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Profinite detection of 3-manifold decompositions

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ABSTRACT

The profinite completion of the fundamental group of a closed, orientable 3-manifold determines the Kneser–Milnor decomposition. If M is irreducible, then the profinite completion determines the Jaco–Shalen–Johannson decomposition of M.

When trying to distinguish two compact 3-manifolds M, N, in practice the easiest method is often to compute some finite quotients of their fundamental groups, and notice that there is a finite group Q which is a quotient of $\pi_1 M$, say, but not of $\pi_1 N$. It would be very useful, both theoretically and in practice, to know that this method always works. The set of finite quotients of a group Γ is encoded by the *profinite completion* $\widehat{\Gamma}$ (the inverse limit of the system of finite quotient groups), and so one is naturally led to the following question.

Question 0.1. Let M be a compact, orientable 3-manifold. To what extent is $\pi_1 M$ determined by its profinite completion?

In particular, if M is determined among all compact, orientable 3-manifolds by $\widehat{\pi_1 M}$, then M is said to be *profinitely rigid*. If there are at most finitely many compact, orientable 3-manifolds N with $\widehat{\pi_1 M} \cong \widehat{\pi_1 N}$, then M is said to have *finite genus*. More precise versions of Question 0.1 ask which 3-manifolds are profinitely rigid, which have finite genus and whether various properties of M are determined by $\widehat{\pi_1 M}$.

The results of this paper show that the profinite completion $\widehat{\pi_1 M}$ determines both the Kneser–Milnor and the Jaco–Shalen–Johannson (JSJ) decompositions of M. This complements our previous results showing that $\widehat{\pi_1 M}$ determines the geometry of M [WZ17]. The first theorem concerns the Kneser–Milnor decomposition.

THEOREM A. *Let M, N be closed, orientable 3-manifolds with Kneser–Milnor decompositions $M \cong M_1 \# \cdots \# M_m \# r(S_1 \times S_2)$ and $N \cong N_1 \# \cdots \# N_n \# s(S_1 \times S_2)$. If $\widehat{\pi_1 M} \cong \widehat{\pi_1 N}$ then $m = n$, $r = s$, and, up to re-indexing, the image of $\widehat{\pi_1 M_i}$ is conjugate to $\widehat{\pi_1 N_i}$ for each i.*

In particular, $\widehat{\pi_1 M}$ determines whether or not M is irreducible. While this work was in progress we discovered that a similar result has also been proved in the pro-p setting by Wilkes, using l^2-cohomology [Wil17a, Proposition 6.2.4]. Our proof is different, using the continuous cohomology of the profinite completion, and naturally generalizes to our next theorem, which shows that the profinite completion determines the JSJ decomposition of M.

THEOREM B. *Let M and N be closed, orientable, irreducible 3-manifolds, and suppose $\widehat{\pi_1 M} \cong \widehat{\pi_1 N}$. Then the underlying graphs of the JSJ decompositions of $\pi_1 M, \pi_1 N$ are isomorphic, and corresponding vertex groups have isomorphic profinite completions.*

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See Theorem 4.3 for a more precise statement, phrased in terms of *profinite Bass–Serre trees*. Partial results along the lines of Theorem B have also been obtained by Wilkes [Wil17a, Theorem I].

In [WZ17, Theorem 8.4], it was shown that the profinite completion of the fundamental group of a closed, orientable 3-manifold M determines the geometry of M. As an immediate consequence of Theorem B, we can extend this result to the case with toral boundary. Recall that $\overline{H}$ denotes the closure of a subgroup H in the profinite completion $\overline{G}$.

COROLLARY C. *Let M, N be compact, orientable, irreducible 3-manifolds with non-empty toral boundaries; let $\{P_1, \ldots, P_m\}$ and $\{Q_1, \ldots, Q_n\}$ be conjugacy representatives for the boundary subgroups of $\pi_1 M$ and $\pi_1 N$, respectively. Suppose that $\widehat{\pi_1 M} \cong \widehat{\pi_1 N}$, that $m = n$, and that the isomorphism takes $\overline{P_i}$ to $\overline{Q_i}$ for each i. If M is geometric then N is also geometric, with the same geometry. In particular, M is Seifert fibred if and only if N is Seifert fibred.*

Proof. Let D be the double of M and E be the double of N. Note that $\widehat{\pi_1 D} \cong \widehat{\pi_1 E}$, and also that the isomorphism respects the profinite completions of the boundary subgroups of $\pi_1 M$ and $\pi_1 N$. The result follows from the observation that the geometry of M is reflected in properties of the double D.

Indeed, if M is Seifert fibred then so is D. In this case, if M is homeomorphic to an interval bundle over the torus then D has Euclidean geometry; otherwise, M and D both have $\mathbb{H}^2 \times \mathbb{R}$ geometry. Finally, M is hyperbolic if and only if D has non-trivial JSJ decomposition, and the boundary tori of M are the only JSJ tori of D. Combining these facts with Theorem B and [WZ17, Theorem 8.4], the result follows. □

In light of Theorem B, the next step in addressing Question 0.1 is to consider the pieces of the JSJ decomposition. The Seifert fibred case has been resolved by Wilkes [Wil17b], building on work of Hempel [Hem14]: Seifert fibred 3-manifolds are not profinitely rigid, but do have finite genus, and Wilkes was able to give a complete description of when two such 3-manifold groups have isomorphic profinite completions; he was subsequently able to extend this to a complete answer to Question 0.1 for graph manifolds [Wil18a, Theorem 10.9]. In that paper, Sol-manifolds were not included in the class of graph manifolds. Nevertheless, Sol-manifolds are also well understood: they are not profinitely rigid [Fun13], but do have finite genus [GPS80]. A definitive treatment of the case of Sol-manifolds would be a valuable addition to the literature.

