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SOME REMARKS ON THE NIJENHUIS TENSOR

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A differential form α of degree r on an n-manifold is exact if there exists a form β of degree $r - 1$ such that $\alpha = d\beta$ and is closed if $d\alpha = 0$. Since $d \cdot d = 0$ any exact form is closed. The Poincaré lemma asserts that a closed differential form of positive degree is locally exact. There is also a complex form, proved by Cartan-Grothendieck, of the Poincaré lemma in which the operator d has a decomposition into components ∂ and $\bar{\partial}$. In this paper a Poincaré lemma is established for an operator d_h, where $\mathbf{h}$ is a nonsingular vector 1-form with vanishing Nijenhuis torsion $[\mathbf{h}, \mathbf{h}]$. The operator d_h is an anti derivation of degree one and it reduces to d when $\mathbf{h}$ is the identity on differential forms of degree one. The main result (Theorem 3.5) states that if $\mathbf{h}$ is any nonsingular vector 1-form whose Nijenhuis tensor $[\mathbf{h}, \mathbf{h}]$ vanishes identically, and if $d_h\alpha = 0$ for any differential form α of degree r, then locally there exists a differential form β of degree $r - 1$ such that $\alpha = d_h\beta$. In section 4 some applications involving conservation laws and almost complex structures are discussed.

It should be noted that the vanishing of $[\mathbf{h}, \mathbf{h}]$ is a condition which has appeared in various integrability problems on manifolds. A list of some of these problems would include the study of almost complex and complex manifolds, Kähler manifolds, G-structures, Sasakian structures, and f-structures.

The operator d_h is an example of a derivation of "type d_*". These derivations were studied by A. Frölicher and A. Nijenhuis in [**1**], which deals with the theory and basic properties of vector valued differential forms (i.e. differential forms whose values are tangent vectors). In their paper the theory is developed by starting with the graded ring Φ of C^∞ differential forms (scalar forms) over a C^∞ manifold. The ring is commutative in the sense that $\varphi_p \wedge \psi_q = (-1)^{pq}\psi_q \wedge \varphi_p$, where the subscripts denote the degrees of the elements. The subring Φ_0 of elements of degree zero consists of the C^∞ functions. Vector forms then arise in connection with derivations in Φ. A mapping $D: \Phi \rightarrow \Phi$ is a derivation of degree r if $D(\Phi_p) \subset \Phi_{p+r}$ and $D(\varphi + \psi) = D\varphi + D\psi$. Their analysis showed that there are two special kinds of derivations, namely those of "type i_*" and those of "type d_*", and every derivation of degree r is a sum of derivations of these two types. A derivation D of type d_* is determined by its action on Φ_0, and the requirement that $Dd = (-1)^r dD$. Examples of derivations of type d_* of degree one are exterior differentiation d, and the operator d_h which is defined in the next section. The Lie derivative provides an example of a derivation of type d_* and degree zero.

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2. Preliminaries. Let M be a C^∞ manifold of dimension n and A the ring of C^∞ functions on M. The module of C^∞ differential forms of degree p on M is denoted by $\Lambda^p\mathscr{E}$ and so $\Lambda^0\mathscr{E} = A$ and $\Lambda^1\mathscr{E} = \mathscr{E}$. A C^∞ vector 1-form $\mathbf{h}$ on M is an element of $\mathrm{Hom}_A(\mathscr{E}, \mathscr{E})$. The vector 1-form $\mathbf{h}$ induces homomorphisms $h^{(p)}$, $p = 0, 1, \ldots, q$,

$$\Lambda^q\mathscr{E} \xleftarrow{h^{(p)}} \Lambda^q\mathscr{E}$$

which are defined (as in [**5**]) by the equations

$$(2.1)\quad h^{(p)}(\varphi_1 \wedge \ldots \wedge \varphi_q) = \frac{1}{(q-p)!p!}\sum_\pi |\pi|\{\mathbf{h}\varphi_{\pi(1)} \wedge \ldots \wedge \mathbf{h}\varphi_{\pi(p)}\} \wedge \varphi_{\pi(p+1)} \wedge \ldots \wedge \varphi_{\pi(q)}$$

where $\varphi_i \in \mathscr{E}$, π runs through all permutations of $(1, 2, \ldots, q)$, and the signature of π is denoted by $|\pi|$. The transformation $h^{(0)}$ is the identity on $\Lambda^q\mathscr{E}$. Observe that $h^{(1)}$ defines a derivation and note also that for $q = 2$.

