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Estimates of Hausdorff Dimension for Non-wandering Sets of Higher Dimensional Open Billiards

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Abstract. This article concerns a class of open billiards consisting of a finite number of strictly convex, non-eclipsing obstacles K. The non-wandering set M_0 of the billiard ball map is a topological Cantor set, and its Hausdorff dimension has been previously estimated for billiards in $\mathbb{R}^2$ using well-known techniques. We extend these estimates to billiards in $\mathbb{R}^n$ and make various refinements to the estimates. These refinements also allow improvements to other results. We also show that in many cases, the non-wandering set is confined to a particular subset of $\mathbb{R}^n$ formed by the convex hull of points determined by period 2 orbits. This allows more accurate bounds on the constants used in estimating Hausdorff dimension.

1 Introduction

A billiard is a dynamical system in which a single pointlike particle moves at constant speed in some domain $Q \subset \mathbb{R}^D$, $D \geq 2$ and reflects off the boundary ∂Q according to the classical laws of optics [Ch]. We describe a particle in the billiard by $x_t = (q_t, v_t)$, where $q_t \in Q$ is the position of the particle and $v_t \in \mathbb{S}^{D-1}$ is its velocity at time t. Then for as long as the particle stays inside Q, it satisfies

$$(q_{t+s}, v_{t+s}) = \varphi_s(x_t) = (q_t + sv_t, v_t).$$

Collisions with the boundary are described by

$$v^+ = v^- - 2\langle v^-, n\rangle n,$$

where n is the normal vector (into Q) of ∂Q at the point of collision, v^- is the velocity before reflection, and v^+ is the velocity after reflection.

Open billiards are a class of billiard in which the domain Q is unbounded. We consider open billiards in which $Q = \mathbb{R}^D \backslash K$, where $K = K_1 \cup \cdots \cup K_u$ is a union of pairwise disjoint, compact and strictly convex sets with C^2 boundary for some integer $u \geq 3$. The K_i are called *obstacles*. We assume that the *no-eclipse condition* (**H**) holds. That is, for any nonequal i, j, k, the convex hull of $K_i \cup K_j$ does not intersect K_k. This condition ensures that the non-wandering set (defined later) does not include trajectories that are tangent to the boundary.

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We denote by $n = n_K(q)$ the outward normal vector of ∂K at q. Let

$$\widehat{Q} = \{(q, v) \in Q \times \mathbb{S}^{D-1} | q \in \operatorname{int}(Q) \text{ or } \langle n, v\rangle \geq 0\}$$

be the phase space of the billiard flow φ_t, with canonical projection $\pi\colon \widehat{Q} \to Q$. Let $M = \{(q, v) \in \partial K \times \mathbb{S}^{D-1} | \langle n, v\rangle \geq 0\}$ be the boundary of $\widehat{Q}$.

Let $t_j(x) \in [-\infty, \infty]$ denote the time of the j-th reflection of $x \in \widehat{Q}$, and let $d_j(x) = t_j(x) - t_{j-1}(x)$. Let $\widehat{Q}' = t_1^{-1}(0, \infty)$; this is the set of trajectories that collide with the billiard at least once in the forward direction. Let $M' = M \cap \widehat{Q}'$ and define the *billiard ball map* as $B\colon M' \to M, x \mapsto \varphi_{t_1(x)}(x)$. Then B is invertible and C^2 (in general B is at least as smooth as the boundaries of the obstacles), except where the direction v is tangent to K at Bx.

The *non-wandering set* $\Omega(\varphi)$ of the flow is the set of points whose trajectories never escape to infinity *i.e.,* the set of points x such that the full trajectory $\{\varphi_t(x) : t \in \mathbb{R}\}$ is bounded. Its restriction to the boundary $M_0 = \Omega \cap (\partial K \times \mathbb{S}^{D-1})$ is the non-wandering set of the billiard ball map, that is, $M_0 = \{x \in \widehat{Q}' : B^j x \in \widehat{Q}' \text{ for all } j \in \mathbb{Z}\}$. The restriction of B to M_0 is a bijection. M_0 is the non-wandering set of the billiard ball map; this non-wandering set is the main focus of this paper.

2 Main Theorem

The main result of this paper is in three parts.

Theorem 2.1 *Let $K = K_1 \cup \cdots \cup K_u \subset \mathbb{R}^D$ be disjoint, compact, and strictly convex sets with smooth boundary satisfying the condition* $(\mathbf{H})$ *for some integer $u \geq 3$. Let B be the billiard ball map in $Q = \mathbb{R}^D \backslash K$. Let $\lambda_1^{-1} = 1 + d_{\max} g_{\max}$ and $\mu_1^{-1} = 1 + d_{\min} g_{\min}$, where $d_{\min}$, $d_{\max}$, $g_{\min}$, and $g_{\max}$ are constants depending on the billiard, defined in Sections 3 and 11. Then the Hausdorff dimension of the non-wandering set M_0 of B satisfies the following inequalities:*

(i) *If $D = 2$, then*

$$\frac{-2\ln(u-1)}{\ln \lambda_1} \leq \dim_H M_0 \leq \frac{-2\ln(u-1)}{\ln \mu_1}. \tag{2.1}$$

(ii) *If $D \geq 3$, and the obstacles K_i are sufficiently far apart that $\lambda_1^{d_{\max}} < \mu_1^{2d_{\min}}$, then* (2.1) *holds.*

(iii) *We always have*

$$\alpha \frac{-2\ln(u-1)}{\ln \lambda_1} \leq \dim_H M_0 \leq \alpha^{-1} \frac{-2\ln(u-1)}{\ln \mu_1},$$

where $\alpha = \frac{2d_{\min} \ln \mu_1}{d_{\max} \ln \lambda_1}$ is a particular Hölder constant, calculated in Section 10.

Remark 2.2 The billiard constants $g_{\min}$ and $g_{\max}$ are calculated differently in the higher dimensional case. This affects the dimension estimates and the condition that $\lambda_1^{d_{\max}} < \mu_1^{2d_{\min}}$.

Remark 2.3 Hassleblatt and Schmeling present a conjecture in [HS] that would imply that $\alpha = 1$ for any billiard, making the above theorem much stronger. This will be discussed in Section 9.

Part (i) was essentially proved in [Ke], except that the improvements to estimates in Section 11 can be applied. We deal with the higher dimensional case here.

3 Properties of Open Billiards

The following lemma is well known (see, for example, [Sto1]).

Lemma 3.1 *If K satisfies the no-eclipse condition* (**H**)*, then for any finite sequence of indices $1 \le i_1, \dots, i_n \le u$ $(n \ge 3)$ such that $i_j \ne i_{j+1}$ for all j, let*

$$F\colon K_{i_1} \times \cdots \times K_{i_n} \to \mathbb{R}, (q_1, \dots, q_n) \mapsto \sum_{j=1}^{n} \|q_j - q_{j+1}\|,$$

where we denote $q_{n+1} = q_1$. Then F achieves its unique (strict) minimum at some $(p_1, \dots, p_n)$ such that $p_j \in \partial K_{i_j}$ for all j. Specifically, the p_j are the successive reflection points of a periodic billiard trajectory in Q with $p_{j+1} = Bp_j$ and $p_1 = Bp_n$.

3.1 Billiard Constants

Definition 3.2 At each point on a hypersurface M, the shape operator or second fundamental form (s.f.f.) $S_q\colon T_q(M) \to T_q(M)$ is defined by $S_q(v) = -\nabla_v n_M(q)$. The curvature of M at q in the direction of a unit vector $\widehat{u} \in T_q(M)$ is $k_q(\widehat{u}) = S_q(\widehat{u}) \cdot \widehat{u}$.