The case of hyperbolic 3-manifolds remains an important open problem. The complement of the figure-eight knot was shown to be profinitely rigid by Boileau and Friedl [BF15] and by Bridson and Reid [BR15]; see also [BCR16] for analogous results for Fuchsian groups and [BRW17] for once-punctured-torus bundles.

Both [BF15, BR15], as well as [BRW17], rely on results showing that fibredness is a profinite invariant in certain contexts. This has recently been proved in full generality by Jaikin-Zapirain [Jai17]. Ueki also recently showed that the profinite completion of a knot group determines the Alexander polynomial of the knot [Uek18].

The results of this paper are proved by considering profinite Poincaré duality groups. The main difficulty in the above theorems is to show that profinite completions of 3-manifold groups do not admit unexpected splittings which are not induced by splittings of the underlying group. It is well known that non-splitting theorems for discrete Poincaré duality groups follow from the Mayer–Vietoris sequence. As a result of the work of Agol, Wise and others on the virtual Haken conjecture [Ago13, Wis12], 3-manifold groups are known to be *good* in the sense of Serre, meaning

that the cohomology of the profinite completion is isomorphic to the ordinary cohomology (with coefficients in finite modules). Furthermore, a version of the Mayer–Vietoris sequence is known for efficient decompositions of profinite completions. The main idea of the proofs of Theorems A and B is to prove the analogues for profinite completions of the non-splitting theorems from the discrete case.

1. Preliminaries on profinite groups

1.1 Profinite trees

A *graph* Γ is a disjoint union $E(\Gamma) \cup V(\Gamma)$ of sets, with two maps $d_0, d_1 : \Gamma \to V(\Gamma)$ that are the identity on the set of vertices $V(\Gamma)$. For an element e of the set of edges $E(\Gamma)$, $d_0(e)$ is called the *initial* and $d_1(e)$ the *terminal* vertex of e.

DEFINITION 1.1. A *profinite graph* Γ is a graph such that:

 (i) Γ is a profinite space (i.e. an inverse limit of finite discrete spaces);
 (ii) $V(\Gamma)$ is closed; and
 (iii) the maps d_0 and d_1 are continuous.

Note that $E(\Gamma)$ is not necessary closed.

A *morphism* $\alpha : \Gamma \longrightarrow \Delta$ of profinite graphs is a continuous map with $\alpha d_i = d_i \alpha$ for $i = 0, 1$.

By [ZM88, Proposition 1.7] or [Rib17, Proposition 2.1.4] every profinite graph Γ is an inverse limit of finite quotient graphs of Γ.

For a profinite space X that is the inverse limit of finite discrete spaces X_j, $[[\widehat{\mathbb{Z}}X]]$ is defined to be the inverse limit of $[\widehat{\mathbb{Z}}X_j]$, where $[\widehat{\mathbb{Z}}X_j]$ is the free $\widehat{\mathbb{Z}}$-module with basis X_j. For a pointed profinite space $(X, *)$ that is the inverse limit of pointed finite discrete spaces $(X_j, *)$, $[[\widehat{\mathbb{Z}}(X, *)]]$ is the inverse limit of $[\widehat{\mathbb{Z}}(X_j, *)]$, where $[\widehat{\mathbb{Z}}(X_j, *)]$ is the free $\widehat{\mathbb{Z}}$-module with basis $X_j \backslash \{*\}$ [RZ10, ch. 5.2].

For a profinite graph Γ, define the pointed space $(E^*(\Gamma), *)$ as $\Gamma/V(\Gamma)$ with the image of $V(\Gamma)$ as a distinguished point $*$, and denote the image of $e \in E(\Gamma)$ by $\bar{e}$.

DEFINITION 1.2. A *profinite tree* Γ is a profinite graph such that the sequence

$$0 \to [[\widehat{\mathbb{Z}}(E^*(\Gamma), *)]] \xrightarrow{\delta} [[\widehat{\mathbb{Z}}V(\Gamma)]] \xrightarrow{\epsilon} \widehat{\mathbb{Z}} \to 0$$

is exact, where $\delta(\bar{e}) = d_1(e) - d_0(e)$ for every $e \in E(\Gamma)$ and $\epsilon(v) = 1$ for every $v \in V(\Gamma)$.

If v and w are elements of a profinite tree T, we denote by $[v, w]$ the smallest profinite subtree of T containing v and w and call it a *geodesic* (cf. [ZM88, 1.19] or [Rib17, Proposition 2.4.9]).

By definition a profinite group G *acts* on a profinite graph Γ if we have a continuous action of G on the profinite space Γ that commutes with the maps d_0 and d_1.

We shall need the following lemma; its proof is contained in the first eight lines of the proof of [Zal90, Lemma 2.3].