$$h^{(1)}(\varphi_1 \wedge \varphi_2) = \mathbf{h}\varphi_1 \wedge \varphi_2 + \varphi_1 \wedge \mathbf{h}\varphi_2,$$
$$h^{(2)}(\varphi_1 \wedge \varphi_2) = \mathbf{h}\varphi_1 \wedge \mathbf{h}\varphi_2.$$

The operators $h^{(p)}$ can be expressed in terms of $h^{(1)}, h^{(2)}, \ldots, h^{(p-1)}$ by the formula

$$(2.2)\quad [(h^p)^{(1)} - (h^{p-1})^{(1)}h^{(1)} + \ldots + (-1)^{p-1}h^{(1)}h^{(p-1)}]\beta = \begin{cases} 0, & p > q \\ (-1)^{p-1}ph^{(p)}\beta, & p \leqq q \end{cases}$$

where $\beta \in \Lambda^q\mathscr{E}$. Formula (2.2) is established in [**5**]. Note that the case $p = 2$ is simply the statement that $(h^2)^{(1)} - h^{(1)}h^{(1)} = -2h^{(2)}$ on q-forms, when $q \geqq 2$.

It is not difficult to prove that

$$(2.3)\quad \det[h^{(p)}] = (\det[\mathbf{h}])^{\binom{n-1}{p-1}}$$

where $1 \leqq p \leqq n$ and where $h^{(p)}$ in formula (2.3) is interpreted as an operator on p-forms. In particular, $h^{(p)}$ regarded as an operator on p-forms is invertible if and only if $\mathbf{h}$ is invertible.

The Nijenhuis tensor $[\mathbf{h}, \mathbf{h}]$ of $\mathbf{h}$ can be defined on p-forms α by the equation

$$(2.4)\quad [\mathbf{h}, \mathbf{h}]\alpha = -h^{(2)}d\alpha + h^{(1)}dh^{(1)}\alpha - d(h^{(2)} + (h^2)^{(1)})\alpha.$$

Observe that if $p = 1$, and $\varphi \in \mathscr{E}$, then $[\mathbf{h}, \mathbf{h}]\varphi = -h^{(2)}d\varphi + h^{(1)}d\mathbf{h}\varphi - d\mathbf{h}^2\varphi$, which is the usual expression for $[\mathbf{h}, \mathbf{h}]$ on 1-forms. It is not hard to establish that

$$(2.5a)\quad [\mathbf{h}, \mathbf{h}](\varphi_p \wedge \psi_q) = [\mathbf{h}, \mathbf{h}]\varphi_p \wedge \psi_q + (-1)^p\varphi_p \wedge [\mathbf{h}, \mathbf{h}]\psi_q$$

$$(2.5b)\quad [\mathbf{h}, \mathbf{h}]f\varphi_p = f[\mathbf{h}, \mathbf{h}]\varphi_p,$$

for any $\varphi_p \in \Lambda^p \mathscr{E}$, $\psi_q \in \Lambda^q \mathscr{E}$, $f \in A$. Thus $[\mathbf{h}, \mathbf{h}]$ is a derivation (of degree 1) and is an element of $\text{Hom}_A(\Lambda^r \mathscr{E}, \Lambda^{r+1} \mathscr{E})$.

An operator $d_h : \Lambda^p \mathscr{E} \to \Lambda^{p+1} \mathscr{E}$ may be defined by setting

$$d_h \alpha = h^{(1)} d\alpha - dh^{(1)} \alpha \tag{2.6}$$

for any $\alpha \in \Lambda^p \mathscr{E}$, and any non-negative integer p. In particular, if $f \in A$ then $d_h f = \mathbf{h} df$. Observe also that $d_I = d$, where I is the identity on $\mathscr{E}$. The following propositions are easy to establish by direct calculations.

