construct the stable family étale-locally; uniqueness then implies that we get a family over S.

Pick a point $s \in S$. By the arguments here, we have the stable irreducible components $C^i_s \subset C_s$. Pick nonnodal points $p^i \in C^i_s$ and let $D^i \subset C_S$ be sections that meet C_s only at p^i. (Usually this needs an étale base change.) Set $L_S := \mathscr{O}_{C_S}(\sum D^i)$. Then Step 3 shows that, for $m \gg 1$,
Step 5 $R^1 g_* L^m = 0$, $g_* L^m$ is locally free, and maps C_S onto $C^{\rm stab}_S$. □

Warning Note that Step 2.b works only for semistable curves. As an example, let $C = C_1 \cup C_2$ be a curve with a single node p with $g(C_1) \geq 2$ and $C_2 \simeq \mathbb{P}^1$. Then we have a **non**-finitely generated ring

$$\oplus_{m\geq 0} H^0(C, \mathscr{O}_C(mK_C)) = \oplus_{m\geq 0} H^0(C_1, \mathscr{O}_{C_1}(mK_{C_1} + (m-1)[p])).$$

Definition 1.50 (Stabilization functor) Trying to generalize (1.49) to higher dimensions, the best would be to have a functor from proper, flat locally stable families to stable families, that agrees with $X \to X^{\rm c}$ on smooth varieties of general type. One can further restrict the singularities of the fibers and talk about stabilization functors for families of smooth varieties, simple normal crossing varieties (p.xvi), and so on.

We see here that such a stabilization functor does exist for smooth families, but not for more complicated singularities. We discuss this phenomenon in

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detail in Section 5.2; see especially (5.11). This is another reason why the moduli theory of higher dimensional varieties is much more complicated.

Theorem 1.51 (Stabilization functor for surfaces)

(1.51.1) *For smooth, projective surfaces of general type, $S \mapsto S^c$ is a stabilization functor.*

(1.51.2) *For projective surfaces of general type with quotient singularities, $S \mapsto S^c$ is **not** a stabilization functor.*

(1.51.3) *For projective surfaces with normal crossing singularities, $S \mapsto S^c$ is **not** a stabilization functor.*

(1.51.4) *For irreducible projective surfaces with normal crossing singularities, S^c does not even make sense in general.*

Proof For the first part, see (1.36) and (2.48). As in (1.49), more work is needed for nonreduced bases.

For (2) and (3), we run into problems even for families over smooth curves.

Consider the simplest case when we have a flat, projective morphism $p : X \to \mathbb{A}^1$ to a smooth curve such that K_X is $\mathbb{Q}$-Cartier, and the fibers are surfaces with quotient singularities only. Then we get the stable model $p^{\mathrm{stab}} : X^{\mathrm{stab}} \to \mathbb{A}^1$ as the relative canonical model (11.28).

We claim that as soon as the process involves a flip, we have an example for (2): the canonical ring of X_0 is strictly larger than the canonical ring of $(X^{\mathrm{c}})_0$.

The flip is a diagram

$$\begin{tikzcd} (C \subset X) \arrow[rr, dashed, "\phi"] \arrow[dr, "\pi"'] & & (C^+ \subset X^+) \arrow[dl, "\pi^+"] \\ & Z & \end{tikzcd} \tag{1.51.5}$$

where $-K_X$ is π-ample and K_{X^+} is π^+-ample.

Restricting it to the fiber over $0 \in \mathbb{A}^1$ we get a similar looking diagram of surfaces with quotient singularities

$$\begin{tikzcd} (C \subset X_0) \arrow[rr, dashed, "\phi_0"] \arrow[dr, "\pi_0"'] & & (C^+ \subset X_0^+) \arrow[dl, "\pi_0^+"] \\ & Z_0 & \end{tikzcd} \tag{1.51.6}$$

where K_{X_0} is π_0-ample and $K_{X_0^+}$ is π_0^+-ample. The difference is that now the exceptional *curves* C, C^+ of (1.51.5) are exceptional *divisors.*

Using (1.51.8) we get the following.

Problem 1.51.7 $X_0 \mapsto X_0^+$ is **not** a step of the MMP for X_0. In fact, the canonical ring of X_0^+ is strictly smaller than the canonical ring of X_0.

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Taking a suitable resolution shows that similar examples happen for families with simple normal crossing fibers.

Claim (1.51.4) is not a precise assertion, but we expect that, even over algebraically closed fields, there is no "sensible" way to associate a stable surface to every projective, normal crossing surface. For example, Kollár (2011c) constructs irreducible, projective surfaces S with normal crossing singularities for which the canonical ring $\oplus_{m\geq 0} H^0(S, \mathscr{O}_S(mK_S))$ is **not** finitely generated. We present a similar example in (1.53). $\square$

Claim 1.51.8 Let $p\colon Y \to T$ be a proper, birational morphism of normal surfaces and $E := \operatorname{Ex}(p)$. Let D be a Cartier divisor on Y and set $D_T := p(D)$. The following are easy to see.

(a) If $-D|_E$ is ample then $p_*\mathscr{O}_Y(mD) = \mathscr{O}_T(mD_T)$ for $m \geq 1$.
(b) If $D|_E$ is ample then $p_*\mathscr{O}_Y(mD) \subsetneq \mathscr{O}_T(mD_T)$ for $m \gg 1$.
(c) If D is ample then $H^0(Y, \mathscr{O}_Y(mD)) \subsetneq H^0(T, \mathscr{O}_T(mD_T))$ for $m > 1$. $\square$

We saw that if $p\colon X \to \mathbb{A}^1$ is a flat, projective family of surfaces with quotient singularities, then the relative canonical model (11.28) gives a stable family, although this is not a fiber-wise construction. The next example shows that, for families over nodal curves, there may not be any stable family.

Example 1.52 Consider any family $X \to \mathbb{A}^1_u$ as in (1.51.7), and glue it to the trivial family $q\colon Y := X_0 \times \mathbb{A}^1_v \to \mathbb{A}^1_v$ along the central fibers to get a locally stable family $r\colon X \amalg_{X_0} Y \to (uv = 0)$. Then

(1.52.1) $p^{\mathrm{stab}}\colon X^{\mathrm{stab}} \to \mathbb{A}^1_u$ and $q^{\mathrm{stab}}\colon Y^{\mathrm{stab}} \to \mathbb{A}^1_v$ both exists, yet
(1.52.2) their central fibers $(X^{\mathrm{c}})_0$ and $(Y^{\mathrm{c}})_0$ are **not** isomorphic, so
(1.52.3) $r\colon X \amalg_{X_0} Y \to (uv = 0)$ does **not** have a stable model.

Example 1.53 Following Kollár (2011c), we give an example of a projective, normal crossing surface whose canonical ring is not finitely generated. The key point is the following observation.

Let T be a projective surface, $C_1, C_2 \subset T$ disjoint smooth curves, and $\tau : C_1 \to C_2$ an isomorphism. Assume that $T \setminus C_1$ is smooth, T has a single node at a point $p_1 \in C_1$, and $K_T + C_1 + C_2$ is ample. Let $T/(\tau)$ be obtained from T by identifying C_1 with C_2 using τ.

Claim 1.53.1 The canonical class of $T/(\tau)$ is not $\mathbb{Q}$-Cartier. Thus its canonical ring is not finitely generated.

Proof T is smooth along C_2, hence the usual adjunction gives that

$$(K_T + C_1 + C_1)|_{C_1} = K_{C_1}.$$

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T has a node along C_1. This modifies the adjunction formula to

$$(K_T + C_1 + C_2)|_{C_1} = K_{C_1} + \tfrac{1}{2}[p_1];$$

see Kollár (2013b, 4.3) for this computation. This means that we cannot match up local generating sections of the sheaf $\mathscr{O}_T(mK_T + mC_1 + mC_2)$ at the points p_1 and $\tau(p_1)$; see Kollár (2013b, 5.12) for the precise statement and proof. This easily implies that finite generation fails; see Kollár (2010, exc.97). □

This is almost what we want, except that T is not a normal crossing surface at the image of p_1. So next we construct a normal crossing surface and check that trying to construct its minimal model leads to a surface as needed.

We start with a smooth plane curve $C \subset \mathbb{P}^2$ of degree d and a line L intersecting C transversally. Let $c \in C \cap L$ be one of the intersection points. Fix distinct points $p, q \in \mathbb{P}^1$. In $\mathbb{P}^2_p := \{p\} \times \mathbb{P}^2$ we get C_p, L_p, and similarly for $C_q, L_q \subset \mathbb{P}^2_q$. We have the "identity" $\tau\colon C_p \simeq C_q$.