LEMMA 1.3. *Suppose that a profinite group G acts on a profinite tree T and does not fix any vertex. Then there exists an open normal subgroup U of G that is not generated by its vertex stabilizers.*

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When we say that $\mathcal{G}$ is a *finite graph of profinite groups* we mean that $\mathcal{G}$ contains the data of the underlying finite graph, the edge profinite groups, the vertex profinite groups and the attaching continuous maps. More precisely, let Δ be a connected finite graph. The data of a graph of profinite groups $(\mathcal{G}, \Delta)$ over Δ consists of a profinite group $\mathcal{G}(m)$ for each $m \in \Delta$, and continuous monomorphisms $\partial_i : \mathcal{G}(e) \longrightarrow \mathcal{G}(d_i(e))$ for each edge $e \in E(\Delta)$.

The definition of the profinite fundamental group of a connected profinite graph of profinite groups is quite involved (see [ZM89] or [Rib17, ch. 6]). However, the profinite fundamental group $\Pi_1(\mathcal{G}, \Gamma)$ of a finite graph of finitely generated profinite groups $(\mathcal{G}, \Gamma)$ can be defined as the profinite completion of the abstract (usual) fundamental group $\Pi_1^{\mathrm{abs}}(\mathcal{G}, \Gamma)$ (using here the fact that every subgroup of finite index in a finitely generated profinite group is open [NS07, Theorem 1.1]). The fundamental profinite group $\Pi_1(\mathcal{G}, \Gamma)$ has the following presentation:

$$\Pi_1(\mathcal{G}, \Gamma) = \langle\, \mathcal{G}(v), t_e \mid rel(\mathcal{G}(v)), \partial_1(g) = \partial_0(g)^{t_e}, g \in \mathcal{G}(e), t_e = 1 \text{ for } e \in T \,\rangle, \qquad (1)$$

where T is a maximal subtree of Γ and $\partial_0 : \mathcal{G}(e) \longrightarrow \mathcal{G}(d_0(e))$, $\partial_1 : \mathcal{G}(e) \longrightarrow \mathcal{G}(d_1(e))$ are monomorphisms.

In contrast to the abstract case, the vertex groups of $(\mathcal{G}, \Gamma)$ do not always embed in $\Pi_1(\mathcal{G}, \Gamma)$. If they do embed, $(\mathcal{G}, \Gamma)$ is called *injective*. If $(\mathcal{G}, \Gamma)$ is not injective the edge and vertex groups can be replaced by their images in $\Pi_1(\mathcal{G}, \Gamma)$, and after this replacement $(\mathcal{G}, \Gamma)$ becomes injective (see [Rib17, § 6.4]).

The profinite fundamental group $\Pi_1(\mathcal{G}, \Gamma)$ acts on the standard profinite tree T (defined analogously to the abstract Bass–Serre tree) associated to it, with vertex and edge stabilizers being conjugates of vertex and edge groups, and such that $\Pi_1(\mathcal{G}, \Gamma) \backslash T = \Gamma$ [ZM88, Proposition 3.8] or [Rib17, Theorem 6.3.5]. In particular, this applies to the cases of an amalgamated free product $G = G_1 \amalg_H G_2$ (Γ is an edge with two vertices) and an HNN extension $G = G_1 \amalg_H$ (Γ is a loop); if $(\mathcal{G}, \Gamma)$ is injective and, in the case of an amalgamated free product, $G_1 \neq H \neq G_2$, we say that G *splits* over H.

Example 1.4. If $G = \pi_1(\mathcal{G}, \Gamma)$ is the fundamental group of a finite graph of (abstract) groups then one has the induced graph of profinite completions of edge and vertex groups $(\widehat{\mathcal{G}}, \Gamma)$ and a natural homomorphism $G = \pi_1(\mathcal{G}, \Gamma) \longrightarrow \Pi_1(\widehat{\mathcal{G}}, \Gamma)$. It is an embedding if $\pi_1(\mathcal{G}, \Gamma)$ is residually finite. In this case $\Pi_1(\widehat{\mathcal{G}}, \Gamma)$ is simply the profinite completion $\widehat{G}$. Moreover, if the edge groups $\mathcal{G}(e)$ are separable in G then the standard tree T_G naturally embeds in the standard profinite tree $\widehat{T}_G$ (see [Cot13, Proposition 2.5]). In particular, this is the case if edge groups are finitely generated and G is subgroup separable.

1.2 Profinite Poincaré duality groups

In this section we collect the facts about profinite groups that we will need. The following results are all profinite analogues of well-known results in the setting of discrete groups. Let $\mathbb{Z}_p$ denote the ring of p-adic integers.

DEFINITION 1.5 [SW00]. Let p be a prime. A profinite group G of type p-FP_∞ is called a *Poincaré duality group* at p of dimension n if $cd_p(G) = n$ and

$$H^i(G, \mathbb{Z}_p[[G]]) = 0 \quad \text{if } i \neq n,$$
$$H^n(G, \mathbb{Z}_p[[G]]) \cong \mathbb{Z}_p \quad \text{(as abelian groups)}.$$

We say that such a group G is a *profinite PD^n-group at p*.

If G is a profinite group with $cd_p(G) < \infty$ and U is an open subgroup of G, then G is a profinite PD^n-group at p if and only if U is a profinite PD^n-group at p (see [SW00, Remark 4.2.9]).

The proofs of our main results rely on the following lemma. In the discrete case, the corresponding result is Strebel's theorem [Str77]. In the profinite case, this is from [Ser97, Exercise 5(b) on p. 44]. The reader is referred to [RZ10, § 2.3] for the definition of *supernatural numbers*.