PROPOSITION 2.1. *The operator d_h is an anti-derivation of degree* 1.

PROPOSITION 2.2. $d[\mathbf{h}, \mathbf{h}] + [\mathbf{h}, \mathbf{h}]d = d_h d_h$.

A p-form α is said to be $\mathbf{h}$-closed if $d_h \alpha = 0$; similarly α is $\mathbf{h}$-exact if there exists a $p - 1$ form β such that $\alpha = d_h \beta$. Hence, as a consequence of Proposition 2.2, if $[\mathbf{h}, \mathbf{h}] = 0$ any $\mathbf{h}$-exact form is $\mathbf{h}$-closed.

3. A Poincaré lemma for d_h.

It is clear from the definitions (2.4) and (2.6) of $[\mathbf{h}, \mathbf{h}]$ and d_h that the Nijenhuis tensor can be written in the alternative form

$$[\mathbf{h}, \mathbf{h}]\alpha = -h^{(2)} d\alpha + d_h h^{(1)} \alpha + dh^{(2)} \alpha \tag{3.1}$$

for any $\alpha \in \Lambda^p \mathscr{E}$. In particular, if $\varphi \in \mathscr{E}$ then equation (3.1) can be written as $[\mathbf{h}, \mathbf{h}]\varphi = -h^{(2)} d\varphi + d_h \mathbf{h} \varphi$.

The relationship between closed forms and $\mathbf{h}$-closed forms (degree 1) is expressed by the following proposition.

PROPOSITION 3.1. *Let $[\mathbf{h}, \mathbf{h}] = 0$ and suppose $\mathbf{h}$ is non-singular. A differential form $\varphi \in \mathscr{E}$ is closed if and only if $\mathbf{h}\varphi$ is $\mathbf{h}$-closed.*

Proof. If φ is closed, then equation (3.1) implies $d_h \mathbf{h} \varphi = 0$. Conversely, if $d_h \mathbf{h} \varphi = 0$, then $h^{(2)} d\varphi = 0$ and since $h^{(2)}$ is invertible on 2-forms φ is closed.

A special case of the main result (Theorem 3.5) is the following theorem, which is proven here to illustrate the idea of the proof of the main theorem.

THEOREM 3.2. *Let $[\mathbf{h}, \mathbf{h}] = 0$ and suppose that $\mathbf{h}$ is non-singular. If $d_h \varphi = 0$ for any $\varphi \in \mathscr{E}$ then locally there exists a function $f \in A$ such that $\varphi = d_h f$.*

Proof. If $d_h \varphi = 0$, then $h^{(2)} d\mathbf{h}^{-1} \varphi = 0$ and hence there exists a function $f \in A$ such that $h^{-1} \varphi = df$. Thus $\varphi = \mathbf{h} df = d_h f$ and the theorem is established.

An operator $\{\mathbf{h}, \mathbf{h}\}$ which takes p-forms into $(p + 1)$-forms is defined by setting

$$\{\mathbf{h}, \mathbf{h}\}\alpha = -h^{(p+1)} d\alpha + d_h h^{(p)} \alpha \tag{3.2}$$

for any $\alpha \in \Lambda^p \mathscr{E}$. Observed that $\{\mathbf{h}, \mathbf{h}\} f\alpha = f\{\mathbf{h}, \mathbf{h}\}\alpha$ for any $f \in A$, and moreover that $\{\mathbf{h}, \mathbf{h}\} = [\mathbf{h}, \mathbf{h}]$ when $p = 1$. The following lemma relates $\{\mathbf{h}, \mathbf{h}\}$ to $[\mathbf{h}, \mathbf{h}]$ when $p > 1$.