Let $\bar{S} \subset \mathbb{P}^1 \times \mathbb{P}^2$ be a surface of bidegree $(e, d+1)$ such that $\bar{S} \cap \mathbb{P}^2_p = C_p \cup L_p$ and $\bar{S} \cap \mathbb{P}^2_q = C_q \cup L_q$. We can further arrange that $\bar{S}$ is smooth, except for an ordinary node at $c_p \in C_p \cap L_p$.

Let $\bar{S}' \to \bar{S}$ be obtained by blowing up c_p and c_q. We get exceptional curves E'_p, E'_q and birational transforms C'_p and C'_q. Note that $\bar{S}'$ is smooth and $E'_p + E'_q + C'_p + C'_q$ is an snc divisor. We can now glue C'_p to C'_q using the "identity" $\tau'\colon C'_p \simeq C'_q$ to obtain the nonnormal surface $S' := \bar{S}'/(\tau')$. It has normal crossing self-intersection along a curve $C \simeq C'_p \simeq C'_q$. Note that $K_{S'} + E_p + E_q$ is a Cartier divisor.

Claim 1.53.2 The projective, normal crossing pair $(S', E_p + E_q)$ does not have a canonical model.

Proof The normalization of $(S', E_p + E_q)$ is $(\bar{S}', E'_p + E'_q + C'_p + C'_q)$, thus the only "sensible" thing to do is to construct its canonical model, and then glue the images of C'_p and C'_q together. We compute that

$$(K_{\bar{S}'} + E'_p + E'_q + C'_p + C'_q) \cdot E'_p = -1 \quad \text{and} \quad (K_{\bar{S}'} + E'_p + E'_q + C'_p + C'_q) \cdot E'_q = -1.$$

Thus we need to contract E'_p and E'_q to get $(\bar{S}, C_p + C_q)$. Note that

$$\mathscr{O}_{\bar{S}}(K_{\bar{S}}) \simeq \mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(e-2, d+1-3)|_{\bar{S}},$$

which is ample for $e \ge 3, d \ge 3$. This shows that if $d \ge 4$, then $K_{\bar{S}} + C_p + C_q$ is ample. Therefore, the only possible choice for the canonical model of $(S', E'_p + E'_q)$ is $\bar{S}/(\tau)$. Now (1.53.1) shows that the canonical ring is not finitely generated. □

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1.6 Examples of Bad Moduli Problems

Now we turn to a more general overview of moduli problems. The aim of this section is to present examples of moduli problems that seem quite reasonable at first sight, but turn out to have rather bad properties. We start with the moduli of hypersurfaces.

The Chow and Hilbert varieties describe families of hypersurfaces in a fixed projective space $\mathbb{P}^n$. For many purposes, it is more natural to consider the moduli functor of hypersurfaces modulo isomorphisms. We consider what kind of "moduli spaces" one can obtain in various cases.

Definition 1.54 (Hypersurfaces modulo linear isomorphisms) Over an algebraically closed field k, we consider hypersurfaces $X \subset \mathbb{P}^n_k$ where $X_1, X_2 \subset \mathbb{P}^n_k$ are considered isomorphic if there is an automorphism $\phi \in \mathrm{Aut}(\mathbb{P}^n_k)$ such that $\phi(X_1) = X_2$.

Over an arbitrary base scheme S, we consider pairs $(X \subset P)$ where P/S is a $\mathbb{P}^n$-bundle for some n and $X \subset P$ is a closed subscheme, flat over S such that every fiber is a hypersurface. There are two natural invariants: the relative dimension of P and the degree of X. Thus for any given n, d we get a functor

$$\mathcal{H}yp\mathcal{S}ur_{n,d}(S) := \left\{ \begin{array}{c} \text{Flat families } X \subset P \\ \text{such that } \dim_S P = n, \deg X = d, \\ \text{modulo isomorphisms over } S. \end{array} \right\}$$

One can also consider subfunctors, where we allow only reduced, normal, canonical, log canonical, or smooth hypersurfaces; these are indicated by the superscripts red, norm, c, lc, or sm.

Our aim is to investigate what the "coarse moduli spaces" of these functors look like. Our conclusion is that in many cases there cannot be any scheme or algebraic space that is a coarse moduli space: any "coarse moduli space" would have to have very strange topology.

Let $\mathcal{H}yp\mathcal{S}ur^*_{n,d}$ be any subfunctor of $\mathcal{H}yp\mathcal{S}ur_{n,d}$, and assume that it has a coarse moduli space $\mathrm{HypSur}^*_{n,d}$. By definition, the set of k-points of $\mathrm{HypSur}^*_{n,d}$ is $\mathcal{H}yp\mathcal{S}ur^*_{n,d}(\operatorname{Spec} k)$. We can also get some idea about the Zariski topology of $\mathrm{HypSur}^*_{n,d}$ using various families of hypersurfaces.

For instance, we can study the closure $\bar{U}$ of a subset $U \subset \mathrm{HypSur}^*_{n,d}(\operatorname{Spec} k)$ using the following observation:

- Assume that there is a flat family of hypersurfaces $\pi\colon X \to S$ and a dense open subset $S^\circ \subset S$ such that $[X_s] \in U$ for every $s \in S^\circ(k)$. Then $[X_s] \in \bar{U}$ for every $s \in S(k)$.

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Next we write down flat families of hypersurfaces $\pi\colon X \to \mathbb{A}^1$ in $\mathcal{H}yp\mathcal{S}ur^*_{n,d}$ such that for $t \neq 0$ the fibers X_t are isomorphic to each other, but X_0 is not isomorphic to them. Such a family corresponds to a morphism $\tau\colon \mathbb{A}^1 \to \mathrm{HypSur}^*_{n,d}$ such that $\tau(\mathbb{A}^1 \setminus \{0\}) = [X_1]$, but $\tau(\{0\}) = [X_0]$. This implies that the point $[X_1]$ is not closed, and its closure contains $[X_0]$.

This is not very surprising in a scheme, but note that X_1 itself is defined over our base field k, so $[X_1]$ is supposed to be a k-point. On a k-scheme, k-points are closed. Thus we conclude that if there is any family as listed, the moduli space $\mathrm{HypSur}^*_{n,d}$ cannot be a k-scheme, not even a quasi-separated algebraic space (Stacks, 2022, tag 08AL).

The simplest way to get such families is by the following construction.

Example 1.55 (Deformation to cones I) Let $f(x_0, \dots, x_n)$ be a homogeneous polynomial of degree d and $X := (f = 0)$ the corresponding hypersurface. For some $0 \leq i < n$ consider the family of hypersurfaces

$$\mathbf{X} := (f(x_0, \dots, x_i, tx_{i+1}, \dots tx_n) = 0) \subset \mathbb{P}^n \times \mathbb{A}^1_t \tag{1.55.1}$$

with projection $\pi\colon \mathbf{X} \to \mathbb{A}^1_t$. If $t \neq 0$ then the substitution

$$x_j \mapsto x_j \quad \text{for } j \leq i, \text{ and} \quad x_j \mapsto t^{-1}x_j \quad \text{for } j > i$$

shows that the fiber X_t is isomorphic to X. If $t = 0$ then we get the cone over $X \cap (x_{i+1} = \cdots = x_n = 0)$:

$$X_0 = (f(x_0, \dots, x_i, 0, \dots, 0) = 0) \subset \mathbb{P}^n.$$

This is a hypersurface iff $f(x_0, \dots, x_i, 0, \dots, 0)$ is not identically 0.

More generally, any algebraic variety has a similar deformation to a cone over its hyperplane section, see (2.36).

Already these simple deformations show that various moduli spaces of hypersurfaces have very few closed points.

Corollary 1.56 *The sole closed point of* $\mathrm{HypSur}_{d,n}$ *is* $[(x_0^d = 0)]$.

Proof Take any $X = (f = 0) \subset \mathbb{P}^n$. After a general change of coordinates, we can assume that x_0^d appears in f with nonzero coefficient. For $i = 0$ consider the family (1.55.1).

Then $X_0 = (x_0^d = 0)$, hence $[X]$ cannot be a closed point unless $X \simeq X_0$. It is quite easy to see that if $X \to S$ is a flat family of hypersurfaces whose generic fiber is a d-fold plane, then every fiber is a d-fold plane. This shows that $[(x_0^d = 0)]$ is a closed point. □

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Corollary 1.57 *The only closed points of* $\mathrm{HypSur}^{\mathrm{red}}_{d,n}$ *are* $[(f(x_0,x_1)=0)]$ *where* f *has no multiple roots.*

Proof If X is a reduced hypersurface of degree d, there is a line that intersects it in d distinct points. We can assume that this is the line $(x_2=\cdots=x_n=0)$. For $i=1$, consider the family (1.55.1).