LEMMA 1.6. *Let G be a profinite PD^n group at p and H a closed subgroup of G such that the supernatural number p^∞ divides $[G : H]$. Then $cd_p(H) < n$.*

The following theorem is the profinite analogue of the well-known fact in the discrete setting that PD^n groups cannot split over groups of much smaller cohomological dimension [DD89, Proposition V.7.4].

THEOREM 1.7. *Suppose that G is a profinite PD^n group at every prime p. If G acts on a profinite tree T with edge stabilizers of $cd(G_e) < n - 1$, then G fixes a vertex.*

Proof. By [ZM88, paragraph 2.7],

$$\operatorname{cd} G \leqslant \sup\{\operatorname{cd} G_v, \operatorname{cd} G_e + 1 \mid v \in V(T), e \in E(T)\}.$$

Suppose that G acts on T without fixing a vertex. We now argue that there exists p such that the supernatural number p^∞ divides $[G : G_v]$ for every $v \in V(\Gamma)$, and deduce a contradiction from Lemma 1.6.

By [Zal90, Lemma 1.5] or [Rib17, Proposition 2.4.12] we may assume that the action of G on T is irreducible (i.e. does not contain proper G-invariant subtrees). If K is the kernel of the action then G/K acts faithfully on T. Hence by [Zal90, Proposition 2.10 and Lemma 2.7] or [Rib17, Theorem 4.2.10] G/K contains a free pro-p subgroup acting freely on T and therefore so does G, whence $p^\infty \mid [G : G_v]$ as claimed. □

We will apply these results to discrete groups Γ such that the cohomology of Γ is closely intertwined with the cohomology of the profinite completion $\widehat{\Gamma}$. Serre called such groups 'good' [Ser97, I.2.6].

DEFINITION 1.8. A discrete group Γ is *good (in the sense of Serre)* if, for any finite Γ-module M, the natural map to the profinite completion $\Gamma \to \widehat{\Gamma}$ induces an isomorphism $H^*(\Gamma, M) \cong H^*(\widehat{\Gamma}, M)$ (where the cohomology of the profinite group $\widehat{\Gamma}$ is defined using the continuous Hom functor).

It has been noticed in various places (e.g. [Cav12, AFW15]; cf. [GJZ08]) that 3-manifold groups are good. For convenience, we record the result here.

THEOREM 1.9. *If M is a closed 3-manifold then $\pi_1 M$ is good.*

Proof. Since goodness passes to finite extensions, we may assume that M is orientable. By [WZ10, Proposition 4.3] and the usual Kneser–Milnor and JSJ decompositions, it suffices to prove that Seifert fibred and hyperbolic 3-manifold groups are good. The Seifert fibred case is [WZ10, Proposition 4.2], and the case of closed hyperbolic 3-manifolds follows from the virtually fibred theorem [Ago13], by [GJZ08, Lemmas 3.2 and 3.3]. □

The next result is the subject of [KZ08, Theorem 4.1] for PD^3-groups, and for general n the proof can be repeated replacing 3 by n.

THEOREM 1.10. *If Γ is a good PD^n group, then $\widehat{\Gamma}$ is PD^n at every p.*

We immediately obtain a profinite non-splitting result for good Poincaré duality groups.

COROLLARY 1.11. *Let G be PD^n group which is good in the sense of Serre. Then any action of $\widehat{G}$ on a profinite tree with edge stabilizers of cohomological dimension $n-2$ has a global fixed point.*

Proof. Since G is good, $\widehat{G}$ is a profinite PD^n group at p for every p by Theorem 1.10, so the result follows from Theorem 1.7. □

Remark 1.12. The combined hypotheses of goodness and PD^n apply to many examples in dimensions 2 and 3, but are restrictive in higher dimensions.

Combining all of the above results, we obtain the following fact, which will be extremely useful to us in what follows.

COROLLARY 1.13. *If M is a closed, orientable, irreducible 3-manifold then any action of $\widehat{\pi_1 M}$ on a profinite tree with procyclic edge stabilizers has a global fixed point.*

Proof. By the sphere theorem, irreducible 3-manifolds either have finite fundamental group or are aspherical (see, for instance, [AFW15, (C.1)]). In the first case $\widehat{\pi_1 M}$ is finite, and the result follows from [ZM88, Theorem 2.10] or [Rib17, Theorem 4.1.8]. In the second case, $\pi_1 M$ is PD^3, and the result follows from Theorem 1.9 and Corollary 1.11. □

2. The Kneser–Milnor decomposition

As a warm-up, we show that the profinite completion of a 3-manifold group determines its Kneser–Milnor decomposition. As noted above, this result can also be obtained using methods from l^2-cohomology [Wil17a]. Recall that a closed 3-manifold M is *irreducible* if every embedded 2-sphere bounds a 3-ball; equivalently, $\pi_1 M$ does not admit a non-trivial splitting over the trivial subgroup.

