A note on Lagrangian submanifolds in symplectic $4m$-manifolds

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Abstract. This short paper shows a topological obstruction of the existence of certain Lagrangian submanifolds in symplectic $4m$-manifolds.

1 Introduction

Let (X, ω) be a symplectic manifold. A submanifold $L \subset X$ is called Lagrangian if $\omega|_L \equiv 0$ and $\dim L = \frac{1}{2} \dim X$. See the textbook and papers [1, 12, 16] for basic properties, general backgrounds, and developments in the study of Lagrangian submanifolds. One question is to understand topological obstructions of embedding manifolds into symplectic manifolds as Lagrangian submanifolds, which has been intensively studied (cf. [1, 3, 4, 6, 11, 13, 14, 17, 18]).

In [19], a phenomenon has been noticed while searching for obstructions of the existence of positive scalar curvature Riemannian metrics on symplectic 4-manifolds. More precisely, Theorem 2.3 of [19] should be interpreted as a simple topological consequence of the presence of certain Lagrangian submanifolds in symplectic 4-manifolds, and this topological constraint is a well-known obstruction of positive scalar curvature metrics by quoting Taubes' theorem on the Seiberg–Witten invariant. The goal of this paper is to reformulate this result in a more suitable context, and to generalize it to the case of $4m$-dimensional symplectic manifolds, which might have some independent interests.

If X is a compact oriented $4m$-dimensional manifold, the intersection pairing is the symmetric nondegenerated quadratic form

$$Q : H^{2m}(X, \mathbb{R}) \oplus H^{2m}(X, \mathbb{R}) \to \mathbb{R}, \qquad Q([\alpha], [\beta]) = \int_X \alpha \wedge \beta,$$

which is represented by the diagonal matrix $\operatorname{diag}(1, \ldots, 1, -1, \ldots, -1)$ under a suitable basis. Denote $b_{2m}^+(X)$ the number of plus ones in the matrix, and $b_{2m}^-(X)$ the number of minus ones. Both of $b_{2m}^\pm(X)$ are topological invariants, and the $2m$-Betti number $b_{2m}(X) = b_{2m}^+(X) + b_{2m}^-(X)$. If X admits a symplectic form ω, then $[\omega^m] \in H^{2m}(X, \mathbb{R})$, and

$$Q([\omega^m], [\omega^m]) = \int_X \omega^{2m} > 0,$$

which implies

$$b_{2m}^+(X) \geq 1.$$

The main theorem asserts that $b_{2m}^\pm(X)$ provides some constraints of the existence of certain Lagrangian submanifolds.

Theorem 1.1   *Let (X, ω) be a compact symplectic manifold of dimension $4m$, $m \geq 1$, $m \in \mathbb{Z}$, and let L be an orientable Lagrangian submanifold in X.*

(i)   *If the Euler characteristic $\chi(L)$ of L satisfies*

$$(-1)^m \chi(L) > 0, \qquad then \qquad b_{2m}^+(X) \geq 2.$$

(ii)   *If L represents a nontrivial homology class, i.e.,*

$$0 \neq [L] \in H_{2m}(X, \mathbb{R}), \quad and \quad \chi(L) = 0,$$

$$then \qquad b_{2m}^+(X) \geq 2, \quad and \quad b_{2m}^-(X) \geq 1.$$

(iii)   *If*

$$(-1)^m \chi(L) < 0, \qquad then \qquad b_{2m}^-(X) \geq 1.$$

(iv)   *If the $2m$-Betti number $b_{2m}(X) = 1$, then*

$$\chi(L) = 0, \quad and \quad 0 = [L] \in H_{2m}(X, \mathbb{R}).$$

When X is four-dimensional, this theorem is known to experts (Proposition 2.17 of [18]), and (i) and (ii) have appeared in the proof of Theorem 2.3 of [19]. There are some immediate applications of Theorem 1.1 that are surely known to experts (cf. [3, 4, 6, 7, 13, 14]):

(i)   Neither $\mathbb{CP}^m$ nor S^{2m} can be embedded in $\mathbb{CP}^{2m}$ as a Lagrangian submanifold.
(ii)   Only possible orientable Lagrangian submanifolds in $\mathbb{CP}^2$ are 2-tori T^2. Orientable Lagrangian submanifolds in $\mathbb{CP}^1 \times \mathbb{CP}^1$ are either tori T^2 or spheres S^2 by $b_2^+(\mathbb{CP}^1 \times \mathbb{CP}^1) = 1$.
(iii)   There is no Lagrangian sphere S^{4m} in $\mathbb{CP}^{4m-1} \times \Sigma$, where Σ is a compact Riemann surface, since $b_{4m}^+(\mathbb{CP}^{4m-1} \times \Sigma) = 1$.