Every billiard has several associated constants that can be useful in various estimates. The s.f.f. $S_q(\partial K)$ of ∂K at q has $n-1$ eigenvalues or principle curvatures. Let $\kappa_{\min}(q), \kappa_{\max}(q)$ denote the smallest and largest eigenvalues respectively at q. The billiard has minimum and maximum curvatures $\kappa^- = \min_{q \in \pi M_0} \kappa_{\min}(x)$ and $\kappa^+ = \max_{x \in M_0} \kappa_{\max}(x)$. We denote

$$d_{\min} = \min\{d_{ij}^- : 1 \le i, j \le u\} \quad \text{and} \quad d_{\max} = \max\{d_{ij}^- : 1 \le i, j \le u\},$$

where d_{ij}^- and d_{ij}^+ are the respective minimum and maximum of the set $\{d(p,q) : p \in K_i \cap \pi M_0, q \in K_j \cap \pi M_0\}$. For a point $x = (q, v) \in M$, we call $\phi(x) = \arccos\langle v, n(q)\rangle$ the *collision angle*, the acute angle that the j-th reflected ray makes with the outer normal to K. We denote $\phi_j(x) = \phi(B^j x)$. The collision angle can be bounded above by some constant $\phi^+ = \max\{\phi(x) : x \in M_0\}$. It can easily be shown that $\phi^+ \le \arccos(b^-/d_{\max})$, where $b^- = \min_{i,j,k} d(K_j, \operatorname{Cvx}(K_i, K_k))$.

4 Convex Fronts

Let X be a smooth, stricly convex $D-1$ dimensional surface in int Q with outer normal field $\nu(q)$, let $\widehat{X} = \{(q, \nu(q)) : q \in X\}$, $\widehat{X}_0 = \widehat{X} \cap \Omega$, and $X_0 = \pi\widehat{X_0}$, where

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π is the canonical projection. We call $\widehat{X}$ a *convex front*, and $\widehat{X}_0$ is the non-wandering part of the front. Let $\widehat{X}_t = \varphi_t \widehat{X}$; after a collision $\widehat{X}_t$ will not necessarily be convex. Let $\mathcal{J}$ denote the tangent space $T_q(X)$ of the convex front at q. This is simply the codim 1 hyperplane normal to v containing q. The s.f.f. of X at q is given by $\mathcal{B}(q)\colon \mathcal{J} \to \mathcal{J}$, $\mathcal{B}(q)dv = S_q(dv)$.

4.1 Evolution of Fronts

With no collisions, the s.f.f. of a convex front X after time t is given by the well-known formula (see, *e.g.*, [S2])

$$\mathcal{B}(q_t(s)) = (\mathcal{B}(q)^{-1} + tI)^{-1}.$$

Fix a point $x = (q, v)$ at which the front $\widehat{X}_t$ collides with ∂K at time t. Note that v is the reflected direction. Let $\mathcal{T}$ denote the tangent space of ∂K at q and let $\mathcal{J}$ be the tangent space $T_q(X_t)$ of the front just after reflection at q. Again $\mathcal{J}$ is the hyperplane normal to v containing q.

At a collision point, let B^- be the s.f.f. just before the collision and let $\mathcal{B}^+$ be the s.f.f. just after the collision. These are related by the well-known formula (again see [BCST, S2])

$$\mathcal{B}^+ = \mathcal{B}^- + 2\Theta = \mathcal{B}^- + 2\langle n, v\rangle V^* K V,$$

where $V\colon \mathcal{J} \to \mathcal{T}$ is the projection $V dv = dv - \frac{\langle dv, n\rangle}{\langle n, v\rangle} v \in \mathcal{T}$, $K\colon \mathcal{T} \to \mathcal{T}$ is the s.f.f. of ∂K at q, $V^*\colon \mathcal{T} \to \mathcal{J}$ is the projection $V^* dq = dq - \frac{\langle dq, v\rangle}{\langle n, v\rangle} n \in \mathcal{J}$, and $\langle n, v\rangle = \cos\phi$, where $\phi \in [0, \frac{\pi}{2})$ is the collision angle. Note in these projections that while $\langle n, v\rangle$ is always positive, $\langle dv, n\rangle$ and $\langle dq, n\rangle$ may be positive or negative.

4.2 Estimating Θ

Lemma 4.1 *If the dimension n is greater than 2, let $\kappa_{\min}$, $\kappa_{\max}$ be the smallest and largest eigenvalues of the s.f.f. K at q, so that $\kappa_{\min}|dq| \leq \|K dq\| \leq \kappa_{\max}|dq|$. Then*

$$\kappa_{\min} \cos\phi \leq \|\Theta\| \leq \frac{\kappa_{\max}}{\cos\phi}.$$

Proof If $n = v$, then $\langle n, v\rangle = \cos\phi = 1$ so $\Theta = \langle n, v\rangle V^* K V = K$, and the inequality holds. Henceforth we assume $n \neq v$. Let $\mathcal{S} = \mathcal{J} \cap \mathcal{T}$. Any vector $dv \in \mathcal{J}$ can be written in the form $dv = |dv|(\widehat{a}\cos\theta + \widehat{s}\sin\theta)$, where $\widehat{s} \in \mathcal{S}$ and $\widehat{a} \in \mathcal{J}$ are unit vectors, $\widehat{a}$ is perpendicular to $\mathcal{S}$, and $\langle \widehat{a}, n\rangle \geq 0$. Then $\widehat{a}$ is in the plane containing by n and v, so the angle between $\widehat{a}$ and n is $\frac{\pi}{2} - \phi$. Using $dv \perp v$ we get

$$\begin{aligned} \|V dv\| &= \|dv - \frac{\langle |dv|\widehat{s}\sin\theta, n\rangle}{\langle n, v\rangle} v - \frac{\langle |dv|\widehat{a}\cos\theta, n\rangle}{\langle n, v\rangle} v\| \\ &= \|dv - \big(|dv|\tan\phi\cos\theta\big) v\| = \sqrt{1 + \tan^2\phi\cos^2\theta}\,|dv| \end{aligned}$$

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Similarly, write $dq \in \mathcal{T}$ as $dq = |dq|(\widehat{b}\cos\theta' + \widehat{s'}\sin\theta')$ for some unit vectors $\widehat{s'} \in \mathcal{S}$ and $\widehat{b} \in \mathcal{T}$ with $\widehat{b} \perp \mathcal{S}$ and $\langle \widehat{b}, \widehat{\nu} \rangle \geq 0$. Then $\|V^* dq\| = \sqrt{1 + \tan^2\phi\cos^2\theta'}|dq|$. Combining these operator norms and using $0 \leq \cos^2\theta, \cos^2\theta' \leq 1$, we get

$$\begin{aligned}\kappa_{\min}\cos\phi &\leq \cos\phi\sqrt{1+\tan^2\phi\cos^2\theta}\,\kappa_{\min}\sqrt{1+\tan^2\phi\cos^2\theta'}\\ &\leq \|\Theta\| \leq \cos\phi\sqrt{1+\tan^2\phi\cos^2\theta}\,\kappa_{\max}\sqrt{1+\tan^2\phi\cos^2\theta'}\\ &\leq \cos\phi\,\kappa_{\max}(1+\tan^2\phi) \leq \frac{\kappa_{\max}}{\cos\phi}\end{aligned}$$

as required. ∎

Note that in the two dimensional case $\theta = \theta' = 0$, since $\mathcal{S} = \mathcal{T} \cap \mathcal{J} = \{q\}$, and $\kappa_{\min} = \kappa_{\max} = \kappa$ at every point. So $\|\Theta\| = \frac{\kappa}{\cos\phi}$ in this case.