Then $X_0=(f(x_0,x_1,0,\dots,0)=0)$ where $f(x_0,x_1)$ has d distinct roots. Since X_0 is reduced, we see that none of the other hypersurfaces correspond to closed points.

It is not obvious that the points corresponding to $(f(x_0,x_1,0,\dots,0)=0)$ are closed, but this can be established by studying the moduli of d points in $\mathbb{P}^1$; see (Mumford, 1965, chap.3) or (Dolgachev, 2003, sec.10.2). □

A similar argument establishes the normal case:

Corollary 1.58 *The only closed points of* $\mathrm{HypSur}^{\mathrm{norm}}_{d,n}$ *are* $[(f(x_0,x_1,x_2)=0)]$ *where* $(f(x_0,x_1,x_2)=0)\subset\mathbb{P}^2$ *is a nonsingular curve.* □

In these examples the trouble comes from cones. Cones can be normal, but they are very singular by other measures; they have a singular point whose multiplicity equals the degree of the variety. So one could hope that high multiplicity points cause the problems. This is true to some extent as the next theorems and examples show. For proofs, see Mumford (1965, sec.4.2) and Dolgachev (2003, sec.10.1).

Theorem 1.59 *Each of the following functors has a coarse moduli space which is a quasi-projective variety.*

(1.59.1) *The functor of smooth hypersurfaces* $\mathcal{H}yp\mathcal{S}ur^{\mathrm{sm}}_{n,d}$.

(1.59.2) *For* $d\geq n+1$, *the functor* $\mathcal{H}yp\mathcal{S}ur^{\mathrm{c}}_{n,d}$ *of hypersurfaces with canonical singularities.*

(1.59.3) *For* $d>n+1$, *the functor* $\mathcal{H}yp\mathcal{S}ur^{\mathrm{lc}}_{n,d}$ *of hypersurfaces with log canonical singularities.*

(1.59.4) *For* $d>n+1$, *the functor* $\mathcal{H}yp\mathcal{S}ur^{\mathrm{low-mult}}_{n,d}$ *of those hypersurfaces that have only points of multiplicity* $<\frac{d}{n+1}$.

Example 1.60 Consider the family of even degree d hypersurfaces

$$((x_0^{d/2}+t^d x_1^{d/2})x_1^{d/2}+x_2^d+\cdots+x_n^d=0)\subset\mathbb{P}^n\times\mathbb{A}^1_t.$$

For $t\neq 0$ the substitution $(x_0{:}x_1{:}x_2{:}\cdots{:}x_n)\mapsto(tx_0{:}t^{-1}x_1{:}x_2{:}\cdots{:}x_n)$ transforms the equation of X_t to

$$X := ((x_0^{d/2} + x_1^{d/2})x_1^{d/2} + x_2^d + \cdots + x_n^d = 0) \subset \mathbb{P}^n.$$

X has a single singular point which is at $(1{:}0{:}\cdots{:}0)$ and has multiplicity $d/2$.

For $t = 0$ we obtain the hypersurface

$$X_0 := (x_0^{d/2} x_1^{d/2} + x_2^d + \cdots + x_n^d = 0).$$

X_0 has two singular points of multiplicity $d/2$, hence it is not isomorphic to X.

Thus we conclude that $[X]$ is not a closed point of the "moduli space" of those hypersurfaces of degree d that have only points of multiplicity $\leq d/2$.

This is especially interesting when $d \leq n$ since in this case X_0 has canonical singularities (1.33).

Thus we see that for $d \leq n$, the functor $\mathcal{H}yp\mathcal{S}ur^{\mathrm{c}}_{n,d}$ parametrizing hypersurfaces with canonical singularities does not have a coarse moduli space. By contrast, for $d > n$ the coarse moduli scheme $\mathrm{HypSur}^{\mathrm{c}}_{n,d}$ exists and is quasi-projective by (1.59).

Example 1.61 One could also consider hypersurfaces modulo isomorphisms which do not necessarily extend to an isomorphism of the ambient projective space. It is easy to see that smooth hypersurfaces can have such nonlinear isomorphisms only for $(d, n) \in \{(3, 2), (4, 3)\}$. A smooth cubic curve in $\mathbb{P}^2$ has an infinite automorphism group, but only finitely many extend to an automorphism of $\mathbb{P}^2$. Similarly, a smooth quartic surface in $\mathbb{P}^3$ can have an infinite automorphism group as in (1.66), but only finitely many extend to an automorphism of $\mathbb{P}^3$. See (1.66) or Shimada and Shioda (2017); Oguiso (2017) for examples of isomorphisms of smooth quartic surfaces in $\mathbb{P}^3$.

The nonseparated examples produced so far all involved ruled or uniruled varieties. Next we consider some examples where the varieties are not uniruled. The bad behavior is due to the singularities, not to the global structure.

Example 1.62 (Double covers of $\mathbb{P}^1$) Let $f(x, y)$ and $g(x, y)$ be two cubic forms without multiple roots, neither divisible by x or y. Set

$$\begin{aligned} S_1 &:= \big(f(x_1, y_1)g(t^2x_1, y_1) = z_1^2\big) \subset \mathbb{P}(1,1,3) \times \mathbb{A}^1, \quad \text{and} \\ S_2 &:= \big(f(x_2, t^2y_2)g(x_2, y_2) = z_2^2\big) \subset \mathbb{P}(1,1,3) \times \mathbb{A}^1. \end{aligned}$$

Note that $K_{S_i/\mathbb{A}^1}$ is relatively ample and the general fiber of $\pi_1 : S_i \to \mathbb{A}^1$ is a smooth curve of genus 2.

The central fibers are $(f(x_1, y_1)g(0, y_1) = z_1^2)$ resp. $(f(x_2, 0)g(x_2, y_2) = z_2^2)$. By assumption, $g(0, y_1) = a_1y_1^3$ and $f(x_2, 0) = a_2x_2^3$ where the $a_i \neq 0$. Setting $z_1 = a_1^{1/2}w_1y_1$ and $z_2 = a_2^{1/2}w_2x_2$ gives the normalizations. Hence the central fibers are elliptic curves with a cusp. Their normalization is isomorphic to $(f(x_1, y_1)y_1 = w_1^2)$ resp. $(x_2 g(x_2, y_2) = w_2^2)$. These are, in general, not isomorphic to each other.

This also shows that along the central fibers, the only singularities are at $(1{:}0{:}0; 0)$ and at $(0{:}1{:}0; 0)$, with local equations $g(t^2, y_1) = z_1^2$ and $f(x_2, t^2) = z_2^2$. (These are simple elliptic. The minimal resolution contains a single smooth elliptic curve of self intersection -1.) Hence the S_i are normal surfaces, each having one simple elliptic singular point.

Finally, the substitution $(x_1 : y_1 : z_1; t) = (x_2 : t^2 y_2 : t^3 z_2; t)$ transforms $f(x_1, y_1)g(t^2 x_1, y_1) - z_1^2$ into

$$f(x_2, t^2 y_2)g(t^2 x_2, t^2 y_2) - t^6 z_2^2 = t^6(f(x_2, t^2 y_2)g(x_2, y_2) - z_2^2),$$

thus the two families are isomorphic over $\mathbb{A}^1 \setminus \{0\}$.

Let us end our study of hypersurfaces with a different type of example. This shows that the moduli problem for hypersurfaces usually includes smooth limits that are not hypersurfaces. These pose no problem for the general theory, but they show that it is not always easy to see what schemes one needs to include in a moduli space.

Example 1.63 (Smooth limits of hypersurfaces) (Mori, 1975) Fix integers $a, b > 1$ and $n \geq 2$. We construct a family of smooth n-folds X_t such that X_t is a smooth hypersurface of degree ab for $t \neq 0$ and X_0 is not isomorphic to a smooth hypersurface.

It is not known if similar examples exist for $n \geq 3$ and $\deg X$ a prime number; see Ottem and Schreieder (2020) for the cases $\deg X \leq 7$.

Start with the weighted projective space $\mathbb{P}(1^{n+1}, a)_{\mathbf{x},\mathbf{z}}$. Let f_a, g_{ab} be general homogeneous forms of degree a (resp. ab) in $x_0, \dots, x_n$. Consider the family of complete intersections

$$X_t := (tz - f_a(x_0, \dots, x_n) = z^b - g_{ab}(x_0, \dots, x_n) = 0) \subset \mathbb{P}(1^{n+1}, a).$$

For $t \neq 0$, we can eliminate z to obtain a degree ab smooth hypersurface

$$X_t \simeq (f_a^b(x_0, \dots, x_n) = t^b g_{ab}(x_0, \dots, x_n)) \subset \mathbb{P}^{n+1}.$$

For $t = 0$, we see that $\mathcal{O}_{X_0}(1)$ is not very ample, but realizes X_0 as a b-fold cyclic cover (11.24) of the degree a smooth hypersurface $(f_a(x_0, \dots, x_n) = 0)$. In particular, X_0 is not isomorphic to a smooth hypersurface.