PROPOSITION 2.1. *Suppose that M_1, M_2 are closed, orientable 3-manifolds. If $\widehat{\pi_1 M_1} \cong \widehat{\pi_1 M_2}$ and M_1 is irreducible then so is M_2.*

Proof. If M_2 were reducible then $\pi_1 M_2$ would act on a tree with trivial edge stabilizers and without a global fixed point, and $\widehat{\pi_1 M_1} \cong \widehat{\pi_1 M_2}$ would act on a profinite tree with trivial edge stabilizers and without a global fixed point. This contradicts Corollary 1.13. □

Non-irreducible 3-manifolds admit non-trivial *Kneser–Milnor decompositions*. If M is a closed, oriented 3-manifold then the Kneser–Milnor decomposition decomposes M as a connect sum

$$M \cong N_1 \# \cdots \# N_m \# F_r,$$

where each N_i is irreducible and F_r is a connect sum of copies of $S^1 \times S^2$. The N_i are uniquely determined, in an appropriate sense. In particular, the conjugacy classes of the subgroups $\pi_1 N_i$ are unique up to reordering, and the integer r is also unique. The reader is referred to [AFW15, Theorem 1.2.1] for details.

Theorem 2.2 (Profinite Kneser–Milnor). *Consider closed, orientable 3-manifolds with Kneser–Milnor decompositions $M = N_1 \# \cdots \# N_m \# F_r$ and $M' = N_1' \# \cdots \# N_{m'}' \# F_{r'}$, where each N_i and N_j' is irreducible and F_r and $F_{r'}$ are connect sums of $S^1 \times S^2$. If $\widehat{\pi_1 M} \cong \widehat{\pi_1 M'}$ then $m = m'$, $r = r'$, and, up to reordering, $\widehat{\pi_1 N_i}$ is conjugate to $\widehat{\pi_1 N_i'}$ for each i.*

Proof. Let S be the Bass–Serre tree of the corresponding decomposition of $\pi_1 M$, and let $\widehat{S}$ be the corresponding profinite tree for $\widehat{\pi_1 M}$ on which $\pi_1 M$ acts with trivial edge stabilizers. By Corollary 1.13, each profinite completion $\widehat{\pi_1 N_i'}$ fixes a vertex of $\widehat{S}$, and hence is conjugate into some $\widehat{\pi_1 N_j}$. By symmetry, each $\widehat{\pi_1 N_i}$ is conjugate into some $\widehat{\pi_1 N_j'}$. Profinite subgroups cannot be conjugate to proper subgroups of themselves, as it would imply the same for some finite image, and for a finite group it is clearly impossible. Therefore, it follows that $m = m'$ and the profinite completions of the vertex groups are conjugate. Factoring the normal closures of these subgroups out, we see that $\widehat{F_r} \cong \widehat{F_{r'}}$ and hence $r = r'$ as claimed. $\qquad\square$

3. Cusped hyperbolic 3-manifolds

An immediate consequence of Corollary 1.11 is that, for a closed 3-manifold M, $\widehat{\pi_1 M}$ does not split over a subgroup of cohomological dimension 0 or 1 (e.g. a profinite free group). In this section we prove some profinite non-splitting results for hyperbolic manifolds with toral boundary. In the hyperbolic case, we will need a fact from [WZ17], describing the non-procyclic abelian subgroups of $\widehat{\pi_1 N}$.

Proposition 3.1. *Let N be a finite-volume hyperbolic 3-manifold and A a closed abelian subgroup of $\widehat{\pi_1 N}$. If A is not procyclic then A is in the closure of a peripheral subgroup of $\pi_1 N$, and this peripheral subgroup is unique up to conjugacy.*

Proof. By [WZ17, Theorem 9.3], A is conjugate into the closure of a peripheral subgroup, and by [WZ17, Lemma 4.5], the conjugacy class of the cusp subgroup is unique. $\qquad\square$

In the classical PD^n setting, one handles manifolds with boundary using the theory of PD^n *pairs* [Dic80]. One of the upshots of this theory is that the fundamental group of an aspherical manifold with aspherical boundary cannot split over a boundary subgroup, relative to the collection of boundary subgroups. (This can be deduced from the results of [KR88].) No doubt the profinite analogue of this statement can be proved by developing the theory of profinite PD^n pairs.[1] We take a quicker route here: we prove the result in the cusped hyperbolic case, using Dehn filling. First, we need to recall the definition of an *acylindrical* splitting.

Definition 3.2. An action of a group Γ on a tree T is *k-acylindrical* (for an integer k) if, for every $\gamma \in \Gamma \smallsetminus 1$, the subtree fixed by γ is either empty or of diameter at most k. Likewise, an action of a profinite group G on a profinite tree $\widehat{T}$ is k-acylindrical if the subtree fixed by $\hat{\gamma}$ is either empty or of diameter at most k, for every $\hat{\gamma} \in G$. Such an action is called *acylindrical* if it is k-acylindrical for some k.

Acylindricity gives useful control over non-cyclic abelian subgroups, via the following lemma. This was proved in [WZ17, Theorem 5.2]. (The discrete version of this fact is left as an instructive exercise to the reader.)

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[1] This suggestion was carried out in [Wil18b].

Lemma 3.3. *If A is an abelian, profinite, non-procyclic group, and A acts acylindrically on a profinite tree T, then A fixes a vertex.*

We are now ready to prove the non-splitting result for hyperbolic manifolds with cusps.

Lemma 3.4. *If N is a compact, orientable, hyperbolic 3-manifold with toral boundary and $\widehat{\pi_1 N}$ acts on a profinite tree T with each edge stabilizer either procyclic or conjugate into a peripheral subgroup, then $\widehat{\pi_1 N}$ fixes a vertex.*

Proof. First, note that if $\widehat{\pi_1 N}$ acts on T without fixed points then, by Lemma 1.3, after passing to a proper open subgroup we may assume that $\widehat{\pi_1 N}$ is not generated by vertex stabilizers.