4.3 Estimating k_j

Let $Y : q(s), s \in [0,1]$ be a C^3 curve on X with outer normal field parametrised by $\nu(s) = \nu(q(s))$. Let $Y_0 = Y \cap X_0$, $\widehat{X}_t = \varphi_t(\widehat{X})$, $X_t = \pi\widehat{X}_t$, $\widehat{Y}_t = \varphi_t(\widehat{Y})$, $Y_t = \pi\widehat{Y}_t$, and $t_j(s) = t_j(q(s), n(s))$. Where defined, let $q_j(s) = \pi B^j(q(s), \nu(s))$ be the j-th reflection point of $(q(s), \nu(s))$, then let $d_j(s) = t_j(s) - t_{j-1}(s)$, and $\phi_j(s) = \phi_j(q(s), \nu(s))$. This section follows the definitions in [Sto3]. Let $u_j(s) = \lim_{\tau \downarrow t_j(s)} \frac{d}{ds}\varphi_\tau q(s)$ and let $\widehat{u}_j(s) = \frac{u_j(s)}{\|u_j(s)\|}$ be the unit tangent vector of Y_t at $q(s)$. Let $\mathcal{B}_j$ be the s.f.f. of $\varphi_{t_j(s)}X$ at $q_j(s)$. Define $\ell_j(s) > 0$ by

$$[1 + d_j(s)\ell_j(s)]^2 = \|\widehat{u}_j(s) + d_j(s)\mathcal{B}_j\widehat{u}_j(s)\|^2.$$

Then set $\delta_j(s) = \frac{1}{1+d_j(s)\ell_j(s)}$.

Proposition 4.2 *Fix a point $x_0 = (q_0, \nu_0) \in \widehat{X}$, a positive integer m and some τ with $t_m(x_0) < \tau < t_{m+1}(x_0)$. Let $Y : [0,a] \to X$ be a C^3 curve with $q(0) = q_0$ with a small enough that for every $s \in [0,a]$ we have $t_m(x(s)) < \tau < t_{m+1}(x(s))$, where $x(s) = (q(s), \nu_X(q(s)))$, and that for all $j = 1, \ldots, m$ the points $q_j(s) \in \partial K_{i_j}$ for all $s \in [0,a]$. Then $p(s) = \pi\varphi_t(x(s))$ is a C^3 curve on X_t. For all $s \in [0,1]$ we have*

$$\|q'(s)\| = \frac{\|p'(s)\|}{1 + (\tau - t_m(s))k_m(s)}\delta_0(s)\delta_1(s)\ldots\delta_m(s).$$

Proof See [Sto3, Sto2]. The same result can be derived from [BCST] and is also proved for completeness (in two dimensions only) in [Ke]. ∎

Now the curvature of the convex front after j reflections in the direction $\widehat{u}_j$ is $k_j = \langle \mathcal{B}_j\widehat{u}_j, \widehat{u}_j \rangle$, so

$$1/\delta_j(s)^2 = 1 + 2d_j(s)k_j(s) + d_j(s)^2\|\mathcal{B}_j\widehat{u}_j(s)\|^2.$$

Let $q \in X$ and let $x = (q, \nu_X(q))$. Let $\mu_j(s)$ and $\lambda_j(s)$ be the minimum and maximum eigenvalues of $\mathcal{B}_j(q(s))$ respectively.

Recall that $\mathcal{B}_{j+1} = \mathcal{B}_{j+1}^{-} + 2\Theta = (\mathcal{B}_j^{-1} + d_j I)^{-1} + 2\Theta$. Now $\mathcal{B}_j$ is always positive definite, so μ_j and λ_j are always positive. Note that if λ is an eigenvalue of $\mathcal{B}(q(s))$, then $\frac{\lambda}{1+t\lambda}$ is an eigenvalue of $\mathcal{B}(q_t(s))$. So we have

$$\lambda_{j+1} = \frac{\lambda_j}{1+d_j\lambda_j} + \frac{2\kappa_{\max}(x_j)}{\cos\phi_j(x)} \quad \text{and} \quad \mu_{j+1} = \frac{\mu_j}{1+d_j\mu_j} + 2\kappa_{\max}(x_j)\cos\phi_j(x).$$

For all $j \geq 0$, $\mu_j(s) \leq k_j(s) \leq \lambda_j(s)$, so we get that $k_{j+1}(s)$ belongs to the interval

$$\left[\frac{k_j(s)}{1+d_j(s)k_j(s)} + 2\kappa_{\min}(x_j(s))\cos\phi_j(s), \frac{k_j(s)}{1+d_j(s)k_j(s)} + \frac{2\kappa_{\max}(x_j(s))}{\cos\phi_j(s)}\right]. \tag{4.1}$$

5 Coding M_0 and X_0

For each $x \in M_0$ we have a bi-infinite sequence of indices $\alpha = \{\alpha_i\}_{i=-\infty}^{\infty}$, $\alpha_i \in \{1,\dots,u\}$ such that $\pi B^i x \in \partial K_{\alpha_i}$. Since each K_i is convex, $\alpha_i \neq \alpha_{i+1}$ for all i, so define the symbol spaces Σ and Σ^+ as

$$\Sigma = \left\{ (\alpha_i)_{i=-\infty}^{\infty} : \alpha_i \in \{1,\dots,u\}, \alpha_i \neq \alpha_{i+1} \text{ for all } i \in \mathbb{Z} \right\},$$
$$\Sigma^+ = \left\{ (\alpha_i)_{i=1}^{\infty} : \alpha_i \in \{1,\dots,u\}, \alpha_i \neq \alpha_{i+1} \text{ for all } i \geq 0 \right\}.$$

Let $f\colon M_0 \to \Sigma, x \mapsto \alpha$ denote the representation map. The two-sided subshift $\sigma\colon \Sigma \to \Sigma, \alpha_i \mapsto \alpha_{i+1}$ is continuous under the following metric d_θ for any $\theta \in (0,1)$:

$$d_\theta(\alpha,\beta) = \begin{cases} 0 & \text{if } \alpha_i = \beta_i \text{ for all } i \in \mathbb{Z} \\ \theta^n & \text{if } n = \max\{j \geq 0 : \alpha_i = \beta_i \text{ for all } |i| < j\}. \end{cases}$$

We define a similar metric on Σ^+:

$$d_\theta(\alpha,\beta) = \begin{cases} 0 & \text{if } \alpha_i = \beta_i \text{ for all } i \geq 0 \\ \theta^n & \text{if } n = \max\{j \geq 0 : \alpha_i = \beta_i \text{ for all } 0 \leq i \leq j\}. \end{cases}$$

Lemma 5.1 ([Ke, Theorem 2.3]) *If $u \geq 2$ and $\theta \in (0,1)$, then f is a homeomorphism of M_0 (with the topology induced by M) onto (Σ, d_θ), and the shift σ is topologically conjugate to B; that is $B = f^{-1} \circ \sigma \circ f$.*

Assuming $u \geq 3$, M_0 is a compact topological Cantor set; B is topologically transitive on M_0, and its periodic points are dense in M_0. Then B is hyperbolic on M_0, and M_0 is a basic set for B [KH].