The next example shows that seemingly equivalent moduli problems may lead to different moduli spaces.

Example 1.64 We start with the moduli space P_{n+1} of $n + 1$ points in $\mathbb{C}$ up to translations. We can view such a point set as the zeros of a unique polynomial of degree $n + 1$ whose leading term is x^{n+1}. We can use a translation to kill the coefficient of x^n, and the universal polynomial is then given by

$$x^{n+1} + a_2 x^{n-1} + \cdots + a_{n+1}.$$

Thus $P_{n+1} \simeq \mathbb{C}^n$ with coordinates $a_2, \ldots, a_{n+1}$.

Let us now look at those point sets where n of the points coincide. There are two ways to formulate this as a moduli problem:

(1.64.1) unordered point sets $p_0, \ldots, p_n \in \mathbb{C}$ where at least n of the points coincide, up to translations, or

(1.64.2) unordered point sets $p_0, \ldots, p_n \in \mathbb{C}$ plus a point $q \in \mathbb{C}$ such that $p_i = q$ at least n-times, up to translations.

If $n \geq 2$ then q is uniquely determined by the points $p_0, \ldots, p_n$, so it would seem that the two formulations are equivalent. We claim, however, that the two versions have nonisomorphic moduli spaces.

If the n-fold point is at t then the corresponding polynomial is $(x-t)^n(x+nt)$. By expanding it we get that

$$a_i = t^i\Big[(-1)^i\tbinom{n}{i} + (-1)^{i-1}n\tbinom{n}{i-1}\Big] \quad \text{for } i = 2, \ldots, n+1.$$

This shows that the space $R_{n+1} \subset P_{n+1}$ of polynomials with an n-fold root is a cuspidal rational curve given as the image of the map

$$t \mapsto \Big(a_i = t^i\Big[(-1)^i\tbinom{n}{i} + (-1)^{i-1}n\tbinom{n}{i-1}\Big] \colon i = 2, \ldots, n+1\Big).$$

So the moduli space R_{n+1} of the first variant (1) is a cuspidal rational curve.

By contrast, the space $\bar{R}_{n+1}$ of the second variant (2) is a smooth rational curve, the isomorphism given by

$$(p_0, \ldots, p_n; q) \mapsto \textstyle\sum_i (p_i - q) \in \mathbb{C}.$$

Not surprisingly, the map that forgets the n-fold root gives $\pi\colon \bar{R}_{n+1} \to R_{n+1}$ which is the normalization map.

Next we have two examples of moduli functors that are not representable (1.18). They suggest that varieties whose canonical class is not ample present special challenges.

Example 1.65 Let $S \subset \mathbb{P}^3$ be a smooth surface of degree 4 over $\mathbb{C}$, with an infinite discrete automorphism group, for example as in (1.66).

Let $\mathbf{S} \to W$ be the universal family of smooth degree 4 surfaces in $\mathbb{P}^3$. The isomorphisms classes of the pairs $(S, \mathscr{O}_S(1))$ correspond to the $\mathrm{Aut}(\mathbb{P}^3)$-orbits

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in W. We see that the fibers isomorphic to S form countably many $\mathrm{Aut}(\mathbb{P}^3)$-orbits.

For any $g \in \mathrm{Aut}(S)$, $g^*\mathscr{O}_S(1)$ gives another embedding of S into $\mathbb{P}^3$. Two such embeddings are projectively equivalent iff $g^*\mathscr{O}_S(1) \simeq \mathscr{O}_S(1)$, that is, when $g \in \mathrm{Aut}(S, \mathscr{O}_S(1))$. The latter can be viewed as the group of automorphisms of $\mathbb{P}^3$ that map S to itself. Thus $\mathrm{Aut}(S, \mathscr{O}_S(1))$ is a closed subvariety of $\mathrm{Aut}(\mathbb{P}^3) \simeq \mathrm{PGL}_4$. Since $\mathrm{Aut}(S)$ is discrete, this implies that $\mathrm{Aut}(S, \mathscr{O}_S(1))$ is finite. Hence the fibers of $\mathbf{S} \to W$ that are isomorphic to S lie over countably many $\mathrm{Aut}(\mathbb{P}^3)$-orbits, corresponding to $\mathrm{Aut}(S)/\mathrm{Aut}(S, \mathscr{O}_S(1))$.

Example 1.66 (Surfaces with infinite discrete automorphism group) Let us start with a smooth genus 1 curve E defined over a field K. Any point $q \in E(K)$ defines an involution τ_q where $\tau_q(p)$ is the unique point such that $p + \tau_q(p) \sim 2q$. (Equivalently, we can set q as the origin, then $\tau_q(p) = -p$.) The first formulation shows that if L/K is a quadratic extension, then any $Q \in E(L)$ also defines an involution τ_Q where $\tau_Q(p)$ is the unique point such that $p + \tau_Q(p) \sim Q$.

Given points $q_1, q_2 \in E(K)$, we see that $p \mapsto \tau_{q_2} \circ \tau_{q_1}(p)$ is translation by $2q_1 - 2q_2$. Similarly, given $Q_i \in E(L_i)$, $p \mapsto \tau_{Q_2} \circ \tau_{Q_1}(p)$ is translation by $Q_1 - Q_2$. Usually these translations have infinite order.

Let $g \colon S \to C$ be a smooth, minimal, elliptic surface. Then, any section or double section of g gives an involution of S, and two involutions usually generate an infinite group of automorphisms of S.

As a concrete example, let $S \subset \mathbb{P}^3$ be a smooth quartic that contains three lines L_i. The pencil of planes through L_1 gives an elliptic fibration with L_2, L_3 as sections. Thus these K3 surfaces usually have an infinite automorphism group.

1.7 Compactifications of M_g

Here we consider what happens if we try to define other compactifications of M_g. First, we give a complete study of a compactified moduli functor of genus 2 curves that uses only irreducible curves.

Definition 1.67 Working over $\mathbb{C}$, let $\mathcal{M}_2^{\mathrm{irr}}$ be the moduli functor of flat families of irreducible curves of arithmetic genus 2 that are either

(1.67.1) smooth,

(1.67.2) nodal,

(1.67.3) rational with two cusps, or

(1.67.4) rational with a triple point whose complete local ring is isomorphic to $\mathbb{C}[[x, y, z]]/(xy, yz, zx)$.

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The aim of this subsection is to prove the following; see Mumford (1965, chap.3) or Dolgachev (2003, sec.10.2) for the relevant background on GIT quotients.

Proposition 1.68 *Let $\mathcal{M}_2^{irr}$ be the moduli functor defined at (1.67). Then*

(1.68.1) *the coarse moduli space M_2^{irr} exists and equals the geometric invariant theory quotient (8.59) of the symmetric power* $\operatorname{Sym}^6 \mathbb{P}^1 /\!/ \operatorname{Aut}(\mathbb{P}^1)$, *but*

(1.68.2) *$\mathcal{M}_2^{irr}$ is a very bad moduli functor.*

Proof A smooth curve of genus 2 can be uniquely written as a double cover $\tau\colon C \to \mathbb{P}^1$, ramified at six distinct points $p_1, \dots, p_6 \in \mathbb{P}^1$, up to automorphisms of $\mathbb{P}^1$. Thus, M_2 is isomorphic to the space of six distinct points in $\mathbb{P}^1$, modulo the action of $\operatorname{Aut}(\mathbb{P}^1)$. If some of the six points coincide, we get singular curves as double covers.

It is easy to see the following; see Mumford (1965, chap.3), Dolgachev (2003, sec.10.2).

(1.68.3) A point set is semistable iff it does not contain any point with multiplicity ≥ 4. Equivalently, if the genus 2 cover has only nodes and cusps.

(1.68.4) The properly semistable point sets are of the form $3p_1 + p_2 + p_3 + p_4$ where the p_2, p_3, p_4 are different from p_1, but may coincide with each other. Equivalently, the corresponding genus 2 cover has at least one cusp.

(1.68.5) Point sets $2p_1+2p_2+2p_3$, where the p_1, p_2, p_3 are different from each other. The double cover is reducible, with two smooth rational components meeting each other at three points.

In the properly semistable case, generically the double cover is an elliptic curve with a cusp over p_1. As a special case, we can have $3p_1 + 3p_2$, giving as double cover a rational curve with two cusps. Note that the curves of this type have a one-dimensional moduli (the cross ratio of the points p_1, p_2, p_3, p_4 or the j-invariant of the elliptic curve), but they all correspond to the same point in $\operatorname{Sym}^6 \mathbb{P}^1 /\!/ \operatorname{Aut}(\mathbb{P}^1)$. (See (1.62) for an explicit construction.) Our definition (1.67) aims to remedy this nonuniqueness by always taking the most degenerate case; a rational curve with two cusps (1.67.3).

