THE mod $\mathfrak{C}$ SUSPENSION THEOREM

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1. Introduction. Our aim in this paper is to prove the general mod $\mathfrak{C}$ suspension theorem: *Suppose that X and Y are* CW-*complexes, $\mathfrak{C}$ is a class of finite abelian groups, and that*

(i) $\pi_i(Y) \in \mathfrak{C}$ *for all* $i < n$,

(ii) $H_*(X; Z)$ *is finitely generated*,

(iii) $H^i(X; Z) \in \mathfrak{C}$ *for all* $i > k$.

Then the suspension homomorphism

$$E: [S^r X,\ Y] \rightarrow [S^{r+1}X,\ SY]$$

is a (mod $\mathfrak{C}$) *monomorphism for* $2 \leqq r \leqq 2n - k - 2$ (*when* $r = 1$, ker E *is a finite group of order* d, *where* $Z_d \in \mathfrak{C}$) *and is a* (mod $\mathfrak{C}$) *epimorphism for* $2 \leqq r \leqq 2n - k - 1$.

The proof is basically the same as the proof of the regular suspension theorem. It depends essentially on (mod $\mathfrak{C}$) versions of the Serre exact sequence and of the Whitehead theorem.

In the first part of this paper we construct the (mod $\mathfrak{C}$) Serre sequence. A certain amount of (mod $\mathfrak{C}$) algebra is required. As much as possible is carried over from (**5**) sometimes without explicit mention. However, the usual definition of exactness (mod $\mathfrak{C}$) is inconvenient and a slightly more general definition is adopted. I believe that this is justified in a remark after Corollary 3.1. Some of this algebra will also be useful in later work.

The (mod $\mathfrak{C}$) Hurewicz and Whitehead theorems are proved here simply because they follow so easily from the (mod $\mathfrak{C}$) Serre sequence. The suspension theorem for homotopy groups is now an easy consequence, but some of the work summarized in Theorem 6 is still necessary in order to pass to the general (mod $\mathfrak{C}$) suspension theorem. (All of Theorem 6 is used in the sequel.)†

2. Some definitions for (mod $\mathfrak{C}$) **algebra.** If $\mathfrak{C}$ is a class of abelian groups $\bar{\mathfrak{C}}$, the non-abelian closure of $\mathfrak{C}$ is defined to be a family of groups satisfying the following conditions:

(i) if $G \in \mathfrak{C}$, then $G \in \bar{\mathfrak{C}}$;

(ii) if $G \in \bar{\mathfrak{C}}$ and H is a normal subgroup of G, then $H \in \bar{\mathfrak{C}}$;

(iii) if $G \in \bar{\mathfrak{C}}$ and H is a quotient group of G, then $H \in \bar{\mathfrak{C}}$;

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†*Added in proof*. See pp. 702–711, 712–729 of this issue of Can. J. Math.

(iv) if G' and G'' are in $\bar{\mathfrak{C}}$ and $0 \to G' \to G \to G'' \to 0$ is exact, then G is in $\bar{\mathfrak{C}}$;
(v) if F is another family of groups satisfying (i)–(iv), then $F \supseteq \bar{\mathfrak{C}}$.

It is clear that any class of abelian groups $\mathfrak{C}$ has a unique non-abelian closure. $\bar{\mathfrak{C}}$ is simply the intersection of all families of groups satisfying conditions (i)–(iv).

A class $\mathfrak{C}$ of finite abelian groups is characterized by a sequence of primes as follows:

(i) a prime p is in the sequence if and only if Z_p is in $\mathfrak{C}$;
(ii) given the sequence $(p_1, p_2, \ldots)$, an abelian group G is in $\mathfrak{C}$ if and only if ord(order)$G = p_1^{n_1} p_2^{n_2} \ldots$.

A class $\mathfrak{C}$ characterized by the primes $(p_1, p_2, \ldots)$ will sometimes be denoted by $\mathfrak{C}(p_1, p_2, \ldots)$. Then $\bar{\mathfrak{C}}$ is the family of all solvable groups G such that ord $G = p_1^{n_1} p_2^{n_2} \ldots$. If $\mathfrak{C}$ is the class of all finitely generated abelian groups, then $\bar{\mathfrak{C}}$ is the family of all groups G with the following property: there exists a finite sequence of subgroups of G,

$$G = G_n \supset G_{n-1} \supset \ldots \supset G_1 \supset G_0 = 0,$$

such that G_{i-1} is normal in G_i and G_i/G_{i-1} is cyclic. In both cases, if G is abelian and $G \in \bar{\mathfrak{C}}$, then $G \in \mathfrak{C}$.