Given the convex front X, the intersection $X_0 = X \cap \Omega$ can also be coded by sequences. Define the representation map $\Upsilon\colon X_0 \to \Sigma^+$ in the same way as $f\colon M_0 \to \Sigma$. Define an equivalence relation $\sim_m$ $(m \geq 0)$ by $\alpha \sim_m \beta \Leftrightarrow \alpha_i = \beta_i$ for all $1 \leq i \leq m$, and $\alpha \sim_0 \beta$ for any $\alpha, \beta \in \Sigma^+$. We call the equivalence classes $[\alpha]_m$ *cylinders*. Define another relation (not an equivalence relation) $\approx_m$ by $\alpha \approx_m \beta$ if $\alpha \sim_m \beta$ and $\alpha_{m+1} \neq \beta_{m+1}$.

The following lemma on Hausdorff dimension $\dim_H$ and upper packing dimension $\overline{\dim}_p$ is the result of direct calculations (see, for example, [Ed, Ke]).

Lemma 5.2 *For any $\alpha \in \Sigma^+$ and $N \in \mathbb{N}$,*

$$\overline{\dim}_p([\alpha]_N, d_\theta) = \dim_H([\alpha]_N, d_\theta) = \frac{-\ln(u-1)}{\ln \theta}.$$

We find upper and lower bounds $g_{\min}$ and $g_{\max}$ such that for some $N \in \mathbb{N}$, $k_j(s) \in [g_{\min}, g_{\max}]$ for all $j \geq N$.

6 Estimating $\delta_j(s)$

Section 4.1 of [Ke] contains a significant improvement to the dimension estimate (2.1) using the continued fraction for $k_j(s)$. We can do the same using the bounds in (4.1).

The map $f_{\gamma,\theta}\colon (0,\infty) \to \mathbb{R}, x \mapsto \frac{x}{1+\theta x} + 2\gamma$ has one positive fixed point $g(\gamma,\theta) = \gamma + \sqrt{\gamma^2 + 2\gamma/\theta}$. This function is non-decreasing in γ and strictly decreasing in θ.

The natural domain for g is $[\kappa_{\min}\cos\phi^+, \frac{\kappa_{\max}}{\cos\phi^+}] \times [d_{\min}, d_{\max}]$ for the arguments of g. On this domain, the minimum and maximum values of g are $g(\kappa_{\min}, d_{\max})$ and $g(\frac{\kappa_{\max}}{\cos\phi^+}, d_{\min})$ respectively. While this domain is an obvious choice, it is not the strictest or most useful domain. We will use a smaller domain $\mathbb{D}$ defined in Section 11.

We write $g_{\min} = \max_{(\gamma,\theta)\in\mathbb{D}} g(\gamma,\theta)$ and $g_{\max} = \max_{(\gamma,\theta)\in\mathbb{D}} g(\gamma,\theta)$. The values that maximise and minimise g are denoted $(\gamma_{\max}, \theta_{\min})$ and $(\gamma_{\min}, \theta_{\max})$ respectively.

Parametrise the surface X by $q(t) = q(t_1, \dots, t_{D-1})$, where each $t_i \in [0,1]$ and D is the dimension of the billiard. Let

$$UT(X) = \left\{ (q, \widehat{u}) : q \in X, \|\widehat{u}\| = 1, \widehat{u} \text{ tangent to } X \text{ at } q \right\}$$

denote the unit tangent bundle of X, and parametrise $UT(X)$ by $x(s) = x(t, \widehat{u})$, where $s \in S = [0,1]^{D-1} \times \mathbb{S}^{D-2}$. Consider any $s = (t, \widehat{u}) \in S$ such that $q(t) \in X_0$. Let $k_0(s) = \mathcal{B}_0(t)(\widehat{u}) \cdot \widehat{u}$ be the curvature of X at $q(t)$ in the direction $\widehat{u}$, and inductively define $k_{j+1}(s) = f_{\gamma_j,\theta_j}(k_j(s))$ for $0 \leq j \leq n-1$. Then let $\theta_j = d_j(s)$ and

$$\gamma_j = k_{j+1} - \frac{k_j(s)}{1 + d_j(s)k_j(s)} \in \left[2\kappa_{\min}(x_j(s))\cos\phi_j(s), \frac{2\kappa_{\max}(x_j(s))}{\cos\phi_j(s)} \right].$$

Then the sequence $(\gamma_j, \theta_j)_1^\infty$ is contained in $\mathbb{D}$.

Lemma 6.1 *Let $a < g_{\min}$ and $b > g_{\max}$. Then there exists $n(X) > 0$ such that for all s and $j \geq n(X)$ we have $k_j(s) \in [a,b]$.*

Proof If $k_N(s) \leq g_{\max}$ for some s and some $N \geq 0$, then inductively

$$k_{j+1}(s) = f_{\gamma_j,\theta_j}(k_j(s)) \leq f_{\gamma_{\max},\theta_{\min}}(k_j(s)) \leq f_{\gamma_{\max},\theta_{\min}}(g_{\max}) = g_{\max}$$

for all $j \geq N$. Similarly if $k_N(s) \geq g_{\min}$ for some N, then $k_j(s) \geq g_{\min}$ for all $j \geq N$. For each s, define k_j^- and k_j^+ by $k_0^- = k_0, k_{j+1}^- = f_{\gamma_{\min},\theta_{\max}}(k_j^-)$ and $k_0^+ = k_0, k_{j+1}^+ = f_{\gamma_{\max},\theta_{\min}}(k_j^+)$. Then for all $j \geq 0$ and $s \in S$ we have $k_j^-(s) \leq k_j(s) \leq k_j^+(s)$,

$\lim_{j\to\infty} k_j^-(s) = g_{\min}$ and $\lim_{j\to\infty} k_j^+(s) = g_{\max}$. There must be some integer $j_0(s) \geq 0$ such that $k_j(s) \in [a,b]$ for all $j \geq j_0(s)$.

Since TX is compact, $k_0(s)$ has an infimum $k_{0,\min} = k_0(s_{\min})$ and a supremum $k_{0,\max} = k_0(s_{\max})$. Let $n(X) = \max\{j_0(s_{\min}), j_0(s_{\max})\}$. Then for $j \geq n(X)$,

$$a \leq k_j(s_{\min}) \leq k_j(s) \leq k_j(s_{\max}) \leq b,$$

so $j_0(s) \leq n(X)$ for all $s \in S$. Thus we have $k_j(s) \in [a,b]$ for all $j \geq n(X)$ as required. ∎

For any $\tau \geq 0$, $n(X) \geq n(\varphi_\tau X)$. So by taking a finite number of convex fronts X_i whose image under φ_τ covers Ω, we can get a global constant $n_0 = n(a,b) = \max\{n(X_i) : M \subset \bigcup_i X_i\}$ that depends only on a, b and the billiard itself.

Now $k_j(s) \in (a,b)$ for all $s \in q^{-1}(X)$ and $j > n_0$. So for these values,

$$\delta_j(s) \in \Big(\frac{1}{1+d_{\max}b}, \frac{1}{1+d_{\min}a}\Big).$$

Define $\lambda = 1/(1+d_{\max}b)$ and $\mu = 1/(1+d_{\min}a)$ for now. For $0 \leq j < n_0$, we can still find bounds for $\delta_j(s)$, and $k_j(s)$ is always bounded below by 0, and we can assume $k_0(s)$ is bounded above by some k_0^+ [S1]. So $\delta_j(s) \in [\delta^-, 1]$ where $\delta^- = 1/(1+d_{\max}k_0^+)$. Furthermore, we have

$$2\kappa^- \cos\phi^+ \leq k_j(s) \leq \frac{1}{d_{\min}} + \frac{2\kappa^+}{\cos\phi^+}$$

for $1 \leq j < n_0$. Thus, $\delta_j(s) \in [\lambda_0, \mu_0]$, where $\lambda_0^{-1} = 1 + d_{\max}(\frac{1}{d_{\min}} + \frac{2\kappa^+}{\cos\phi^+})$ and $\mu_0^{-1} = 1 + 2d_{\min}\kappa^- \cos\phi^+$.