Theorems in [6, 7, 14] assert that Riemannian manifolds with negative sectional curvature do not admit any Lagrangian embedding into uniruled symplectic manifolds. As a corollary, we show certain constraints of the existence of special metrics on Lagrangian submanifolds.

Corollary 1.2   *Let (X, ω) be a compact symplectic $4m$-dimensional manifold with*

$$b_{2m}^+(X) = 1.$$

(i) *A compact orientable 2m-manifold Y admitting a hyperbolic metric, i.e., a Rie-mannian metric with sectional curvature −1, cannot be embedded into X as a Lagrangian submanifold.*

(ii) *If m = 2, and L is an orientable Lagrangian submanifold in X admitting an Einstein metric g, i.e., the Ricci tensor*

$$\mathrm{Ric}(g) = \lambda g$$

for a constant $\lambda \in \mathbb{R}$, then a finite covering of L is the 4-torus T^4.

We remark that the condition $b_{2m}^+(X) = 1$ is necessary for Corollary 1.2(ii). For example, any symplectic 4-manifold (X, ω) admitting an Einstein metric could be diagonally embedded into the symplectic 8-manifold $(X, \omega) \times (X, -\omega)$ as a Lagrangian.

We prove Theorem 1.1 and Corollary 1.2 by using classical differential topology/geometry techniques in the next section.

2 Proofs

First, we recall basic relevant facts, and provide the detailed proofs for the completeness. Let (X, ω) be a compact symplectic $4m$-dimensional manifold, and let L be an orientable Lagrangian submanifold in X. The symplectic form ω induces a natural orientation of X, i.e., given by ω^{2m}. We fix an orientation on L.

Lemma 2.1

$$[L]^2 = (-1)^m \chi(L),$$

where $[L]^2$ is the self-intersection number of L in X. Consequently, if $\chi(L) \neq 0$, then $[L]^2 \neq 0$, and

$$0 \neq [L] \in H_{2m}(X, \mathbb{R}).$$

Proof The Weinstein neighborhood theorem (cf. [16, 12]) asserts that there is a tubular neighborhood U of L in X diffeomorphic to a neighborhood of the zero section in the cotangent bundle T^*L, and the restriction of ω on U is the canonical symplectic form on T^*L under the identification. More precisely, if $x_1, \ldots, x_{2m}$ are local coordinates on L, then the symplectic form

$$\omega = \sum_{i=1}^{2m} dx_i \wedge dp_i,$$

where $p_1, \ldots, p_{2m}$ are coordinates on fibers induced by dx_j, i.e., any 1-form $\sum_{i=1}^{2m} p_i dx_i$ is given by the numbers p_j. We regard L as the zero section of T^*L, and assume that $dx_1 \wedge \cdots \wedge dx_{2m}$ gives the orientation of L.

If we denote $\varpi_i = dx_i \wedge dp_i$, $i = 1, \ldots, 2m$, then the algebraic relationships are

$$\varpi_i \wedge \varpi_j = \varpi_j \wedge \varpi_i, \quad i \neq j, \quad \text{and} \quad \varpi_i^2 = 0.$$

The orientation inherited from the symplectic manifold is given by

$$\omega^{2m} = (2m)! \varpi_1 \wedge \cdots \wedge \varpi_{2m}.$$

The orientation on L induces an orientation on T^*L given by

$$dx_1 \wedge \cdots \wedge dx_{2m} \wedge dp_1 \wedge \cdots \wedge dp_{2m} = (-1)^m \varpi_1 \wedge \cdots \wedge \varpi_{2m}.$$

If L' is a small deformation of L intersecting with L transversally, the intersection number $L \cdot L'$ with respect to the orientation given by ω^{2m} is the self-intersection number of L in X, i.e., $L \cdot L' = [L]^2$. The Euler characteristic of L equals to the intersection number $L \overset{.}{\cdot} L'$ with respect to the orientation defined by $dx_1 \wedge \cdots \wedge dx_{2m} \wedge dp_1 \wedge \cdots \wedge dp_{2m}$ (cf. [5]), i.e., $L \overset{.}{\cdot} L' = \chi(L)$, where we identify the tangent bundle TL with T^*L via a Riemannian metric. Since $L \cdot L' = (-1)^m L \overset{.}{\cdot} L'$, we obtain $[L]^2 = (-1)^m \chi(L)$. ∎

There are analogous formulae in more general contexts (see, e.g., [15]).

