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THERE ARE $2^{\aleph_\alpha}$ FRIENDSHIP GRAPHS OF CARDINAL $\aleph_\alpha$

BY

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A *friendship graph* is a graph in which every two distinct vertices have exactly one common neighbour. Finite friendship graphs were characterized by Erdős, Rényi, and Sós [1] as those for which the vertices can be enumerated as $u, v_1, \ldots, v_k, w_1, \ldots, w_k$ in such a way that the only edges are uv_i, uw_i, and v_iw_i $(i=1,\ldots,k)$. Thus finite friendship graphs are rather rare. In contrast, we shall show that there are as many nonisomorphic friendship graphs of given infinite cardinal as there are nonisomorphic graphs of that cardinal altogether. In fact, we do a little more.

THEOREM. *Let c be a cardinal with $3 \le c \le \aleph_0$. Then there are $2^{\aleph_\alpha}$ nonisomorphic c-chromatic friendship graphs of cardinal $\aleph_\alpha$.*

This was proved for $\alpha=0$ in [2], and we use here a similar method. The range of c cannot be extended, because firstly every nontrivial friendship graph contains a triangle and is therefore at least 3-chromatic, and secondly every friendship graph is without cycles of length 4 and therefore is at most $\aleph_0$-chromatic by a result of Hajnal [3, Cor. 5.6].

With every graph G, associate a graph G' by introducing one additional vertex w_{uv} for every two distinct vertices u, v of G such that u and v have no common neighbour in G, and joining w_{uv} to both u and v. Let $G''=(G')'$ and so on, and $\operatorname{ext} G = \bigcup_{i=1}^{\infty} G^{(i)}$. We summarize the principal properties of this extension.

LEMMA 1. *If G is infinite then* $\operatorname{ext} G$ *has the same cardinal as G. If G has chromatic number at least* 3 *then* $\operatorname{ext} G$ *has the same chromatic number as G. If G contains no cycle of length* 4 *then* $\operatorname{ext} G$ *is a friendship graph. If H is a finite subgraph of* $\operatorname{ext} G$ *and every vertex of H has degree at least* 3, *then H is in fact a subgraph of G.*

Only the last statement requires proof. Suppose if possible that $H \cap (\operatorname{ext} G \setminus G) \neq \emptyset$, let i be the largest integer such that $H \cap (G^{(i)} \setminus G^{(i-1)})$ is nonempty, and let h be any element of this set. Since h has degree at least 3 in H, and in $\operatorname{ext} G$ is joined by at most two edges to points of $\operatorname{ext} G \setminus \bigcup_{j=i+1}^{\infty} G^{(j)}$, it is joined in H to a point of $\bigcup_{j=i+1}^{\infty} G^{(j)}$. This contradicts the definition of i.

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LEMMA 2. *There exists a finite graph* Γ, *with two distinguished vertices* a *and* b, *such that*

(i) *every vertex of* Γ *has degree at least* 3, *and* Γ *is* 3-*chromatic and contains no cycle of length* 4;

(ii) *there is a* 3-*coloring of* Γ *in which* a *and* b *have the same color, but there exists no automorphism of* Γ *that interchanges* a *and* b;

(iii) *every path from* a *to* b *in* Γ *has length at least* 3.

Such a graph is exhibited, together with an appropriate coloring, in Figure 1. Verifications are left to the reader.

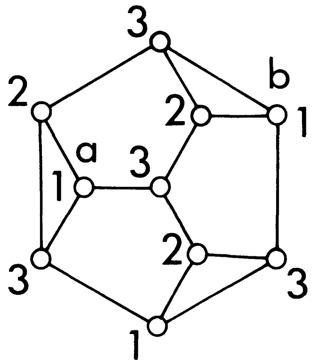

Figure 1

Proof of the Theorem. We first apply the method [4] of obtaining a graph ("šíp-product" [5]) by joining each pair of a binary relation with an "arrow", in this case a copy of Γ. With every ordinal $\xi \geq \omega_0$ associate a graph C_ξ as follows: for each pair of ordinals α, β satisfying $0 \leq \alpha < \beta < \xi$ we take a copy $\Gamma_{\alpha\beta}$ of Γ, and for each ordinal γ with $0 \leq \gamma < \xi$ we identify with γ all the a-vertices of the graphs $\Gamma_{\gamma\beta}$ for $\beta > \gamma$ and all the b-vertices of the graphs $\Gamma_{\alpha\gamma}$ for $\alpha < \gamma$. It follows from Lemma 2 that G_ξ is 3-chromatic and contains no cycle of length 4. If $\xi \neq \zeta$ then G_ξ and G_ζ are nonisomorphic, because from G_ξ we can recover the set of ordinals less than ξ as the set of vertices of infinite degree in G_ξ, and we can recover its standard well-ordering by observing that $\alpha < \beta$ if and only if there exists an isomorphism φ of Γ into G_ξ such that $\varphi(u) = \alpha$, $\varphi(v) = \beta$, and all vertices of $\varphi(\Gamma) \backslash \{\alpha, \beta\}$ have finite degree in G_ζ.

For every nonempty set S of infinite ordinals, let $G(S)$ denote the disjoint union of the graphs G_ξ for $\xi \in S$. Since the graphs G_ξ, G_ζ are nonisomorphic for $\xi \neq \zeta$, it follows that the graphs $G(S)$, $G(T)$ are nonisomorphic for $S \neq T$.

For any integer $c \geq 3$, let H_c be a finite c-chromatic graph of girth at least 5, with all vertices of degree at least 3 (the existence of such graphs is well known [6]); let $H_{\aleph_0}$ denote the disjoint union of the graphs H_c, $3 \leq c < \aleph_0$. For every

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nonempty set S of infinite ordinals, let $G_c(S)$ denote the disjoint union of $G(S)$ and H_c. Clearly each graph $G_c(S)$ is c-chromatic and contains no cycle of length 4; moreover $G_c(S)$, $G_c(T)$ are nonisomorphic for $S \neq T$.

Now let an ordinal α be given, and a cardinal c with $3 \leq c \leq \aleph_0$. For each set S of cardinal $\aleph_\alpha$ consisting of ordinals ξ with $\omega_0 \leq \xi < \omega_{\alpha+1}$, let $F_c(S) = \operatorname{ext} G_c(S)$. Since there are $2^{\aleph_\alpha}$ such sets S, and by Lemma 1 each $F_c(S)$ is a c-chromatic friendship graph of cardinal $\aleph_\alpha$, it remains only to observe that (by Lemma 1) $G_c(S)$ can be recovered from $F_c(S)$ because its vertices are those belonging to finite subgraphs of $F_c(S)$ with all vertices of degree at least 3, and therefore $F_c(S)$, $F_c(T)$ are nonisomorphic for $S = T$.

The referee suggested that it may be worthwhile to get some information whether or not one can get all the friendship graphs through the given construction. In each of the friendship graphs constructed here, there are many vertices of infinite degree. Not all the friendship of cardinality $\aleph_\alpha$ graphs are like that: consider the trivial generalization of finite friendship graphs.

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