In case (5), write the reducible double cover as $C = C_1 + C_2$. The only obvious candidate to get an irreducible curve is to contract one of the two components C_i. We get an irreducible rational curve; denote it by C_j' where $j = 3 - i$. Note that C_j' has one singular point which is analytically isomorphic to the three coordinate axes in $\mathbb{A}^3$. The resulting singular rational curves C_j' are isomorphic to each other. These are listed in (1.67.4).

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Let $p\colon X \to S$ be any flat family of irreducible, reduced curves of arithmetic genus 2. The trace map (Barth et al., 1984, III.12.2) shows that $R^1 p_*\omega_{X/S} \simeq \mathscr{O}_S$. Thus, by cohomology and base change, $p_*\omega_{X/S}$ is locally free of rank 2. Set $P := \mathbb{P}_S(p_*\omega_{X/S})$. Then P is a $\mathbb{P}^1$-bundle over S, and we have a rational map $\pi\colon X \to P$. If X_s has only nodes and cusps, then ω_{X_s} is locally free and generated by global sections, thus π is a morphism along X_s.

If X_s is as in (1.67.4), then ω_{X_s} is not locally free, and π is not defined at the singular point. $\pi|_{X_s}$ is birational and the three local branches of X_s at the singular point correspond to three points on $\mathbb{P}(H^0(X_s, \omega_{X_s}))$.

The branch divisor of π is a degree 6 multisection of $P \to S$, all of whose fibers are stable point sets. Thus we have a natural transformation

$$\mathcal{M}_2^{\text{irr}}(*) \to \operatorname{Mor}(*, \operatorname{Sym}^6 \mathbb{P}^1 /\!/ \operatorname{Aut}(\mathbb{P}^1)).$$

We have already seen that we get a bijection

$$\mathcal{M}_2^{\text{irr}}(\mathbb{C}) \simeq (\operatorname{Sym}^6 \mathbb{P}^1 /\!/ \operatorname{Aut}(\mathbb{P}^1))(\mathbb{C}).$$

Since $\operatorname{Sym}^6 \mathbb{P}^1 /\!/ \operatorname{Aut}(\mathbb{P}^1)$ is normal, we conclude that it is the coarse moduli space. This completes the proof of (1.68.1).

The assertion (1.68.2) is more a personal opinion. There are three main things "wrong" with the functor $\mathcal{M}_2^{\text{irr}}(*)$. Let us consider them one at a time.

1.68.6 (Stable extension questions)

At the set-theoretic level, we have $\mathrm{M}_2^{\text{irr}} = \operatorname{Sym}^6 \mathbb{P}^1 /\!/ \operatorname{Aut}(\mathbb{P}^1)$, but what about at the level of families?

The first indications are good. Let $\pi_B\colon S_B \to B$ be a stable family of genus 2 curves. Assume that no fiber is of type (1.68.5). Let $b_i \in B$ be the points corresponding to fibers with two components of arithmetic genus 1. Let $p\colon A \to B$ be a double cover ramified at the points b_i. Consider the pull-back family $\pi_A\colon S_A \to A$. Set $a_i = p^{-1}(b_i)$ and let $s_i \in \pi_A^{-1}(a_i)$ be the point where the two components meet. Since we took a ramified double cover, each $s_i \in S_A$ is a double point. Thus if we blow up every s_i, the exceptional curves appear in the fiber with multiplicity 1. We can now contract the birational transforms of the elliptic curves to get a family where all these reducible fibers are replaced by a rational curve with two cusps. We have proved the following analog of (1.15.1):

Claim 1.68.6.a Let $\pi\colon S \to B$ be a stable family of genus 2 curves such that no fiber has two smooth rational components. Then, after a suitable double cover $A \to B$, the pull-back $S \times_B A$ is birational to another family where each reducible fiber is replaced by a rational curve with two cusps. □

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This solved our problem for one-parameter families, but, as it turns out, not over higher dimensional bases. In particular, there is no universal family over any base scheme Y that finitely dominates $\operatorname{Sym}^6 \mathbb{P}^1 /\!/ \operatorname{Aut}(\mathbb{P}^1)$, not even locally in any neighborhood of the properly semistable point. Indeed, this would give a proper, flat family of curves of arithmetic genus 2 over a three-dimensional base $\pi\colon X \to Y$ where only finitely many of the fibers (the ones over the unique properly semistable point) have cusps. However, there is no such family.

To see this we use that, by (2.27), every flat deformation of a cusp is induced by pull-back from the two-parameter family

$$\begin{tikzcd} (y^2 = x^3 + ux + v) \arrow[r, hook] \arrow[d, "p"'] & \mathbb{A}^2_{xy} \times \mathbb{A}^2_{uv} \arrow[d] \\ \mathbb{A}^2_{uv} \arrow[r, equal] & \mathbb{A}^2_{uv}. \end{tikzcd} \tag{1.68.6.b}$$

Thus our family π gives an analytic morphism $\tau\colon Y \to \mathbb{A}^2_{uv}$ (defined in some neighborhood of $0 \in Y$), and $C = \tau^{-1}(0,0) \subset Y$ is a curve along which the fiber has a cusp.

1.68.7 (Failure of representability)
Following (1.68.6.b), consider the universal deformation of the rational curve with two cusps. This is given as

$$\begin{tikzcd} (z^2 = (x^3 + uxy^2 + vy^3)(y^3 + syx^2 + tx^3)) \arrow[r, hook] \arrow[d, "p"'] & \mathbb{P}^2(1,1,3) \times \mathbb{A}^4_{uvst} \arrow[d] \\ \mathbb{A}^4_{uvst} \arrow[r, equal] & \mathbb{A}^4_{uvst}. \end{tikzcd}$$

Let us work in a neighborhood of $(0,0,0,0) \in \mathbb{A}^4$, where the two factors $x^3 + uxy^2 + vy^3$ and $y^3 + syx^2 + tx^3$ have no common roots. There are three types of fibers: $p^{-1}(0,0,0,0)$ is a rational curve with two cusps, $p^{-1}(a,b,0,0)$ and $p^{-1}(0,0,a,b)$ are irreducible with exactly one cusp if $(a,b) \neq (0,0)$, and $p^{-1}(a,b,c,d)$ is irreducible with at worst nodes otherwise.

Thus the curves that we allow in our moduli functor $\mathcal{M}_2^{\mathrm{irr}}$ do not form a representable family. Even worse, the subfamily

$$(z^2 = (x^3 + uxy^2 + vy^3)y^3) \to \operatorname{Spec} k[[u,v]]$$

is not allowed in our moduli functor $\mathcal{M}_2^{\mathrm{irr}}$, but the family

$$(z^2 = (x^3 + uxy^2 + vy^3)(y^3 + u^n yx^2 + v^n x^3)) \to \operatorname{Spec} k[[u,v]]$$

is allowed. Over $\operatorname{Spec} k[u,v]/(u^n, v^n)$ the two families are isomorphic. Since deformation theory is essentially a study of families over Artinian rings, this means that the usual methods cannot be applied to understand the functor $\mathcal{M}_2^{\mathrm{irr}}$.

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1.68.8 (Unusual nonseparatedness) A quite different type of problem arises at the curve corresponding to $2p_1 + 2p_2 + 2p_3$.

Write the double cover as $C = C_1 + C_2$. As before, if we contract one of the two components C_i, we get an irreducible rational curve C'_j, where $j = 3 - i$ as in (1.67.4).

Since the curves C'_1 and C'_2 are isomorphic, from the set-theoretic point of view this is a good solution. However, as in (1.27), something strange happens with families. Let $p\colon S \to \mathbb{A}^1$ be a family of stable curves whose central fiber $S_0 := p^{-1}(0)$ is isomorphic to $C = C_1 + C_2$. We have two ways to construct a family with an irreducible central fiber: contract either of the two irreducible components C_i. Thus we get two families

$$S \xrightarrow{\pi_i} S_i \xrightarrow{p_i} \mathbb{A}^1 \quad \text{with } p_i^{-1}(0) \simeq C'_{3-i}.$$

Over $\mathbb{A}^1 \setminus \{0\}$ the two families are naturally isomorphic to $S \to \mathbb{A}^1$, hence to each other, yet this isomorphism does not extend to an isomorphism of S_1 and S_2. Indeed, the closure of the graph of the resulting birational map is given by the image $(\pi_1, \pi_2)\colon S \to S_1 \times_{\mathbb{A}^1} S_2$. Thus the corresponding moduli functor is not separated.