Let the family of peripheral subgroups of $\pi_1 N$ be $P_1, \ldots, P_n$. By Thurston's hyperbolic Dehn filling theorem (see, for instance, [Ago10, LM13] for modern improvements), we may choose slopes $c_i \in P_i$ so that the resulting Dehn filled manifold $N(c_1, \ldots, c_n)$ is a closed, hyperbolic (in particular, aspherical) manifold. Therefore,

$$\widehat{\pi_1 N}/\overline{\langle\!\langle c_1, \ldots, c_n \rangle\!\rangle} \cong \widehat{\pi_1 N}(c_1, \ldots, c_n)$$

is a profinite PD^3 group. Since $\overline{\langle\!\langle c_1, \ldots, c_n \rangle\!\rangle}$ is generated by vertex stabilizers, $\widehat{\pi_1 N}(c_1, \ldots, c_n)$ acts on a profinite tree $\overline{\langle\!\langle c_1, \ldots, c_n \rangle\!\rangle} \backslash T$ (see [ZM88, Proposition 2.5] or [Rib17, Proposition 4.1.1]) and still does not fix a vertex. The edge stabilizers of the latter action are procyclic. This contradicts Corollary 1.11. □

4. The JSJ decomposition

In this section we show that, as well as the Kneser–Milnor decomposition, the JSJ decomposition is also determined by the profinite completion. In order to avoid ambiguity, we start by stating the form of the JSJ decomposition we consider. In a nutshell, it is the minimal decomposition along tori such that the complementary pieces are geometric.

Definition 4.1. Let M be a closed, orientable, irreducible 3-manifold which is not a torus bundle over the circle. Let $\mathcal{T} \subseteq M$ be an embedded disjoint union of essential tori such that the connected components of $M \smallsetminus \mathcal{T}$ are each geometric – that is, either Seifert fibred or admitting hyperbolic or Sol-geometry. Such a union $\mathcal{T}$ with the smallest number of connected components is called the *JSJ decomposition of M*.

The existence of the JSJ decomposition follows from the work of Jaco, Shalen and Johannson together with Perelman's proof of the geometrization conjecture; see [AFW15, §1.6, §1.7] for details. The tori are unique up to isotopy. We follow Wilkes's elegant terminology [Wil18a], and use the term *minor* to denote those components of $M \smallsetminus \mathcal{T}$ that are homeomorphic to the twisted interval bundle over the Klein bottle; the remaining components we call *major*. If two minor components are adjacent then their union is virtually a torus bundle over a circle, and so admits either Euclidean, Nil- or Sol-geometry, which contradicts the hypothesis that $\mathcal{T}$ was minimal. Therefore, every edge adjoins at least one major vertex.

The submanifold $\mathcal{T}$ induces a graph-of-spaces decomposition of M, and hence a graph-of-groups decomposition of $\pi_1 M$ and a profinite graph-of-groups decomposition of $\widehat{\pi_1 M}$ (see Example 1.4). The Bass–Serre trees of the latter are denoted by T_M and $\widehat{T}_M$, respectively. Crucially, these trees turn out to be acylindrical, in the sense of Definition 3.2.

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Proposition 4.2. *For M a closed, orientable, irreducible 3-manifold, the JSJ tree T_M and the profinite JSJ tree $\widehat{T}_M$ are both 4-acylindrical.*

Proof. In [WZ10] the authors showed that the corresponding decomposition of $\pi_1 M$ is 4-acylindrical and fits into Example 1.4. In [HWZ13], the authors showed with Hamilton that the corresponding profinite decomposition of $\widehat{\pi_1 M}$ is a 4-acylindrical injective graph of profinite groups (see also [WZ17, Lemma 4.5]). □

We are now ready to state our main theorem.

Theorem 4.3. *If M, M' are closed, orientable, irreducible 3-manifolds and*

$$f : \widehat{\pi_1 M} \overset{\cong}{\to} \widehat{\pi_1 M'}$$

is an isomorphism, then there is an f-equivariant isomorphism

$$\phi : \widehat{T}_M \to \widehat{T}_{M'}$$

of the corresponding profinite Bass–Serre trees. In particular, the underlying graphs of the JSJ decompositions of M and M' are isomorphic, as are the profinite completions of the fundamental groups of the corresponding pieces.

Consider a vertex space N of M. The next three lemmas show that $\widehat{\pi_1 N}$ must act with a fixed point on $\widehat{T}_{M'}$. We start with the hyperbolic case.

Lemma 4.4. *Consider N a compact, hyperbolic 3-manifold with (possibly empty) toral boundary. If $\widehat{\pi_1 N}$ acts acylindrically on a profinite tree $\widehat{S}$ with abelian edge stabilizers then $\widehat{\pi_1 N}$ fixes a unique vertex.*

Proof. If N is closed then every abelian subgroup of $\widehat{\pi_1 N}$ is procyclic [WZ17, Theorem D] and the result follows from Corollary 1.13.

Suppose therefore that N has non-empty toroidal boundary. By Proposition 3.1 every edge stabilizer is either procyclic or conjugate into a peripheral subgroup, and therefore $\widehat{\pi_1 N}$ fixes a vertex by Lemma 3.4.