7 Hausdorff Dimension of $\widehat{X}_0$

Proposition 7.1 *Let $[a,b] \supset [g_{\min}, g_{\max}]$, $\lambda = 1/(1+d_{\max}b)$, $\mu = 1/(1+d_{\min}a)$, and $n_0 = n(a,b)$ as defined above and let $\widehat{X}_0$ be the non-wandering part of $\widehat{X}$. There exist constants c, C depending only on the billiard, such that for any integer $n \geq n_0$ and $x_1, x_2 \in \widehat{X}_0$ such that $x_1 \approx_n x_2$, we have*

$$c\lambda^{n-n_0} \leq \|\pi x_1 - \pi x_2\| \leq C\mu^{n-n_0}.$$

Proof Let $n \geq n_0$ and let $x_1, x_2 \in \widehat{X}_0$ with $x_1 \approx_n x_2$. Without loss of generality assume that $t_n(x_1) < t_n(x_2)$ and let $\tau = t_n(x_2)$. Let $y_1 = \varphi_\tau x_1$, $y_2 = \varphi_\tau x_2$. Now let $p(s)$ parametrise (by arc length) the shortest curve $\Gamma \subset \varphi_\tau X$ between y_1 and y_2. Let $q(s) = \varphi_{-\tau}(p(s))$ paramatrize the curve $Y = \varphi_{-\tau}\Gamma$. This curve will not be the shortest curve between its endpoints x_1 and x_2, in fact for large n it can be much longer. We have

$$
\begin{aligned}
\|\pi x_1 - \pi x_2\| &= \left\| \int_Y q'(s) ds \right\| \le \int_Y \|q'(s)\| ds \\
&= \int_\Gamma \frac{\|p'(s)\|}{1 + (\tau - t_n(s))k_n(s)} \Big(\prod_{j=0}^{n-1} \delta_j(s) \Big) ds \\
&\le \mu^{n-n_0} \mu_0^{n_0} \int_\Gamma ds \le C\mu^{n-n_0}.
\end{aligned}
$$

Here we used Proposition 4.2, $(\tau - t_n(s))k_n(s) \ge 0$, $\delta_j(s) < \mu_0$ for $0 \le j \le n_0$, $\delta_j(s) < \mu$ for $j > n_0$. Since the curve Γ is the shortest curve between two points on a surface with bounded curvature (see [S1]) and is confined to a bounded set (*e.g.*, a ball containing K), its arc length $\int_\Gamma ds$ can be bounded above by a constant independent of X. Then μ, μ_0, and n_0 all depend only on the billiard.

Now we find an estimate for $\|x_1 - x_2\|$ from below using different curves. Let $q(s)$ parametrise the shortest curve Y in X between x_1 and x_2. Now let $[s_1, s_2] \subseteq [0, 1]$ such that $s = s_1, s_2$ are the only values for which $(q(s), n(s))$ has an $(n{+}1)$-st reflection. Let $y_1 = q_{n+1}(s_1)$, $y_2 = q_{n+1}(s_2)$. Without loss of generality assume $t_{n+1}(s_1) < t_{n+1}(s_2)$ and let $\tau = t_{n+1}(s_1)$, $z = \varphi_\tau(q(s_2))$. Then $p(s) = \varphi_\tau q(s)$ parametrises the curve $\varphi_\tau \widehat{Y}$.

Since Y has bounded curvature, we have constants C_1 and C_2 depending only on the billiard such that

$$
\begin{aligned}
\|\pi x_1 - \pi x_2\| &\ge C_1 \int_X \|q'(s)\| ds \ge C_1 \int_{s_1}^{s_2} \|q'(s)\| ds \\
&= C_1 \int_{s_1}^{s_2} \frac{\|p'(s)\|}{1 + (\tau - t_n(s))k_n(s)} \Big(\prod_{j=0}^{n-1} \delta_j(s) \Big) \\
&\ge C_1 C_2 \lambda_0^{n_0} \lambda^{n-n_0} \int_{s_1}^{s_2} \|p'(s)\| ds.
\end{aligned}
$$

Clearly z is in the convex hull of the two obstacles containing $q_n(s_2)$ and y_2 respectively, and y_1 is in a third obstacle. Thus we have $\int_{s_1}^{s_2} \|p'(s)\| \ge \|y_1 - z\| \ge b^-$, where b^- is the minimum distance between K_k and Cvx $(K_i \cup K_j)$ for any nonequal i, j, k. Letting $c = C_1 C_2 \lambda_0^{n_0} b^-$ (these factors all depend only on the billiard), we have $c\lambda^{n-n_0} \le \|\pi x - \pi y\| \le C\mu^{n-n_0}$ as required. ∎

Proposition 7.2 *Let $0 < n_0 \le n$. Suppose there are constants $c, C > 0$ such that $c\lambda^{n-n_0} \le \|\pi x - \pi y\| \le C\mu^{n-n_0}$ whenever $x, y \in \widehat{Y}_0$ with $x \approx_n y$. Then $\Upsilon\colon \widehat{Y}_0 \to \Sigma^+$ is injective and a Lipschitz homeomorphism from $\widehat{Y}_0$ to the metric space $(\Upsilon(\widehat{Y}_0), d_\lambda)$, and Υ^{-1} is a Lipschitz homeomorphism from $(\Upsilon(\widehat{Y}_0), d_\mu)$ onto $\widehat{Y}_0$.*

Proof For any $x \in X_0$ with sufficiently large $n \ge n_0$, there is some $z \in X_0$ such that $z \approx_n x$, so if $\Upsilon(x) = \Upsilon(y)$, then $\|x - y\| \le \|x - z\| + \|y - z\| \le 2C\mu^n \to 0$ as $n \to \infty$. So Υ^{-1} is well defined and Υ is injective.

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Let $x \approx_n y \in X_0$. Then $d_\lambda(\Upsilon x, \Upsilon y) = \lambda^n \leq \frac{1}{c}\|x-y\|$, so Υ is Lipschitz.

Similarly, for distinct $\alpha, \beta \in \Upsilon(X_0)$, $x \in \Upsilon^{-1}(\alpha)$, $y \in \Upsilon^{-1}(\beta)$, and n such that $x \approx_n y \in X_0$, we have $\|\Upsilon^{-1}(\alpha) - \Upsilon^{-1}(\beta)\| \leq C\mu^n = Cd_\mu(\alpha,\beta)$. Finally, since the identity $I\colon (\Upsilon(X_0), d_\lambda) \to (\Upsilon(X_0), d_\mu)$ is continuous, the maps $\Upsilon\colon X_0 \to (\Upsilon(X_0), d_\mu)$ and $\Upsilon^{-1}\colon (\Upsilon(X_0), d_\lambda) \to X_0$ are also continuous. ■

The following theorem is well known (see [Fa]).

Theorem 7.3 *Let $f\colon A \to B$ be a Lipschitz map and let $F \subset A$. Then* $\dim_H f(F) \leq \dim_H F$.

For some $\alpha \in \Sigma^+$ and sufficiently large $n \geq n_0$ the cylinder $[\alpha]_n \subset \Upsilon(\widehat{Y}_0)$. It follows that $\dim_H(\Upsilon(\widehat{Y}_0), d_\lambda) \leq \dim_H \widehat{Y}_0 \leq \dim_H(\Upsilon(\widehat{Y}_0), d_\mu)$.