We claimed in (1.68.1) that, by contrast, the coarse moduli space is $\overline{\mathrm{M}}_2$, hence separated. A closer study reveals the source of this discrepancy: we have been thinking of schemes instead of algebraic spaces. The occurrence of such problems in moduli theory was first observed by Artin (1974). The aim of the next paragraph is to show how such examples arise.

1.68.9 (Bug-eyed covers) (Artin, 1974); (Kollár, 1992a) A non-separated scheme always has "extra" points. The typical example is when we take two copies of a scheme $X \times \{i\}$ for $i = 0, 1$, an open dense subscheme $U \subsetneq X$, and glue $U \times \{0\}$ to $U \times \{1\}$ to get $X \amalg_U X$. The non-separatedness arises from having two points in $X \amalg_U X$ for each point in $X \setminus U$.

By contrast, an algebraic space can be nonseparated by having no extra points, only extra tangent directions. The simplest example is the following.

On $\mathbb{A}^1_t$ consider two equivalence relations. The first is $R_1 \rightrightarrows \mathbb{A}^1$ given by

$$(t_1 = t_2) \cup (t_1 = -t_2) \subset \mathbb{A}^1_{t_1} \times \mathbb{A}^1_{t_2}.$$

Then $\mathbb{A}^1_t / R_1 \simeq \mathbb{A}^1_u$ where $u = t^2$.

The second is the étale equivalence relation $R_2 \rightrightarrows \mathbb{A}^1$ given by

$$\mathbb{A}^1 \xrightarrow{(1,1)} \mathbb{A}^1 \times \mathbb{A}^1 \quad \text{and} \quad \mathbb{A}^1 \setminus \{0\} \xrightarrow{(1,-1)} \mathbb{A}^1 \times \mathbb{A}^1.$$

(Note that we take the disconnected union of the two components, instead of their union as two lines in $\mathbb{A}^1 \times \mathbb{A}^1$ intersecting at the origin.)

One can also obtain $\mathbb{A}^1_t/R_2$ by taking the quotient of the nonseparated scheme $\mathbb{A}^1 \amalg_{\mathbb{A}^1\setminus\{0\}} \mathbb{A}^1$ by the (fixed point free) involution that interchanges $(t, 0)$ and $(-t, 1)$.

The morphism $\mathbb{A}^1_t \to \mathbb{A}^1_t/R_2$ is étale, thus $\mathbb{A}^1_t/R_2 \neq \mathbb{A}^1_t/R_1$. Nonetheless, there is a natural morphism $\mathbb{A}^1_t/R_2 \to \mathbb{A}^1_t/R_1$ which is one-to-one and onto on closed points. The difference between the two spaces is seen by the tangent spaces. The tangent space of $\mathbb{A}^1_t/R_2$ at the origin is spanned by $\partial/\partial t$ while the tangent space of $\mathbb{A}^1_t/R_1$ at the origin is spanned by $\partial/\partial u = (2t)^{-1}\partial/\partial t$.

1.69 Our attempt to replace the moduli functor of stable curves of genus 2 with another one that parametrizes only irreducible curves was not successful, but some of the problems seemed to have arisen from the symmetry that forced us to make artificial choices. We can avoid such choices for other values of the genus using the following observation.

Let $\pi\colon S \to B$ be a flat family of curves with smooth general fiber and reduced special fibers. If $C_b := \pi^{-1}(b)$ is a singular fiber and C_{bi} are the irreducible components of its normalization, then

$$\textstyle\sum_i h^1(C_{bi}, \mathscr{O}_{C_{bi}}) \leq h^1(C_b, \mathscr{O}_{C_b}) = h^1(C_{gen}, \mathscr{O}_{C_{gen}}),$$

where C_{gen} is the general smooth fiber. In particular, there can be at most one irreducible component with geometric genus $> \frac{1}{2}g(C_{gen})$.

From this it is easy to prove the following:

Claim 1.69.1 Let B be a smooth curve and $S^\circ \to B^\circ$ a smooth family of genus g curves over an open subset of B. Then there is at most one normal surface $S \to B$ extending S° such that every fiber of $S \to B$ is irreducible and of geometric genus $> \frac{1}{2}g(C_{gen})$.

Moreover, if $S_{stab} \to B$ is a stable family extending S° and every fiber of $S_{stab} \to B$ contains an irreducible curve of geometric genus $> \frac{1}{2}g(C_{gen})$, then we obtain S from S_{stab} by contracting all connected components of curves of geometric genus $< \frac{1}{2}g(C_{gen})$ that are contained in the fibers. (It is not hard to show that $S \to B$ exists, at least as an algebraic space.)

In fact, this way we obtain a partial compactification $\mathrm{M}_g \subset \mathrm{M}'_g$ such that

- M'_g parametrizes smoothable irreducible curves of arithmetic genus g and geometric genus $> \frac{1}{2}g$.
- Let $\mathrm{M}_g \subset \mathrm{M}''_g \subset \overline{\mathrm{M}}_g$ be the largest open subset parametrizing curves that contain an irreducible component of geometric genus $> \frac{1}{2}g$. Then there is a natural morphism $\mathrm{M}''_g \to \mathrm{M}'_g$.

So far so good, but, as we see next, we cannot extend M'_g to a compactification in a geometrically meaningful way. This happens for every $g \geq 3$; the following example with $g = 13$ is given by simple equations.

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This illustrates a general pattern: one can easily propose partial compactifications that work well for some families, but lead to contradictions for some others. (See Schubert, 1991; Hassett and Hyeon, 2013; Smyth, 2013 for a search for geometrically meaningful compactifications of M_g.)

Example 1.70 Consider the surface $F := (x^8 + y^8 + z^8 = t^2) \subset \mathbb{P}^3(1, 1, 1, 4)$ and on it the curve $C := F \cap (xyz = 0)$. C has three irreducible components $C_x = (x = 0), C_y = (y = 0), C_z = (z = 0)$, which are smooth curves of genus 3. C itself has arithmetic genus 13.

We work with a three-parameter family of deformations

$$T := (xyz - ux^3 - vy^3 - wz^3 = 0) \subset F \times \mathbb{A}^3_{uvw}. \tag{1.70.1}$$

For general $uvw \neq 0$ the fiber of the projection $\pi\colon T \to \mathbb{A}^3$ is a smooth curve of genus 13. If one of the u, v, w is zero, then generically we get a curve with two nodes, hence with geometric genus 11.

If two of the coordinates are zero, say $v = w = 0$, then we have a family

$$T_x := (x(yz - ux^2) = 0) \subset F \times \mathbb{A}^1_u.$$

For $u \neq 0$, the fiber $C_{(u,0,0)}$ has two irreducible components. One is $C_x = (x = 0)$, the other is $(yz - ux^2 = 0)$ which is a smooth genus 7 curve.

Thus the proposed rule says that we should contract $C_x \subset C_{(u,0,0)}$.

Similarly, by working over the v and the w-axes, the rule tells us to contract $C_y \subset C_{(0,v,0)}$ for $v \neq 0$ and $C_z \subset C_{(0,0,w)}$ for $w \neq 0$.

It is easy to see that over $\mathbb{A}^3 \setminus \{(0, 0, 0)\}$ these contractions can be performed (at least among algebraic spaces). Thus we obtain

$$\begin{tikzcd} T \setminus \{\pi^{-1}(0,0,0)\} \arrow[r, "p^\circ"] \arrow[d] & S^\circ \arrow[d, "\tau^\circ"'] \\ \mathbb{A}^3 \setminus \{(0,0,0)\} \arrow[r, equal] & \mathbb{A}^3 \setminus \{(0,0,0)\} \end{tikzcd} \tag{1.70.2}$$

where τ° is flat with irreducible fibers.

Claim 1.70.3 There is no proper family of curves $\tau\colon S \to \mathbb{A}^3$ that extends τ°. (We do not require τ to be flat.)

Proof Assume to the contrary that $\tau\colon S \to \mathbb{A}^3$ exists, and let $\Gamma \subset T \times_{\mathbb{A}^3} S$ be the closure of the graph of p°. Since p° is a morphism on $T \setminus \{\pi^{-1}(0, 0, 0)\}$, we see that the first projection $\pi_1 : \Gamma \to T$ is an isomorphism away from $\pi^{-1}(0, 0, 0)$. Since $T \times_{\mathbb{A}^3} S \to \mathbb{A}^3$ has two-dimensional fibers, we conclude that

$\dim \pi_1^{-1}(\pi^{-1}(0,0,0)) \le 2$. T is, however, a smooth 4-fold, hence the exceptional set of any birational map to T has pure dimension 3. Thus $\Gamma \simeq T$ and so p° extends to a morphism $p\colon T \to S$.

