Local Electromagnetic Fields.

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§ 1. *The radiation of electromagnetic charges from a moving point.*

Let the electric charge associated with a moving point Q at time s be $f(s)$. This charge is supposed to vary on account of the radiation of electric charges from Q in a variable direction, which, at time s, has direction cosines $l(s)$, $m(s)$, $n(s)$ Using $\xi(s)$, $\eta(s)$, $\zeta(s)$ to denote the coordinates of Q at time s, and defining the effective time τ by means of the equation

$$[x - \xi(\tau)]^2 + [y - \eta(\tau]^2 + [z - \zeta(\tau)]^2 = c^2(t - \tau)^2, \ \tau \leq t,$$

the electromagnetic field may be specified by means of the potentials *

$$A_x = \frac{1}{4\pi}\int_{-\infty}^{\tau} f'(s)\frac{l(s)\,ds}{L(x, y, z, t, s)} - f(\tau)\frac{\xi'(\tau)}{4\pi M}$$

$$A_y = \frac{1}{4\pi}\int_{-\infty}^{\tau} f'(s)\frac{m(s)\,ds}{L(x, y, z\ t, s)} - f(\tau)\frac{\eta'(\tau)}{4\pi M}$$

$$A_z = \frac{1}{4\pi}\int_{-\infty}^{\tau} f'(s)\frac{n(s)\,ds}{L(x, y, z, t, s)} - f(\tau)\frac{\zeta'(\tau)}{4\pi M}$$

$$\Phi = \frac{1}{4\pi}\int_{-\infty}^{\tau} f'(s)\frac{ds}{L(x, y, z, t, s)} - f(\tau)\frac{c}{4\pi M}$$

where

$$L(x, y, z, t, s) = [x - \xi(s)]\,l(s) + [y - \eta(s)]\,m(s) + [z - \zeta(s)]\,n(s) - c(t - s),$$

$$M = [x - \xi(\tau)]\,\xi'(\tau) + [y - \eta(\tau)]\,\eta'(\tau) + [z - \zeta(\tau)]\,\zeta'(\tau) - c^2(t - \tau).$$

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* This field was obtained by the author when attempting to solve another problem proposed to him by Professor P. Ehrenfest when the latter was in Pasadena in 1924.

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In these expressions x, y, z are the coordinates of the point P at which the field is required, t is the time and c the velocity of light which is supposed to be constant. The electric vector $\mathbf{E}$ and the magnetic vector $\mathbf{H}$ are derived by the usual formula

$$\mathbf{H} = curl\, \mathbf{A}, \quad \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\Phi.$$

If Q has a variable magnetic charge $g(s)$ and magnetic charges are radiated in the same directions as the electric charges, we must add to the field just specified a second field derived from potentials B_x, B_y, B_z, Ψ, by means of the equations

$$\mathbf{E} = -curl\, \mathbf{B}, \quad \mathbf{H} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} - \nabla\Psi.$$

The expressions for these potentials are of the same forms as those which have already been given, except that $g(s)$ takes the place of $f(s)$.

From this general type of field other fields with various properties may be derived by superposition or differentiation.

§ 2. *A special case.*

When Q is permanently at the origin of coordinates and the charges are radiated along the axis of z, we have simply

$$\tau = t - \frac{r}{c}, \quad M = -cr, \quad L = z - c(t - s)$$

and the field takes the form*

$$E_x = \frac{x}{r^3}F(\tau) - \frac{xz}{cr^2(r-z)}F'(\tau) + \frac{y}{cr(r-z)}G'(\tau)$$

$$E_y = \frac{y}{r^3}F(\tau) - \frac{yz}{cr^2(r-z)}F'(\tau) - \frac{x}{cr(r-z)}G'(\tau)$$

$$E_z = \frac{z}{r^3}F(\tau) + \frac{r+z}{cr^2}F'(\tau)$$

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* This is a simple generalisation of the field discussed in *Physical Review*, Vol. 17 (1921), p. 64.

$$H_x = \frac{x}{r^3} G(\tau) - \frac{xz}{cr^2(r-z)} G'(\tau) - \frac{y}{cr(r-z)} F'(\tau)$$

$$H_y = \frac{y}{r^3} G(\tau) - \frac{yz}{cr^2(r-z)} G'(\tau) + \frac{x}{cr(r-z)} F'(\tau)$$

$$H_z = \frac{z}{r^3} G(\tau) + \frac{r+z}{cr^2} G'(\tau)$$

where

$$F(\tau) = \frac{1}{4\pi} f(\tau), \quad G(\tau) = \frac{1}{4\pi} g(\tau).$$

If we differentiate with respect to x, we get the case in which electromagnetic dipoles are radiated along the axis of z, the axis of a dipole being parallel to the axis of x. If we differentiate again, we get the case in which quadrupoles are radiated along the axis of z. If we operate with $\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2}$ we obtain a field in which pairs of quadrupoles with axes parallel to the axes of x and y are radiated along the axis of z. The field vectors in this case have components of types

$$E_x = \Big[-\frac{3xz}{cr^5} F'(\tau) - \frac{3xz}{c^2 r^4} F''(\tau) - \frac{xz}{c^3 r^3} F'''(\tau) + \frac{y}{c^2 r^3} G''(\tau) + \frac{y}{c^3 r^2} G'''(\tau)$$