LEMMA 3.3. *For* $p = 1, 2, \ldots, n$, *and* $\varphi_i \in \mathscr{E}$, $1 \leqq i \leqq p$,

(3.3) $\{\mathbf{h}, \mathbf{h}\}(\varphi_1 \wedge \ldots \wedge \varphi_p)$

$$= \sum_{i=1}^{p} (-1)^{i-1} \mathbf{h}\varphi_1 \wedge \ldots \wedge \mathbf{h}\varphi_{i-1} \wedge [\mathbf{h}, \mathbf{h}]\varphi_i \wedge \mathbf{h}\varphi_{i+1} \wedge \ldots \wedge \mathbf{h}\varphi_p.$$

Proof. The formula (3.3) is established by a direct computation. From equation (3.2),

$$\begin{aligned}
&\{\mathbf{h}, \mathbf{h}\}(\varphi_1 \wedge \ldots \wedge \varphi_p) \\
&= -h^{(p+1)} d(\varphi_1 \wedge \ldots \wedge \varphi_p) + d_h h^{(p)}(\varphi_1 \wedge \ldots \wedge \varphi_p) \\
&= (-h^{(p+1)}[d\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_p - \varphi_1 \wedge d\varphi_2 \wedge \ldots \wedge \varphi_p \wedge + \wedge \ldots \wedge + \\
&\quad + (-1)^{p-1} \varphi_1 \wedge \ldots \wedge \varphi_{p-1} \wedge d\varphi_p]) \\
&\quad + d_h(\mathbf{h}\varphi_1 \wedge \mathbf{h}\varphi_2 \wedge \ldots \wedge \mathbf{h}\varphi_p) \\
&= (-h^{(2)} d\varphi_1 \wedge \mathbf{h}\varphi_2 \wedge \ldots \wedge \mathbf{h}\varphi_p + \mathbf{h}\varphi_1 \wedge h^{(2)} d\varphi_2 \wedge \mathbf{h}\varphi_3 \wedge \ldots \wedge \mathbf{h}\varphi_p - \ldots \\
&\quad + (-1)^p \mathbf{h}\varphi_1 \wedge \ldots \wedge \mathbf{h}\varphi_{p-1} \wedge d\varphi_p) + (d_h \mathbf{h}\varphi_1 \wedge \mathbf{h}\varphi_2 \wedge \ldots \wedge \mathbf{h}\varphi_p \\
&\quad - \mathbf{h}\varphi_1 \wedge d_h \mathbf{h}\varphi_2 \wedge \mathbf{h}\varphi_3 \wedge \ldots \wedge \mathbf{h}\varphi_p + \ldots \\
&\quad + (-1)^{p-1} \mathbf{h}\varphi_1 \wedge \ldots \wedge \mathbf{h}\varphi_{p-1} \wedge d_h \mathbf{h}\varphi_p) \\
&= [\mathbf{h}, \mathbf{h}]\varphi_1 \wedge \mathbf{h}\varphi_2 \wedge \ldots \wedge \mathbf{h}\varphi_p - \mathbf{h}\varphi_1 \wedge [\mathbf{h}, \mathbf{h}]\varphi_2 \wedge \ldots \wedge \mathbf{h}\varphi_q + \ldots \\
&\quad + (-1)^{p-1} \mathbf{h}\varphi_1 \wedge \ldots \wedge \mathbf{h}\varphi_{p-1} \wedge [\mathbf{h}, \mathbf{h}]\varphi_p.
\end{aligned}$$

COROLLARY. *If* $[\mathbf{h}, \mathbf{h}] = 0$, *then* $\{\mathbf{h}, \mathbf{h}\} = 0$.

The proofs of the following proposition and theorem are a consequence of the preceding corollary and they parallel the proofs of Proposition 3.1 and Theorem 3.2. The fact that $h^{(p)}$ is invertible on p-forms if and only if $\mathbf{h}$ is invertible on 1-forms is also needed in the proofs.

PROPOSITION 3.4. *If* $[\mathbf{h}, \mathbf{h}] = 0$ *and* $\mathbf{h}$ *is non-singular, then* α *is a closed* p*-form if and only if* $h^{(p)}\alpha$ is $\mathbf{h}$*-closed.*

THEOREM 3.5. *Let* $[\mathbf{h}, \mathbf{h}] = 0$ *and suppose that* $\mathbf{h}$ *is non-singular. If* $d_h\alpha = 0$ *for any* $\alpha \in \Lambda^p \mathscr{E}$, *then locally there exists a form* $\beta \in \Lambda^{p-1}\mathscr{E}$ *such that* $\alpha = d_h\beta$.