8 Hausdorff Dimension of M_0

We now relate $\dim_H X_0$ to $\dim_H M_0$. Let $x \in M_0$ and let $\widehat{X} = \varphi_\tau(W_\theta^{(u)}(x))$ be the image of the local unstable manifold $W_\theta^{(u)}(x)$ under φ_t. Let $X_0 = X \cap M_0$. Define $d^{(s)} = \dim_H(W_\theta^{(s)}(x) \cap M_0)$ and $d^{(u)} = \dim_H(W_\theta^{(u)}(x) \cap M_0)$. Then using Lemma 5.2 and Proposition 7.2, we get

$$d^{(u)} = \dim_H X_0 \in \Big[\frac{-\ln(u-1)}{\ln\lambda}, \frac{-\ln(u-1)}{\ln\mu}\Big].$$

We can use the same estimate for $d^{(s)}$, since $W_\theta^{(u)} = \operatorname{Refl} W^{(s)}(\operatorname{Refl}(x))$, where $\operatorname{Refl}\colon \widehat{Q} \to \widehat{Q}$ is a bi-Lipschitz involution given by

$$\operatorname{Refl}(q,v) = \begin{cases} (q,-v) & \text{for } q \in \operatorname{int}(Q), \\ (q, 2\langle n_K(q), v\rangle n_K(q) - v\rangle) & \text{for } q \in \partial K. \end{cases}$$

If E, F are Borel sets, the following inequalities are well known (see [Fa]):

$$\dim_H E + \dim_H F \leq \dim_H(E \times F) \leq \dim_H E + \overline{\dim}_p F.$$

Lemma 5.2 gives $\overline{\dim}_p(\Sigma^+, d_\theta) = \dim_H(\Sigma^+, d_\theta)$. Let V be a neighbourhood of M_0 and let $U \subset V$ be a neighbourhood of x. Let ε be small enough that $W_\varepsilon^{(u)}(x)$, $W_\varepsilon^{(s)}(x) \subset U$, and let $h\colon W_\varepsilon^{(u)}(x) \times W_\varepsilon^{(s)}(x) \to R$ be the usual local product map, where R is an open neighbourhood of x. The local product map is at least Hölder continuous. Let α be the Hölder constant of h, then using basic properties of Hausdorff dimension [Fa] we have

$$\alpha(d^{(s)} + d^{(u)}) \leq \dim_H(R \cap M_0) \leq \alpha^{-1}(d^{(s)} + d^{(u)}). \tag{8.1}$$

If $\alpha = 1$, we have

$$\dim_H(R \cap M_0) = d^{(s)} + d^{(u)}. \tag{8.2}$$

Theorem 8.1 $\lambda_1 = 1/(1 + d_{\max} g_{\max}), \mu_1 = 1/(1 + d_{\min} g_{\min})$. *Assume that* $\alpha = 1$. *Then*

$$\frac{-2\ln(u-1)}{\ln \lambda_1} \leq \dim_H M_0 \leq \frac{-2\ln(u-1)}{\ln \mu_1}. \tag{8.3}$$

Proof For any $a < g_{\min}$ and $b > g_{\max}$, letting

$$\lambda(b) = \frac{1}{(1 + d_{\max} b)}, \quad \mu(a) = \frac{1}{(1 + d_{\min} a)}$$

we have

$$\dim_H M_0 = \dim_H(R \cap M_0) = d^{(s)} + d^{(u)} \in \Big[\frac{-2\ln(u-1)}{\ln \lambda(b)}, \frac{-2\ln(u-1)}{\ln \mu(a)}\Big].$$

Taking limits $a \to g_{\min}$ and $b \to g_{\max}$, we get the result. ∎

9 Dimension Product Structure

In this section we discuss what is currently known about the local product map h. The map is always Lipshitz if the diffeomorphism B is *conformal* on both the stable and unstable manifolds (see [B] and [P, §7]). This is the case for the billiard ball map B in $\mathbb{R}^2$ but not in higher dimensions. To see this, suppose one of the obstacles is the unit sphere centered on the origin, and consider an unstable manifold containing the points $(0, 0, 10)$, $(\frac{1}{2}, 0, 10)$, $(0, \frac{1}{2}, 10)$, each with a ray in a direction sufficiently close to $(0, 0, -1)$ that the rays collide with the sphere. These points form a right angle, but their image under B does not, so B does not always preserve angles on unstable manifolds and is not conformal.

However Stoyanov [Sto3] showed that a class of billiards satisfy a pinching condition, which would imply that the stable and unstable manifolds are C^1. In the notation of this paper, a billiard satisfies the $\frac{1}{4}$*-pinching condition* if $\lambda_0^{d_{\max}} < \mu_0^{2d_{\min}}$, where

$$\lambda_0^{-1} = 1 + d_{\max}\Big(\frac{1}{d_{\min}} + \frac{2\kappa^+}{\cos\phi^+}\Big) \quad \text{and} \quad \mu_0^{-1} = 1 + 2d_{\min}\kappa^- \cos\phi^+.$$

In fact we will show that it holds when $\lambda(a)^{d_{\max}} < \mu(b)^{2d_{\min}}$.

Hasselblatt and Schmeling [HS] proposed the conjecture that equation (8.2) holds generically or under mild hypotheses, even for non-conformal diffeomorphisms and non-Lipschitz local product maps. They proved this conjecture for a class of Smale solonoids. If the conjecture is shown to be true, at least in the case of dynamical billiards, then we recover equation (8.3). If not, then the result still holds for the class of billiards in [Sto3]. We now calculate the constant α to get an estimate in terms of constants related to the billiard.

10 Calculating the Hölder Constant

A combination of arguments from [Sto3, H] and Section 11 can be used to calculate the Hölder constant α for the local product map. The open billiard flow φ_t is an example of an Axiom A flow, with hyperbolic splitting into $TM = E^{su} \oplus E^{ss} \oplus E^{S}$. These are the strong stable manifold, strong unstable manifold, and the direction of the flow S respectively. That is, for some $0 < \eta < 1$ we have $\|d\varphi_t(u)\| \le C\eta^t \|u\|$ for all $u \in E^s(t)$ and $t \ge 0$, and $\|d\varphi_t(u)\| \le C\eta^{-t}\|u\|$ for all $u \in E^u(t)$ and $t \le 0$.

For each point x there exist $\alpha_x < \beta_x < 0 < \alpha'_x < \beta_x$ such that for $v \in E^{ss}(x)$, $u \in E^{su}(x)$, and $t > 0$ we have

$$\frac{1}{C} e^{\alpha_x t}\|v\| \le \|d\varphi_t(x)\cdot v\| \le Ce^{\beta_x t}\|v\|, \text{ and}$$
$$\frac{1}{C} e^{-\alpha'_x t}\|u\| \le \|d\varphi_{-t}(x)\cdot u\| \le Ce^{-\beta'_x t}\|u\|.$$

In the case of billiards, the reflection property $W^{(u)}_\theta = \mathrm{Refl}\, W^{(s)}(\mathrm{Refl}(x))$ implies that $\alpha_x = -\alpha'_x$ and $\beta_x = -\beta'_x$. The Hölder constant α is then given by the *bunching constant*

$$\alpha = B^u(\varphi) = \inf_{x\in M_0} \frac{\beta_x - \beta'_x}{\alpha_x} = \inf_{x\in M_0} \frac{2\beta_x}{\alpha_x},$$

see [H]. The system is said to satisfy the $\frac{1}{4}$-*pinching condition* if there exist $0 < \alpha_0 \le \beta_0$ such that $0 \le \alpha_0 \le \alpha'_x \le \beta'_x \le \beta_0$ and $2\alpha_x - \beta_x \ge \alpha_0$ for all $x \in M_0$.