Now the rule lands us in a contradiction over the origin $(0,0,0)$. Here all three components $C_x, C_y, C_z \subset C_{(0,0,0)} = C$ should be contracted. This is impossible to do since this would give that the central fiber of $S \to \mathbb{A}^3$ is a point. □

1.8 Coarse and Fine Moduli Spaces

As in (1.7), let $\mathbf{V}$ be a "reasonable" class of projective varieties and $\mathcal{V}arieties_{\mathbf{V}}$ the corresponding functor. The aim of this section is to study the difference between coarse and fine moduli spaces, mostly through a few examples. We are guided by the following:

Principle 1.71 *Let* $\mathbf{V}$ *be a "reasonable" class as above, and assume that it has a coarse moduli space* $Moduli_{\mathbf{V}}$. *Then* $Moduli_{\mathbf{V}}$ *is a fine moduli space iff* $\mathbf{Aut}(V)$, *the group of automorphisms of* V *(8.63), is trivial for every* $V \in \mathbf{V}$.

From the point of view of algebraic stacks, a precise version is given in Laumon and Moret-Bailly (2000, 8.1.1). In positive characteristic, one should pay attention to the scheme structure of $\mathbf{Aut}(V)$. Our construction of the moduli spaces shows that this principle is true for polarized varieties, see Section 8.7, but a precise version needs careful attention to the difference between schemes and algebraic spaces.

Let L be a field and $X_L \in \mathbf{V}$ an L-variety. Let $[X] \in \mathrm{Moduli}_{\mathbf{V}}$ be the corresponding point with residue field $K := k([X])$. If $\mathrm{Moduli}_{\mathbf{V}}$ is fine, then the resulting map $\operatorname{Spec} K \to \mathrm{Moduli}_{\mathbf{V}}$ corresponds to a K-variety X_K such that $X_L \simeq X_K \times \operatorname{Spec} L$. Moreover, X_K is the unique K-variety with this property.

If $\mathrm{Moduli}_{\mathbf{V}}$ is not a fine moduli space, then it is not clear how to define this field K. X_K may not be unique and may not exist. We study these questions, mostly through examples.

1.72 (Field of moduli) Let $X \subset \mathbb{P}^n$ be a projective variety defined over an algebraically closed field K. Any set of defining equations involves only finitely many elements of K, thus X can be defined over a finitely generated subfield of K. It is a natural question to ask: Is there a smallest subfield $F \subset K$ such that X can be defined by equations over F? There are two variants of this question.

1.72.1 (Embedded version) Fix coordinates on $\mathbb{P}^n_K$ and view X as a specific subvariety. In this case a smallest subfield F exists; see Weil (1946, sec.I.7) or Kollár et al. (2004, sec.3.4). This is a special case of the existence of Hilbert schemes (1.5). More generally, the same holds if $\mathbb{P}^n$ is replaced by any $\mathbb{Z}$-scheme. We can also think of this as a Galois invariance property. If $\sigma \in \operatorname{Aut}(K)$ then $\sigma(X) = X$ iff σ is the identity on F. If $\operatorname{char} K = 0$, this property characterizes F, but otherwise only its purely inseparable closure F^{ins}.

1.72.2 (Absolute version) No embedding of X is fixed. Thus we are looking for a field $F \subset K$ and an F-variety X_F such that $X \simeq (X_F)_K$. It turns out that there is no smallest field in general. As a first approximation, we call the intersection of all such fields F the *field of moduli* of X. As the examples (1.76) show, this naive version can be unexpectedly small.

The situation is better if K_X is ample, but in (1.75) we construct a hyperelliptic curve whose field of moduli is $\mathbb{Q}$, yet it cannot be defined over $\mathbb{R}$. The first such examples are in Earle (1971); Shimura (1972).

To get the right notion, we instead look for *isotrivial* families with fiber X, defined over some subfield $F \subset K$. That is, flat, projective morphisms $u : U_Z \to Z$ (defined over F), whose every geometric fiber is isomorphic to X.

We say that $u\colon U_Z \to Z$ is *universal* if every isotrivial family $v\colon U_S \to S$ with fiber X is locally the pull-back of $u\colon U_Z \to Z$. That is, there is an open cover $S = \cup_i S_i$ and morphisms $\sigma_i\colon S_i \to Z$ such that the restriction $v_i\colon U_{S_i} \to S_i$ is isomorphic to the pull-back $U \times_{u,\sigma_i} S_i \to S_i$.

We see in (1.73) that universal isotrivial families exist and they are defined over the same subfield $F_X \subset K$, giving the right notion of field of moduli. How is this connected with moduli theory?

Let $\mathbf{V}$ be a class of varieties with a coarse moduli space $\mathrm{Moduli}_{\mathbf{V}}$. Let $u\colon U_Z \to Z$ be an isotrivial family with fiber X defined over $F \subset K$. By the definition of a coarse moduli space, there is a morphism $Z \to \mathrm{Moduli}_{\mathbf{V}}$, whose image must be the point $[X] \in \mathrm{Moduli}_{\mathbf{V}}$ corresponding to X. In particular, we get an injection of the residue field $k([X])$ into F.

If $\mathrm{Moduli}_{\mathbf{V}}$ is a fine moduli space, then X can be defined over $k([X])$, and (1.73.2) shows that $k([X]) = F_X$.

The construction of the moduli spaces of stable varieties shows that the extension $F_X/k([X])$ is purely inseparable, hence trivial in characteristic 0.

Proposition 1.73 *Let K be an algebraically closed field of characteristic* 0 *and X a projective K-variety with ample canonical class. Then there is a unique smallest field $F_X \subset K$ – called the field of moduli of X – such that there is a geometrically irreducible, universal, isotrivial family $u\colon U \to Z$ with fiber X, defined over F_X. Moreover, $X \simeq X^\sigma$ for every $\sigma \in \operatorname{Gal}(K/F_X)$.*

Proof Fix m such that $|mK_X|$ is very ample, giving an embedding $X \hookrightarrow \mathbb{P}^N$. The image depends on a choice of a basis in $H^0(X, \mathscr{O}_X(mK_X))$, so instead of getting a point in $\mathrm{Chow}(\mathbb{P}^N)$ or $\mathrm{Hilb}(\mathbb{P}^N)$, we get a whole $\mathbf{Aut}(\mathbb{P}^N)$-orbit. Denote it by Z (it depends on X and m). Over it we have a universal family $u\colon U_Z \to Z$, which is isotrivial with fiber X.

The closure of Z is now a closed subvariety of the $\mathbb{Z}$-schemes $\mathrm{Chow}(\mathbb{P}^N)$ or $\mathrm{Hilb}(\mathbb{P}^N)$, thus it has a smallest field of definition by (1.72.1). This is our F_X.

To see that $u\colon U_Z \to Z$ is universal, let $v\colon V_S \to S$ be an isotrivial family with fiber X. Then $v_*\mathscr{O}_{V_S}(mK_{V_S/S})$ is locally free. Choose an open trivializing cover $S = \cup_i S_i$. These define embeddings $V_{S_i} \hookrightarrow \mathbb{P}^N \times S_i$, hence morphisms $\sigma_i : S_i \to Z$.

For a subvariety $X \subset \mathbb{P}^n_K$, let $[X] \in \mathrm{Hilb}(\mathbb{P}^N)$ denote the corresponding point. Then $[X^\sigma] = \sigma[X]$, hence the last claim is a reformulation of the Galois invariance property noted in (1.72.1). □

1.74 (Field of moduli for hyperelliptic curves) Let A be a smooth hyperelliptic curve of genus ≥ 2. Over an algebraically closed field, A has a unique degree 2 map to $\mathbb{P}^1$. Let $B \subset \mathbb{P}^1$ be the branch locus, that is, a collection of $2g+2$ points in $\mathbb{P}^1$. If the base field k is not closed, then A has a unique degree 2 map to a smooth genus 0 curve Q. (One can always think of Q as a conic in $\mathbb{P}^2$.) Thus A is defined over a field k iff the pair $(B \subset \mathbb{P}^1)$ can be defined over k.

The latter problem is especially transparent if A is defined over $\mathbb{C}$, and we want to know if it is defined over $\mathbb{R}$ or if its field of moduli is contained in $\mathbb{R}$.

Up to isomorphism, there are two real forms of $\mathbb{P}^1$. One is $\mathbb{P}^1$, corresponding to the antiholomorphic involution $(x{:}y) \mapsto (\bar{x}{:}\bar{y})$, which, after a coordinate change, can also be written as $\sigma_1\colon (x{:}y) \mapsto (\bar{y}{:}\bar{x})$. (In the latter, the real points are the unit circle.) The other is the "empty" conic, corresponding to the antiholomorphic involution $\sigma_2 : (x{:}y) \mapsto (-\bar{y}{:}\bar{x})$.