Proof. It is clear that if $d_h\alpha = 0$, then there exists a form $\lambda \in \Lambda^{p-1}\mathscr{E}$ such that $\alpha = h^{(p)} d\lambda$. Thus if $\beta \equiv h^{(p-1)}\lambda$, then

$$\begin{aligned}
\alpha = h^{(p)} d\lambda = d_h h^{(p-1)}\lambda &= h^{(1)} d h^{(p-1)}\lambda - d h^{(1)} h^{(p-1)}\lambda \\
&= h^{(1)} d\beta - d h^{(1)}\beta = d_h\beta
\end{aligned}$$

and the theorem is established.

4. Applications. The first application concerns the notion of conservation laws on a manifold. A 1-form $\varphi \in \mathscr{E}$ is a conservation law for $\mathbf{h}$ if both φ and $\mathbf{h}\varphi$ are exact (see [**4**]). This definition of a conservation law on a manifold can be related to the notion of a conservation law in the sense of physics. The following proposition gives an equivalent definition of a conservation law for $\mathbf{h}$ in terms of the operator d_h.

PROPOSITION 4.1. *Let* $[\mathbf{h}, \mathbf{h}] = 0$ *and suppose that* $\mathbf{h}$ *is non-singular. A* 1-*form* φ *is a conservation law for* $\mathbf{h}$ *if and only if* $d_h\varphi = d_h\mathbf{h}\varphi = 0$.

Proof. If $\varphi \in \mathscr{E}$ is a conservation law for $\mathbf{h}$, then formula (2.4) implies $d\mathbf{h}^2\varphi = 0$. Thus $d\varphi = d\mathbf{h}\varphi = d\mathbf{h}^2\varphi = 0$ and hence $d_h\varphi = d_h h\varphi = 0$. Conversely if $d_h\mathbf{h}\varphi = 0$, then $h^{(2)}d\varphi = 0$ and hence φ is locally exact. The condition that $d_h\varphi = 0$ then implies $\mathbf{h}\varphi$ is also locally exact.

Note that if φ is a conservation law for $\mathbf{h}$, so are the forms $\mathbf{h}^i\varphi$ for any positive integer i and hence $d_h\mathbf{h}^i\varphi = 0$. This fact results from a repeated application of formula (2.4).

An almost complex structure is defined if $\mathbf{h}$ satisfies the condition $\mathbf{h}^2 = -I$, where I is the identity on 1-forms. A second application provides an alternative characterization of the integrability of an almost complex structure $\mathbf{h}$. Since $[\mathbf{h}, \mathbf{h}] = -h^{(2)}d + d_h\mathbf{h}$, the condition that $h^2 = -I$ yields the formula (on 1-forms)

$$(\mathbf{h}, \mathbf{h}] = \tfrac{1}{2}\{d_h\mathbf{h} - h^{(1)}d_h + d\}. \tag{4.1}$$

Thus if φ is a conservation law for $\mathbf{h}$, $[\mathbf{h}, \mathbf{h}]\varphi = 0$.

PROPOSITION 4.2. *Let* $\mathbf{h}^2 = -I$; *then* $\mathbf{h}$ *defines a complex structure if and only if there exists a basis of conservation laws for* $\mathscr{E}$.

Proof. If a basis of conservation laws exists then equation (4.1) implies $[\mathbf{h}, \mathbf{h}] = 0$. Conversely, if $[\mathbf{h}, \mathbf{h}] = 0$ then the almost complex structure $\mathbf{h}$ is integrable and hence there exist coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ such that $\mathbf{h}dx^i = -dy^i$ and $\mathbf{h}dy^i = dx^i$, $i = 1, 2, \ldots, n$, and consequently $(dx^1, \ldots, dx^n, dy^1, \ldots, dy^n)$ is a basis of conservation laws.

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