Let $\widehat{X} = \varphi_\tau(W^{(u)}_\theta(x))$ for some small τ, let $t > d_1(x) + \cdots + d_n(x)$ and let $\delta_j(s)$ be defined as in Section 4.3. Then from [Sto3], there are constants c_1, c_2 such that

$$\frac{c_1}{c_2}\frac{\|u\|}{\delta_1(0)\delta_2(0)\cdots\delta_n(0)} \le \|d\varphi_t(x)\cdot u\| \le \frac{c_2}{c_1}\frac{\|u\|}{\delta_1(0)\delta_2(0)\cdots\delta_n(0)},$$
$$\frac{c_1}{c_2}\frac{\|u\|}{\mu_0^{n_0}\mu^{n-n_0}} \le \|d\varphi_t(x)\cdot u\| \le \frac{c_2}{c_1}\frac{\|u\|}{\lambda_0^{n_0}\lambda^{n-n_0}},$$
$$\frac{c_1}{c_2}\Big(\frac{\mu}{\mu_0}\Big)^{n_0}\mu^{-t/d_{\max}}\|u\| \le \|d\varphi_t(x)\cdot u\| \le \frac{c_2}{c_1}\Big(\frac{\lambda}{\lambda_0}\Big)^{n_0}\lambda^{-t/d_{\min}}\|u\|,$$
$$Ae^{-t\ln\mu/d_{\max}}\|u\| \le \|d\varphi_t(x)\cdot u\| \le Be^{-t\ln\lambda/d_{\min}}\|u\|,$$

where $\lambda = \lambda(b) = \frac{1}{1+d_{\max}b}$, $\mu = \mu(a) = \frac{1}{1+d_{\min}a}$, while $A = A(a,b)$ and $B = B(a,b)$ are new global constants that exist for all $a < g_{\min}$ and $b > g_{\max}$ (these are not necessarily bounded above). This inequality holds for all $t \ge t_0$ with t_0 sufficiently large that $m > n_0$, but there must be constants A' and B' such that the same inequality holds for all $0 < t \le t_0$. Taking C large enough that $C > \max\{B, B'\}$ and $\frac{1}{C} < \min\{A, A'\}$, we now have $\alpha_x = -\ln\mu/d_{\max}$ and $\beta_x = -\ln\lambda/d_{\min}$, so the bunching constant is $B^u(\varphi) = 2d_{\min}\ln\mu/d_{\max}\ln\lambda$. This argument improves [Sto3, Proposition 1.2] by replacing $[\mu_0, \lambda_0]$ with the smaller interval $[\mu, \lambda]$ for any $a < g_{\min}$ and $b > g_{\max}$.

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Proposition 10.1 *Let $a < g_{\min}$ and $b > g_{\max}$. Assume that $\lambda(b)^{d_{\max}} < \mu(a)^{2d_{\min}}$, and the boundary ∂K is C^3. Then the open billiard flow in the exterior of K satisfies the $\frac{1}{4}$-pinching condition on its non-wandering set M_0. For any $x \in M_0$ we can choose $\alpha_x = \alpha_0 = \frac{\ln \mu(a)}{d_{\max}}$ and $\beta_x = \beta_0 = \frac{\ln \lambda(b)}{d_{\min}}$.*

We cannot take the limit as $a \to g_{\min}, b \to g_{\max}$ for this proposition, since the constants A and B may not be bounded above. However, when $\lambda^{d_{\max}} < \mu^{2d_{\min}}$ we have $\alpha = 1$, so equations (8.2) and (8.3) hold. Taking limits we can extend this to $\lambda_1^{d_{\max}} < \mu_1^{2d_{\min}}$, which proves part 2 of the main theorem. If (8.2) does not hold, then we have the following general estimate using (8.1):

$$-\frac{4d_{\min} \ln \mu_1 \ln(u-1)}{d_{\max}(\ln \lambda_1)^2} \leq \dim_H M_0 \leq -\frac{d_{\max} \ln \lambda_1 \ln(u-1)}{d_{\min}(\ln \mu_1)^2}.$$

11 Improvement of Estimates

11.1 Convex Hull Conjecture

We propose a conjecture that restricts the non-wandering set to a smaller area. This allows some relaxation of conditions.

Definition 11.1 For any $i \neq j$, let $(p_{ij}, p_{ji}) \in K_i \times K_j$ denote the minimum of $F\colon K_i \times K_j \to \mathbb{R}, (q_1, q_2) \mapsto \|q_1 - q_2\|$. Then each p_{ij} is on the boundary ∂K_i and the vector $p_{ji} - p_{ij}$ is normal to ∂K_i at p_{ij}.

Conjecture 11.2 *Denote the convex hull $Cvx\{p_{ij} : 1 \leq i, j \leq n, i \neq j\}$ by H. Let $1 \leq \alpha_1, \dots, \alpha_n \leq u$ ($n \geq 3$) be a finite sequence of indices and let $(q_1, \dots, q_n)$ be a periodic billiard trajectory such that $q_j \in K_{\alpha_k}$ for each j. Then each q_j is contained in H. Furthermore, the non-wandering set M_0 is contained in H.*

We prove this conjecture for the case of a 3-dimensional billiard in which the obstacles are spheres. A very similar proof will work for all two-dimensional billiards and higher dimensional billiards with hyperspherical obstacles. The general case in higher dimensions may be more difficult.

Proof of the conjecture for spherical obstacles If the obstacles are spheres, then $H \cap Q$ is simply the convex hull of the centres of the spheres intersected with Q. Suppose that $(q_1, \dots, q_n)$ is a periodic trajectory, but that at least one point is outside H. Without loss of generality we can number the points and obstacles such that $q_1 \notin H$ and $\alpha_1 = 1$. Then H is bounded by a number of planes, so $q_1 \in K_1$ is on the outside (*i.e.*, the side not containing H) of one such plane, say $\Pi = \Pi_{123}$, determined by the centres of obstacles K_1, K_2, K_3. Let ν be the outward normal vector of Π and denote $v_j = \frac{q_{j+1} - q_j}{\|q_{j+1} - q_j\|}$, (with the convention that $q_0 = q_n$). Without loss of generality, assume that $v_0 \cdot \nu > 0$.

For each $k \geq 1$ we have $q_{k+1} = q_k + d_k v_k$ and $v_k = v_{k-1} - 2\langle v_{k-1}, n_K(q_k)\rangle n_K(q_k)$. We also have $\langle v_{k-1}, n_K(q_k)\rangle < 0$. We show by induction that $q_k \cdot \nu > q_1 \cdot \nu$ and $v_{k-1} \cdot \nu > v_0 \cdot \nu$ for all $k > 1$.

Suppose $q_k \in \partial K_{\alpha_k}$ is on the outside of Π and $v_{k-1} \cdot \nu > v_0 \cdot \nu$. The centre of ∂K_{α_k} is on the inside of Π, so the normal vector $n(q_k)$ must point away from Π, *i.e.*, $n_K(q_k) \cdot \nu > 0$. So

$$v_k \cdot \nu = v_{k-1} \cdot \nu - 2\big\langle v_{k-1}, n_K(q_k)\big\rangle n_K(q_k) \cdot \nu > v_{k-1} \cdot \nu > v_0 \cdot \nu.$$

Then $q_{k+1} \cdot \nu = q_k \cdot \nu + d_k v_k \cdot \nu > q_k \cdot \nu$. So q_{k+1} is also on the outside.