Uniqueness follows from [ZM88, Corollary 2.9] or [Rib17, Corollary 4.1.6], since $\widehat{\pi_1 N}$ is non-abelian and edge stabilizers are abelian. □

We move on to the major Seifert fibred case.

Lemma 4.5. *Consider N a compact, major Seifert fibred 3-manifold with (possibly empty) toral boundary. If $\widehat{\pi_1 N}$ acts acylindrically on a profinite tree $\widehat{S}$ with abelian edge groups then $\widehat{\pi_1 N}$ fixes a unique vertex.*

Proof. The subgroups of $\widehat{\pi_1 N}$ which are isomorphic to $\widehat{\mathbb{Z}}^2$ each fix a vertex by Lemma 3.3. Thus the maximal procyclic normal subgroup C of $\widehat{\pi_1 N}$ fixes a vertex.

Suppose to the contrary $\widehat{\pi_1 N}$ does not fix a vertex. By [Zal90, Lemma 1.5] or [Rib17, Proposition 2.4.12], there exists a unique minimal $\widehat{\pi_1 N}$-invariant subtree $\widehat{D}$ of $\widehat{S}$, which is infinite. Now by [ZM88, Theorem 2.12] or [Rib17, Proposition 4.2.2], C acts trivially on $\widehat{D}$, which contradicts the acylindricity of the action.

Uniqueness again follows from [ZM88, Corollary 2.9] or [Rib17, Corollary 4.1.6]. □

The case of minor Seifert fibred vertex follows immediately from [WZ17, Theorem 5.2] and [ZM88, Corollary 2.9] or [Rib17, Corollary 4.1.6].

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LEMMA 4.6. *Consider N a minor Seifert fibred 3-manifold. If $\widehat{\pi_1 N}$ acts acylindrically on a profinite tree $\widehat{S}$ with abelian edge groups then $\widehat{\pi_1 N}$ fixes a unique vertex.*

We next classify the fixed point sets of $\widehat{\mathbb{Z}}^2$ subgroups of $\widehat{\pi_1 M}$. First, we need an analysis of their normalizers in vertex stabilizers. We start with the hyperbolic case, in which case normalizers coincide with centralizers.

LEMMA 4.7. *Let N be a compact, orientable, hyperbolic 3-manifold with toral boundary, and $H \cong \widehat{\mathbb{Z}}^2$ a subgroup of $\widehat{\pi_1 N}$. Then $N_{\widehat{\pi_1 N}}(H) = C_{\widehat{\pi_1 N}}(H)$.*

Proof. By [WZ17, Theorem 9.3], H is conjugate into the closure of a cusp subgroup, and by [WZ17, Lemma 4.5], that cusp subgroup is malnormal. The result follows. □

We next treat the case of a major Seifert fibred manifold.

LEMMA 4.8. *Let N be a compact, orientable, major Seifert fibred 3-manifold with toral boundary, and $H \cong \widehat{\mathbb{Z}}^2$ a subgroup of $\widehat{\pi_1 N}$ conjugate to the closure of the fundamental group of a boundary component. Then $N_{\widehat{\pi_1 N}}(H) = C_{\widehat{\pi_1 N}}(H)$.*

Proof. The fundamental group $\pi_1 N$ is torsion-free of the form

$$1 \to Z \to \pi_1 N \to \pi_1 O \to 1,$$

where $\pi_1 O$ is a Fuchsian group and Z is infinite cyclic (and not necessarily central). Since Seifert fibred 3-manifold groups are LERF [Sco78, Sco85] we have a corresponding short exact sequence of profinite completions,

$$1 \to \widehat{Z} \to \widehat{\pi_1 N} \xrightarrow{f} \widehat{\pi_1 O} \to 1.$$

Then $C_{\widehat{\pi_1 N}}(H)$ contains $\widehat{Z}$ and centralizes it. So $f(C_{\widehat{\pi_1 N}}(H)) = C_{\widehat{\pi_1 O}}(f(H))$ and $f(N_{\widehat{\pi_1 N}}(H)) = N_{\widehat{\pi_1 O}}(f(H))$. Hence it suffices to show that $C_{\widehat{\pi_1 O}}(f(H)) = N_{\widehat{\pi_1 O}}(f(H))$. We may assume that H is the closure of the fundamental group of a boundary component; then $f(H)$ is the closure of a peripheral infinite-cyclic subgroup C of $\pi_1 O$. Since Fuchsian groups are conjugacy separable [FR90] we deduce that every-finite index subgroup of $\pi_1 O$ is conjugacy separable. Then $\overline{C_{\pi_1 O}(C)} = C_{\widehat{\pi_1 O}}(f(H))$ by [Min12, Corollary 12.3], and $\overline{N_{\pi_1 O}(C)} = N_{\widehat{\pi_1 O}}(f(H))$ by [CZ13, Lemma 2.3 combined with Theorem 2.14]. But $N_{\pi_1 O}(C) = C_{\pi_1 O}(C)$, so $C_{\widehat{\pi_1 O}}(f(H)) = N_{\widehat{\pi_1 O}}(f(H))$ as required. □

Next, we classify the possible fixed subtrees for $\widehat{\mathbb{Z}}^2$ subgroups of $\widehat{\pi_1 M}$.