Thus let $A \to \mathbb{CP}^1$ be a smooth hyperelliptic curve of genus ≥ 2 over $\mathbb{C}$ with branch locus $B \subset \mathbb{CP}^1$. Then (1.72.5) gives that

(1.74.1) A can be defined over $\mathbb{R}$ iff there is a $g \in \mathrm{Aut}(\mathbb{CP}^1)$ such that gB is invariant under σ_1 or σ_2, and

(1.74.2) the field of moduli of A is contained in $\mathbb{R}$ iff there is an $h \in \mathrm{Aut}(\mathbb{CP}^1)$ such that hB equals B^{σ_1} or B^{σ_2}.

Note that if $(gB)^\sigma = gB$ then $B^\sigma = (g^\sigma)^{-1}gB$ shows that (1) ⇒ (2). Conversely, if $B^\sigma = hB$ and we can write $h = (g^\sigma)^{-1}g$ then $(gB)^\sigma = gB$.

Example 1.75 Here is an example of a hyperelliptic curve C whose field of moduli is $\mathbb{Q}$, but C cannot be defined over $\mathbb{R}$.

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Pick $\alpha = a + ib$ where a, b are rational. Consider the hyperelliptic curve

$$C(\alpha) := \left(z^2 - (x^8 - y^8)(x^2 - \alpha y^2)(\bar{\alpha}x^2 + y^2) = 0\right) \subset \mathbb{P}^2(1,1,6).$$

Its complex conjugate is

$$C(\bar{\alpha}) := \left(z^2 - (x^8 - y^8)(x^2 - \bar{\alpha}y^2)(\alpha x^2 + y^2) = 0\right) \subset \mathbb{P}^2(1,1,6).$$

$C(\alpha)$ and $C(\bar{\alpha})$ are isomorphic, as shown by the substitution $(x, y, z) \mapsto (iy, x, z)$. So, over $\operatorname{Spec}_{\mathbb{Q}} \mathbb{Q}[t]/(t^2 + 1)$ we have a curve

$$C(a,b) := \left(z^2 - (x^8 - y^8)(x^2 - (a + tb)y^2)((a - tb)x^2 + y^2) = 0\right) \subset \mathbb{P}^2(1,1,6)$$

whose geometric fibers are isomorphic to $C(\alpha)$. Thus the field of moduli of $C(\alpha)$ is $\mathbb{Q}$ by (1.72.5).

We claim that, for sufficiently general a, b, the curve $C(\alpha)$ cannot be defined over $\mathbb{Q}$, not even over $\mathbb{R}$. By (1.74) we need to show that there is no anti-holomorphic involution that maps the branch locus to itself. In the affine chart $y \neq 0$, the ramification points of $C(\alpha) \to \mathbb{P}^1$ are:

(1.75.1) the 8th roots of unity corresponding to $x^8 - y^8$, and

(1.75.2) the four points $\pm\beta, \pm i/\bar{\beta}$ where $\beta^2 = \alpha$.

The anti-holomorphic automorphisms of the Riemann sphere map circles to circles. Out of our 12 points, the 8th roots of unity lie on the circle $|z| = 1$, but no other 8 can lie on a circle. Thus any antiholomorphic automorphism that maps our configuration to itself, must fix the unit circle $|z| = 1$ and map the 8th roots of unity to each other.

The only such antiholomorphic involutions are

(1.75.3) reflection on the line $\mathbb{R} \cdot \varepsilon$, where ε is a 16th root of unity, and

(1.75.4) $z \mapsto 1/\bar{z}$ or $z \mapsto -1/\bar{z}$.

A short analysis shows that $C(\alpha)$ is not isomorphic (over $\mathbb{C}$) to a real curve, as long as β^{16} is not a positive real number.

Example 1.76 We give an example of a smooth projective surface S such that if S is defined over a field extension $K/\mathbb{C}$ then $\operatorname{trdeg}_{\mathbb{C}} K = 2$, but the intersection of all such fields of definition is $\mathbb{C}$.

Let X be a projective surface such that $\operatorname{Aut}(X)$ is discrete and contains finite subgroups G_1, G_2 such that $\langle G_1, G_2 \rangle$ has a Zariski dense orbit on X.

One such example is $B_0(E \times E)$, the blow-up of the square of an elliptic curve at a point, as shown by the subgroups generated by the matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

There are also K3 surfaces with infinite automorphism group generated by two involutions (1.66).

Let $\Delta \subset X \times X$ be the diagonal and, using one of the projections, consider the family of smooth varieties $f\colon Y := B_\Delta(X \times X) \to X$. Our example is $K = \mathbb{C}(X)$ and Y_K the generic fiber of $Y \to X$.

Note that $Y \to X$ is the universal family of the varieties of the form $B_x X$ for $x \in X$. This shows that Y_K cannot be obtained by base change from a variety over a field of smaller transcendence degree over $\mathbb{C}$.

Let $G \subset \operatorname{Aut}(X)$ be a finite subgroup. There is an open subset $U_G \subset X$ such that G operates on U_G without fixed points. Thus $f/G\colon Y/G \to X/G$ is a family of smooth varieties over U_G/G and $Y|_{U_G} \simeq Y/G \times_{X/G} U_G$. Thus Y_K can be defined over $\mathbb{C}(X/G) = K^G$ for every finite subgroup $G \subset \operatorname{Aut}(X)$.

On the other hand, the intersection $K^{G_1} \cap K^{G_2}$ is $\mathbb{C}$. Indeed, any function in $K^{G_1} \cap K^{G_2}$ is constant on every G_1-orbit and also on every G_2-orbit, hence on a dense set by our assumptions.

This phenomenon is also connected with the behavior of ample line bundles on $\pi_i\colon X \to X/G_i$. Although both of the X/G_i are projective, there are no ample line bundles L_i on X/G_i such that $\pi_1^* L_1 \simeq \pi_2^* L_2$.

1.77 (Openness of the fine locus) Let $\mathbf{V}$ be a "reasonable" class of varieties with a coarse moduli space $\operatorname{Moduli}_{\mathbf{V}}$.

If $\mathbf{Aut}(X) = \{1\}$ is an open condition in flat families with fibers in $\mathbf{V}$, then there is an open subscheme $\operatorname{Moduli}_{\mathbf{V}}^{\mathrm{rigid}} \subset \operatorname{Moduli}_{\mathbf{V}}$ that is a coarse moduli space for varieties in $\mathbf{V}$ without automorphisms. By (1.71), $\operatorname{Moduli}_{\mathbf{V}}^{\mathrm{rigid}}$ should be a fine moduli space. In many cases, $\operatorname{Moduli}_{\mathbf{V}}^{\mathrm{rigid}}$ is dense in $\operatorname{Moduli}_{\mathbf{V}}$, thus one can understand much about the whole $\operatorname{Moduli}_{\mathbf{V}}$ by studying the fine moduli space $\operatorname{Moduli}_{\mathbf{V}}^{\mathrm{rigid}}$.

Let $X \to S$ be a flat family with fibers in $\mathbf{V}$ and $\pi\colon \mathbf{Aut}_S(X) \to S$ the scheme representing automorphisms of the fibers (8.63). If $\mathbf{V}$ satisfies the valuative criterion of separatedness (1.20), and all automorphism groups are finite, then π is proper. More careful attention to the scheme structure of the automorphism groups shows that in fact $\mathbf{Aut}(X) = \{1\}$ is an open condition.

However, automorphism groups of smooth, projective surfaces can jump unexpectedly. For example, the automorphism group of a general Enriques surface is infinite, but there are special Enriques surfaces with finite automorphism group. A more elementary example is the following:

Example 1.77.1 Let ζ be a primitive mth root of unity. Then $\tau(x{:}y{:}z) = (\zeta x{:}y{:}z)$ defines a $\mathbb{Z}/m$-action on $\mathbb{P}^2$. For $t \neq 0$, let S_t be the surface obtained by blowing up the m points $(\zeta^i t{:}t{:}1)$.

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What should $\lim_{t\to 0} S_t$ be? A natural candidate is to blow up first (0:0:1) and then the m intersection points p_i of the exceptional curve E with the birational transforms of the lines $L_i := (x = \zeta^i y)$. The resulting S_0 has a $\mathbb{Z}/m$-action, but we blew up $m + 1$-times, so there is no family of smooth surfaces with fibers $\{S_t \colon t \in \mathbb{C}\}$.

As in (1.24), for any $j \in \mathbb{Z}/m$ we can get a smooth family of surfaces with central fiber S_0^j, obtained by blowing up first (0:0:1) and then all the p_i for $i \neq j$. These give m distinct families, and we do not have a $\mathbb{Z}/m$-action on any of these S_0^j.