For the orbit to be periodic we must have $q_1 = q_{n+1}$ for some n. So by contradiction, all periodic points must be contained in H. Since H is a closed set and the periodic points are dense in M_0, we have $M_0 \subset H$. ∎

Corollary 11.3 *Given a billiard for which the above conjecture is true, the non-wandering set M_0 is entirely contained in H, which means any change to the billiard outside of H will not have any effect on the non-wandering set, unless it introduces a new periodic point. This means all results in this paper (and perhaps others) apply to billiards that are not smooth or convex, or that violate the no-eclipse condition (**H**), provided that the intersection $K \cap H$ still satisfies these conditions.*

Corollary 11.4 *In cases where the conjecture is true, we can use the set H to find better estimates for billiard constants. For example, we can estimate $d_{\max} \leq \operatorname{diam} H$. The minimum and maximum curvatures over M_0 can be estimated by*

$$\kappa^- \leq \min_{q \in \partial K \cap H} \kappa_{\min}(q) \quad \textit{and} \quad \kappa^+ \leq \min_{q \in \partial K \cap H} \kappa_{\max}(q).$$

11.2 Adjusted Domain of g

Recall that the natural domain for the function g is $[\kappa^- \cos\phi^+, \frac{\kappa^+}{\cos\phi^+}] \times [d_{\min}, d_{\max}]$. This applies for billiards in $\mathbb{R}^D$ with $D > 2$; when $D = 2$ the natural domain is $[\kappa^-, \frac{\kappa^+}{\cos\phi^+}] \times [d_{\min}, d_{\max}]$ (see the end of section 4.2). To cover both cases at once, we let $\iota = 0$ if $D = 2$ and $\iota = 1$ if $D > 2$, so that $\cos^\iota\phi$ is 1 if $D = 2$ and $\cos\phi$ otherwise. Define the adjusted domain by

$$\mathbb{D} = \bigcup_{i,j}\Big[\kappa_i^- \cos^\iota \phi_{ij}^+, \frac{\kappa_i^+}{\cos\phi_{ij}^+}\Big] \times [d_{ij}^-, d_{ij}^+],$$

where κ_i^-, κ_i^+ are the minimum and maximum curvatures on $\partial K_i \cap H$, $d_{ij}^- \geq |p_{ij} - p_{ji}|$, $d_{ij}^+ \leq \max_{k,l} |p_{ik} - p_{jl}|$ are the minimum and maximum distances between $K_i \cap H$ and $K_j \cap H$, and $\phi_{ij} = \max\{\phi(x) : x \in K_i \cap H, Bx \in K_j \cap H\}$ is the maximum collision angle over trajectories from K_i to K_j. These ϕ_{ij} can be estimated by $\cos\phi_{ij} \geq b_{ij}^-/d_{\max}$ where $b_{ij}^- = \min_k d(K_i, Cvx(K_j, K_k))$.

The minimum and maximum values of g over the natural domain may be outside of the adjusted domain. The minimum and maximum values in the adjusted domain are given by

$$g_{\min} = \min\{g(\gamma,\theta) : (\gamma,\theta) \in \mathbb{D}\} = \min\{g(\kappa_i^- \cos^\iota \phi_{ij}, d_{ij}^+), 1 \leq i, j, \leq u\},$$

$$g_{\max} = \max\{g(\gamma,\theta) : (\gamma,\theta) \in \mathbb{D}\} = \max\Big\{g\Big(\frac{\kappa_j^+}{\cos\phi_{ij}}, d_{ij}^-\Big), 1 \leq i, j, \leq u\Big\}.$$

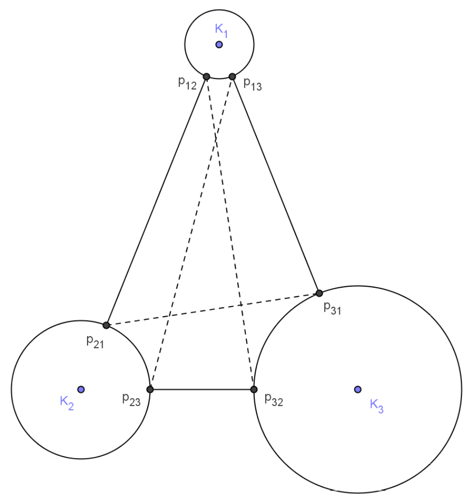

Figure 1: The billiard used in Example 11.6.

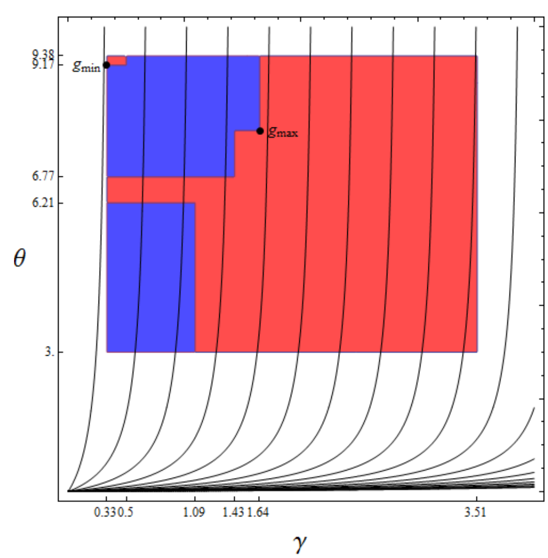

Figure 2: The adjusted domain displayed over the natural domain.

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Lemma 11.5 *For any $x = (q, v) \in M_0$, we have*

$$\Big(\frac{\kappa(q_j)}{\cos\phi_j(x)}, d_j(x)\Big) \in \mathbb{D} \quad \textit{and} \quad \big(\kappa(q_j)\cos^{\iota}\phi_j(x), d_j(x)\big) \in \mathbb{D}$$

for all $j \in \mathbb{Z}$.

Proof Assume $D > 2$. Since $q_j = \pi B^j x \in M_0$ for all $j \in \mathbb{Z}$, we have $\kappa(q_j) \in [\kappa_{i_j}^-, \kappa_{i_j}^+]$, $\phi(B^j x) \in [0, \phi^+_{i_j i_{j+1}}]$, and $d(q_j, q_{j+1}) \in [d^-_{i_j i_{j+1}}, d^+_{i_j i_{j+1}}]$. Hence there exist some integers $1 \le a, b, \le u$ such that

$$\kappa(q_j)\cos\phi(B^j x) \ge \kappa_a^- \cos\phi^+_{ab} \quad \text{and} \quad \frac{\kappa(q_j)}{\cos\phi(B^j x)} \le \frac{\kappa_a^+}{\cos\phi^+_{ab}}.$$

For the same a, b we have $d(q_j) \in [d^-_{ab}, d^+_{ab}]$. The proof for $D = 2$ is analogous. ■

Example 11.6 Consider the billiard displayed in Figure 1, consisting of three disks arranged in an isoceles triangle of height 10 and base length 8. The disks K_1, K_2, K_3 have radii 1, 2, and 3 respectively. The solid lines give the distances d^-_{ij} and the dashed lines give the distances d^+_{ij}. Figure 2 displays the adjusted domain over the natural domain, with contour lines of the function $g(\gamma, \theta)$. The following calculations were obtained using the programs Geogebra and Mathematica.

Using the adjusted domain rather than the natural domain means that the interval $[g_{\min}, g_{\max}]$ is reduced from $[0.760, 7.34]$ to $[0.762, 3.41]$. Using the natural domain we have the estimate $0.326 \le \dim_H M_0 \le 1.167$, but with the adjusted domain we get $0.396 \le \dim_H M_0 \le 1.165$.

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