LEMMA 4.9. *Let M be a closed, orientable, irreducible 3-manifold. Consider the action of a subgroup $H \cong \widehat{\mathbb{Z}}^2$ of $\widehat{\pi_1 M}$ on $\widehat{T}_M$. One of the following holds.*

(i) *The fixed point set of H is a vertex with Seifert fibred stabilizer.*

(ii) *The fixed point set of H consists of exactly one edge.*

(iii) *The fixed subtree of H consists of exactly two edges; the central vertex has a minor Seifert fibred stabilizer, and the other two vertices are major.*

Furthermore, if the centralizer $C_{\widehat{\pi_1 M}}(H)$ is properly contained in the normalizer $N_{\widehat{\pi_1 M}}(H)$, then we are in case (i) or case (iii).

Proof. By Lemma 3.3 and Proposition 4.2, H fixes a non-empty subtree. Recall that every edge of $\widehat{T}_M$ adjoins a major vertex, and that every minor vertex of $\widehat{T}_M$ adjoins exactly two edges. If H stabilizes an edge e and an adjacent major vertex v then, by Lemmas 4.7 and 4.8, e is the unique edge incident at v stabilized by H. It follows that the fixed tree of H is of one of the three claimed forms.

We now prove that, in case (ii), $N_{\widehat{\pi_1 M}}(H) = C_{\widehat{\pi_1 M}}(H)$. Indeed, $N_{\widehat{\pi_1 M}}(H)$ preserves the fixed subtree of $C_{\widehat{\pi_1 M}}(H)$, and so if H fixes a unique edge, $N_{\widehat{\pi_1 M}}(H)$ is contained in an edge stabilizer, hence is abelian, and so $N_{\widehat{\pi_1 M}}(H) = C_{\widehat{\pi_1 M}}(H)$. □

We now have enough information to construct a map ϕ. To start with, it will only be a map of abstract, unoriented graphs.

LEMMA 4.10. *Consider closed, orientable, irreducible 3-manifolds M, M', and let $f : \widehat{\pi_1 M} \to \widehat{\pi_1 M'}$ be an isomorphism. Then there exists an f-equivariant morphism of graphs*

$$\phi : \widehat{T}_M \to \widehat{T}_{M'}.$$

Note that, here, we only claim that ϕ is a map of abstract, non-oriented graphs. This map may in principle send edges to either edges or vertices.

Proof. For brevity, we write $G = \widehat{\pi_1 M}$ and $\widehat{S} = \widehat{T}_{M'}$, and let G act on $\widehat{S}$ via f. Let e be an edge of $\widehat{T}_M$ with stabilizer $G_e \cong \widehat{\mathbb{Z}}^2$. Let u_1, u_2 be the adjacent vertices of $\widehat{T}_M$. Lemmas 4.4–4.6 together guarantee the existence of unique vertices v_1, v_2 of $\widehat{S}$ such that $G_{u_i} \subseteq G_{v_i}$ for both i.

We claim that v_1 and v_2 are either equal or adjacent. Suppose therefore that v_1, v_2 are at distance greater than 2 (possibly infinite). Then G_e stabilizes the geodesic $[v_1, v_2]$ (see [ZM88, Corollary 2.9] or [Rib17, Corollary 4.1.6]) and therefore, by Lemma 4.9, v_1, v_2 are at distance precisely 2, are both adjacent to a minor vertex w, and $C_G(G_e)$ is properly contained in $N_G(G_e)$. Therefore, by Lemma 4.9, e is adjacent to a minor vertex; without loss of generality, we may assume that u_1 is major, $G_{u_1} \subseteq G_{v_1}$, and that u_2 is minor and $G_{u_2} \subseteq G_{v_2}$. But G_w also normalizes G_e, so $G_w \subseteq G_{u_2} \subseteq G_{v_2}$. This implies that G_w stabilizes an edge, which is absurd because edge stabilizers are abelian. Therefore, v_1 and v_2 are either equal or adjacent. If they are equal to a vertex v, we set $\phi(e) = v$. If they are adjacent, we set $\phi(e)$ to be the image of the unique edge joining them. This completes the construction of the map $\phi : \widehat{T}_M \to \widehat{S}$, which is equivariant by construction. □

We are now ready to complete the proof of the main theorem.

Proof of Theorem 4.3. Applying Lemma 4.10 twice, we obtain maps of graphs

$$\phi : \widehat{T}_M \to \widehat{T}_{M'}, \quad \psi : \widehat{T}_{M'} \to \widehat{T}_M,$$

where ϕ is f-equivariant and ψ is f^{-1}-equivariant. Equivariance implies that

$$g\psi \circ \phi(x) = \psi \circ \phi(gx)$$

for all $g \in \widehat{\pi_1 M}$ and $x \in \widehat{T}_M$, whence the stabilizer of x is contained in the stabilizer of $\psi \circ \phi(x)$. Since vertex stabilizers stabilize unique vertices, it follows that $\psi \circ \phi$ is equal to the identity on vertices, and hence on the whole of $\widehat{T}_M$. In particular, ϕ and ψ induce isomorphisms of the finite quotient graphs $\widehat{\pi_1 M} \backslash \widehat{T}_M$ and $\widehat{\pi_1 M'} \backslash \widehat{T}'_M$; we may therefore choose consistent orientations on these graphs, which lift to equivariant orientations on the profinite trees $\widehat{T}_M$ and $\widehat{T}_{M'}$, which are respected by ϕ and ψ; this also implies continuity of ϕ and ψ. □

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Profinite detection of 3-manifold decompositions

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