Let g be a Riemannian metric compatible with ω, i.e., $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$ for an almost complex structure J compatible with ω, and let $*$ be the Hodge star operator of g. The volume form dv_g of g is the symplectic volume form, i.e., $dv_g = \frac{1}{(2m)!} \omega^{2m}$. When $*$ acts on $\wedge^{2m} T^*X$, $*^2 = \text{Id}$ holds. $H^{2m}(X, \mathbb{R})$ is isomorphic to the space of harmonic $2m$-forms by the Hodge theory (cf. [8]). Thus, $H^{2m}(X, \mathbb{R})$ admits the self-dual/anti-self-dual decomposition

$$H^{2m}(X, \mathbb{R}) \cong \mathcal{H}_+(X) \oplus \mathcal{H}_-(X), \quad \text{where}$$

$$\mathcal{H}_\pm(X) = \{\alpha \in C^\infty(\wedge^{2m} T^*X) | d\alpha = 0, \ *\alpha = \pm\alpha\}.$$

Note that $b_{2m}^\pm(X) = \dim \mathcal{H}_\pm(X)$.

Lemma 2.2

$$\omega^m \in \mathcal{H}_+(X).$$

Proof Since $d\omega^m = 0$, we only need to verify that ω^m is self-dual, i.e., $*\omega^m = \omega^m$, which is a pointwise condition. For any point $x \in X$, we choose coordinates $x_1, \ldots, x_{2m}, p_1, \ldots, p_{2m}$ on a neighborhood of x such that on the tangent space $T_x X$,

$$\omega = \sum_{i=1}^{2m} dx_i \wedge dp_i, \quad \text{and} \quad g = \sum_{i=1}^{2m}(dx_i^2 + dp_i^2).$$

The calculation is the same as those in the Kähler case (cf. [8]) because of the locality. If we still write $\varpi_i = dx_i \wedge dp_i$, $i = 1, \ldots, 2m$, then the volume form $dv_g = \varpi_1 \wedge \cdots \wedge \varpi_{2m}$. Denote the multi-index sets $I = \{i_1, \ldots, i_m\}$, $i_1 < \cdots < i_m$, and the complement $I^c = \{1, \ldots, 2m\}\backslash I$. Note that

$$\omega^m = (\varpi_1 + \cdots + \varpi_{2m})^m = m! \sum_{I \subset \{1,\ldots,2m\}} \varpi_I, \quad \varpi_I = \varpi_{i_1} \wedge \cdots \wedge \varpi_{i_m}.$$

By $\varpi_I \wedge *\varpi_I = dv_g = \varpi_I \wedge \varpi_{I^c}$, we obtain $*\varpi_I = \varpi_{I^c}$. Hence, $*\omega^m = \omega^m$. ∎

Now, we are ready to prove the results.

Proof (Theorem 1.1**)**   Note that the assumptions imply $0 \neq [L] \in H_{2m}(X, \mathbb{R})$ in the cases (i)–(iii). If $PU(L) \in H^{2m}(X, \mathbb{R})$ is the Poincaré dual of $[L]$, then

$$PU(L) = \alpha_+ + \alpha_- \neq 0, \quad \text{and} \quad \alpha_\pm \in \mathcal{H}_\pm(X).$$

Since $\int_X \alpha_+ \wedge \alpha_- = 0$,

$$(-1)^m \chi(L) = [L]^2 = \int_X (\alpha_+ + \alpha_-)^2 = \int_X \alpha_+^2 + \int_X \alpha_-^2.$$

If $(-1)^m \chi(L) \geq 0$, then

$$\int_X \alpha_+^2 \geq -\int_X \alpha_-^2 = \int_X \alpha_- \wedge *\alpha_- = \int_X |\alpha_-|_g^2 dv_g \geq 0.$$

If $\alpha_+ = 0$, then $\alpha_- = 0$ and $PU(L) = 0$, which is a contradiction. Thus, $\alpha_+ \neq 0$, and

$$\int_L \alpha_+ = \int_X \alpha_+ \wedge (\alpha_+ + \alpha_-) = \int_X \alpha_+^2 = \int_X |\alpha_+|_g^2 dv_g \neq 0.$$

Since

$$\int_L \omega^m = 0,$$

ω^m and α_+ are linearly independent in $\mathcal{H}_+(X)$. Therefore, we obtain $b_{2m}^+(X) \geq 2$.

If $\chi(L) = 0$, then

$$0 = [L]^2 = \int_X \alpha_+^2 + \int_X \alpha_-^2, \quad \text{and} \quad \int_X |\alpha_-|_g^2 dv_g = \int_X |\alpha_+|_g^2 dv_g.$$

Thus, $\alpha_- \neq 0$ and $b_{2m}^-(X) \geq 1$.

If we assume $(-1)^m \chi(L) < 0$, then

$$0 > \int_X \alpha_+^2 + \int_X \alpha_-^2, \quad \text{and} \quad \int_X |\alpha_-|_g^2 dv_g > \int_X |\alpha_+|_g^2 dv_g \geq 0.$$

We obtain the conclusion $b_{2m}^-(X) \geq 1$.