$$+ \Big[\left(\frac{3x}{r^5} - \frac{15xz^2}{r^7}\right) F(\tau) + \left(\frac{3x}{cr^4} - \frac{12xz^2}{cr^6}\right) F'(\tau) + \left(\frac{x}{c^2 r^3} - \frac{6xz^2}{c^2 r^5}\right) F''(\tau)$$

$$- \frac{xz^2}{c^3 r^4} F'''(\tau) + \frac{3yz}{cr^5} G'(\tau) + \frac{3yz}{c^2 r^4} G''(\tau) + \frac{yz}{c^3 r^3} G'''(\tau) \Big],$$

$$E_z = \Big[\left(\frac{1}{cr^3} - \frac{3z^2}{cr^5}\right) F'(\tau) + \left(\frac{1}{c^2 r^2} - \frac{3z^2}{c^2 r^4}\right) F''(\tau) + \left(\frac{1}{c^3 r} - \frac{z^2}{c^3 r^3}\right) F'''(\tau) \Big]$$

$$+ \Big[\left(\frac{9z}{r^5} - \frac{15z^3}{r^7}\right) F(\tau) + \left(\frac{9z}{cr^4} - \frac{15z^3}{cr^6}\right) F'(\tau) + \left(\frac{4z}{c^2 r^3} - \frac{6z^2}{c\, r^5}\right) F''(\tau)$$

$$+ \left(\frac{z}{c^3 r^2} - \frac{z^3}{c^3 r^4}\right) F'''(\tau) \Big],$$

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the other components being easily written down by analogy. Now this field consists of two parts as indicated by the square brackets. The first part represents the field of a variable electromagnetic dipole whose axis is along the axis of z, while the second part represents the field of a variable electromagnetic quadrupole whose axis is likewise along the axis of x.

Subtracting the fields of the dipole and quadrupole from our field in which pairs of quadrupoles are radiated we are left with simply a "*local field*," *i.e.* a field which vanishes outside the radiated quadrupoles.*

This field bears some resemblance to the type of field which has been used by Professor Whittaker † to adjust Sir J. J. Thomson's theory of light to the classical electromagnetic theory. In Whittaker's field, magnetism alone is radiated, the positive magnetism being within a cylinder and the negative magnetism within the annular region bounded by this cylinder and a coaxial cylinder. A similar type of field may be obtained from ours by simply giving the axis of x all possible directions perpendicular to the direction of radiation and superposing the fields corresponding to the different directions by integration. The additional field needed to produce a local field is still composed of the field of a variable magnetic dipole and the field of a variable quadrupole; consequently we are perhaps justified in saying that the radiated magnetic quadrupoles are produced by the variation of the magnetic dipole and quadrupole at O.

The oscillating magnetic dipole may be regarded as the primary radiator, the presence of the quadrupole meaning simply that the dipole is displaced parallel to the axis of z on account of the reaction of the radiation. The magnetic dipole may, in its turn, be simply a symbol for an electron describing an orbit round a proton, and so the present theory of radiation bears some resemblance also to Bohr's theory.

It may be mentioned that there are many cases of electromagnetic potentials and Hertzian vectors with singularities with

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* A simple example of a local field is mentioned in *Physical Review*, Vol. 23 (1924), p. 782.

† *Proc. Roy. Soc. Edinburgh.* Vol. 46 (1926), p. 116.

the property that the associated electromagnetic field vanishes outside the singularities. The example which has just been given indicates that these fields may be well worth studying in detail.

§ 3. *Some types of local fields.*

If we put $\mathbf{Q} = \mathbf{H} + i\mathbf{E}$ a solution of Maxwell's equations is obtained by writing *

$$\mathbf{Q} = -i \, curl \, \mathbf{L} = -\frac{1}{c}\frac{\partial \mathbf{L}}{\partial t} - \nabla \Lambda$$

$$\mathbf{L} = \frac{1}{c}\frac{\partial \mathbf{G}}{\partial t} + i \, curl \, \mathbf{G} + \nabla K = \mathbf{B} + i\mathbf{A}$$

$$\Lambda = -div \, \mathbf{G} - \frac{1}{c}\frac{\partial K}{\partial t} = \Psi + i\Phi$$

$$div \, \mathbf{L} + \frac{1}{c}\frac{\partial \Lambda}{\partial t} = 0$$

where $\mathbf{G} = \Gamma + i\Pi$ and $K = U + iV$ satisfy the wave equation $\Box \theta = 0$. When we differentiate to obtain the electric and magnetic vectors, K drops out and we find

$$\mathbf{Q} = curl\left(curl \, \mathbf{G} - \frac{i}{c}\frac{\partial \mathbf{G}}{\partial t}\right).$$

A local field may be obtained by making *curl* $\mathbf{G}$ the gradient of a scalar complex function. An interesting type is obtained by writing $K = 0$, $div \, \mathbf{G} = 0$, $curl \, \mathbf{G} = \frac{i}{c}\frac{\partial \mathbf{G}}{\partial t}$. In this case the vectors Γ and Π satisfy equations analogous to Maxwell's equations, and all the singular fields that are known to satisfy these equations may be used to obtain local electromagnetic fields

Local fields may also be obtained by writing

$$L = \nabla\theta, \ \Lambda = -\frac{1}{c}\frac{\partial \theta}{\partial t},$$

where θ is a complex scalar satisfying the wave-equation.

* E. T. Whittaker, *Proc. London Math. Soc.* (2), Vol. 1 (1903). H. Bateman *Electrical and Optical Wave Motion*, p. 7.