If $b_{2m}(X) = 1$, then $b_{2m}^+(X) = 1$, $b_{2m}^-(X) = 0$, and $H^{2m}(X, \mathbb{R}) \cong \mathcal{H}_+(X)$. Hence, $\chi(L) = 0$ and $[L] = 0$. ∎

Proof (Corollary 1.2**)**   Let Y be an orientable compact $2m$-manifold, and let h be a hyperbolic metric on Y. By the Gauss–Bonnet formula of hyperbolic manifolds (cf. [9, 10]),

$$(-1)^m \chi(Y) = \varepsilon_{2m} \mathrm{Vol}_h(Y) > 0,$$

where $\varepsilon_{2m} > 0$ is a constant depending only on m, and $\mathrm{Vol}_h(Y)$ is the volume of the hyperbolic metric. Theorem 1.1 implies (i).

Now, we assume that $m = 2$. If g is a Riemannian metric on L, then the Gauss–Bonnet–Chern formula for 4-manifolds reads

$$\chi(L) = \frac{1}{8\pi^2} \int_L \left(\frac{R^2}{24} + |W^+|_g^2 + |W^-|_g^2 - \frac{1}{2} |\mathrm{Ric}^o|_g^2 \right) dv_g,$$

where R is the scalar curvature, $W^{\pm}$ denotes the self-dual/anti-self-dual Weyl curvature of g, and $\mathrm{Ric}^o = \mathrm{Ric} - \frac{R}{4}g$ is the Einstein tensor (cf. [2]). If g is Einstein, then $\mathrm{Ric}^o \equiv 0$ and thus $\chi(L) \geq 0$. By Theorem 1.1, either $\chi(L) < 0$, or $\chi(L) = 0$ and $[L] = 0$ in $H^4(X, \mathbb{R})$ since $b_4^+(X) = 1$. Therefore, $\chi(L) = 0$, which implies that

$$R \equiv 0, \quad W^{\pm} \equiv 0, \quad \mathrm{Ric} \equiv 0.$$

The curvature tensor of g vanishes, i.e., g is a flat metric. Thus, a finite covering is the torus T^4. ∎

A An alternative proof

by Anonymous Referee

Proposition 2.17 of [18] gives a very simple argument using the light cone lemma (which is a version of the Cauchy–Schwartz inequality) to eventually prove Theorem 1.1(i) and (ii), the harder parts, for $m = 1$. In fact, using the light cone lemma to give topological restriction on Lagrangians is not new (cf. [11, 17]). This argument is easily generalized to prove the whole Theorem 1.1 for arbitrary m as follows.

The light cone lemma asserts that for the light cone $\mathcal{C}$ of signature $(1, n)$ $(n > 0)$, $\mathcal{C} \subset \mathbb{R}^{1,n}$, any two elements in the forward cone $\mathcal{C}_+$ have a nonnegative inner product. Especially, if the inner product is zero, then the two elements are proportional to each other. Note that $\mathcal{C} \backslash \{0\}$ has two connected components. One is the forward cone $\mathcal{C}_+$, and the other is the backward cone $\mathcal{C}_-$. Furthermore, $\mathcal{C}_+ = -\mathcal{C}_-$.

Now, we follow Proposition 2.17 of [18] to prove Theorem 1.1 instead. If $(-1)^m \chi(L) \geq 0$, we have

$$PU(L)^2 \geq 0, \quad [\omega^m]^2 > 0, \quad [\omega^m] \cdot [L] = 0.$$

If we assume $[L] \neq 0$ and $b_{2m}^+(X) = 1$, both classes $PU(L)$ and $[\omega^m] \in H^{2m}(X, \mathbb{R})$ are in the forward cone by choosing a suitable orientation of L. Then the light cone lemma implies that this is a contradiction. That is, if $b_{2m}^+(X) = 1$, then $(-1)^m \chi(L) < 0$, and if $\chi(L) = 0$, then $[L] = 0 \in H_{2m}(X, \mathbb{R})$. Other parts of Theorem 1.1 can be similarly argued and are easier (without using Riemannian metrics and self-dual/anti-self-dual decompositions).

Finally, the same argument of Corollary 1.2(i) also works to show that a manifold admitting a complex hyperbolic metric, $X \cong \mathbb{CH}^m/\Gamma$, cannot be embedded as a Lagrangian since

$$\mathrm{Vol}(X) = \frac{(-4\pi)^m}{(m+1)!} \chi(X)$$

by the traditional Gauss–Bonnet theorem. This in particular implies that fake projective planes cannot be embedded as Lagrangians of eight-dimensional symplectic manifolds with $b_4^+ = 1$.

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