Definition. An element $a \in G$ is in $\mathfrak{C}(p_1, p_2, \ldots)$ if and only if ord $a = p_1^{n_1} p_2^{n_2} \ldots$, or equivalently, if the cyclic group generated by a is in $\mathfrak{C}$. Suppose that G is a finitely generated abelian group. Consider its prime power decomposition. The set of elements in G which are in $\mathfrak{C}(p_1, p_2, \ldots)$ form a subgroup $G_{\mathfrak{C}}$ equal to the direct sum of those cyclic subgroups whose orders are powers of a prime in $\mathfrak{C}$. Thus, this subgroup $G_{\mathfrak{C}}$ is a direct summand of G. $G_{\mathfrak{C}}$ is the largest subgroup of G which is in $\mathfrak{C}$, hence $G \in \mathfrak{C} \Leftrightarrow G = G_{\mathfrak{C}}$.

When the class $\mathfrak{C}$ is fixed we shall write $\bar{G} = G/G_{\mathfrak{C}}$. There exist covariant functors $G \to G_{\mathfrak{C}}$ and $G \to \bar{G}$. $\bar{G}$ will be called the (mod $\mathfrak{C}$) reduction of G.

3. Some lemmas in (mod $\mathfrak{C}$) **algebra.** In the algebra below, all groups considered will be finitely generated abelian groups. Let $\mathfrak{C}$ be a class of finite abelian groups.

Definition.

$$A \xrightarrow{f} B \xrightarrow{g} D$$

is exact (mod $\mathfrak{C}$) if and only if

$$gf(A) \in \mathfrak{C} \quad \text{and} \quad g^{-1}(D_{\mathfrak{C}})/f(A) \in \mathfrak{C}.$$

Definition. $f\colon A \to B$ is a (mod $\mathfrak{C}$) isomorphism if and only if

$$0 \to A \xrightarrow{f} B \to 0$$

is exact (mod $\mathfrak{C}$).

Definition. $A \approx_{\mathfrak{C}} B$ (A is isomorphic to B (mod $\mathfrak{C}$)) if and only if there exists a C such that

$$0 \to A \to C \to 0 \quad \text{and} \quad 0 \to B \to C \to 0$$

are both exact (mod $\mathfrak{C}$). This is an equivalence relation; cf. (**5**, p. 299).

Suppose that

$$0 \to A \xrightarrow{f} B \to 0$$

is exact (mod $\mathfrak{C}$). Let $\bar{A} = A/A_{\mathfrak{C}}$, $\bar{B} = B/B_{\mathfrak{C}}$. Then $A = \bar{A} \oplus A_{\mathfrak{C}}$ and $B = \bar{B} \oplus B_{\mathfrak{C}}$. By looking at the prime power decompositions and counting, one sees that $\bar{A}$ and $\bar{B}$ must have the same cyclic summands, i.e., $\bar{A} \approx \bar{B}$. (Note that the induced homomorphism $\bar{f}$: $\bar{A} \to \bar{B}$ may not be an isomorphism, but it is an isomorphism (mod $\mathfrak{C}$). Furthermore, if $\bar{A} \approx \bar{B}$, we may take $C = A \oplus A_{\mathfrak{C}} \oplus B_{\mathfrak{C}}$, obtaining $A \approx_{\mathfrak{C}} B$. Thus, for finitely generated abelian groups,

$$A \approx_{\mathfrak{C}} B \Leftrightarrow \bar{A} \approx \bar{B}.$$

LEMMA 1. *If $G_k \supset G_{k-1} \supset \ldots \supset G_1 \supset G_0$ and $G_i/G_{i-1} \in \mathfrak{C}$ for all i, then $G_k/G_0 \in \mathfrak{C}$.*

Proof. We use the exact sequence

$$0 \to G_i/G_0 \to G_{i+1}/G_0 \to G_{i+1}/G_i \to 0$$

and induction on i.

LEMMA 2. *If f and g are homomorphisms from A to B and $(f - g)A \in \mathfrak{C}$, then $f(A) \in \mathfrak{C}$ if and only if $g(A) \in \mathfrak{C}$.*

Proof. Let p: $B \to \bar{B}$ be the canonical projection. Then $0 = p(f - g)A = (pf - pg)A$. Thus, $pf(A) = 0$ if and only if $pg(A) = 0$. That is, $f(A) \in \mathfrak{C}$ if and only if $g(A) \in \mathfrak{C}$.

The canonical projection p: $A \to \bar{A}$ has a right inverse i: $\bar{A} \to A$. That is, $pi = 1$ and $(1 - ip)A = A_{\mathfrak{C}}$. Furthermore, p and i are (mod $\mathfrak{C}$) isomorphisms.

LEMMA 3.

$$G \xrightarrow{f} A \xrightarrow{h} B$$

is exact (mod $\mathfrak{C}$) *if and only if*

$$G \xrightarrow{pf} \bar{A} \xrightarrow{hi} B$$

is exact mod $\mathfrak{C}$.

Proof. We have to prove (a) and (b).

(a) $(hf)G \in \mathfrak{C} \leftrightarrow (hipf)G \in \mathfrak{C}$. $(1 - ip)A = A_{\mathfrak{C}} \in \mathfrak{C}$, therefore, $(h(1 - ip)f)G = (hf - hipf)G \in \mathfrak{C}$. This implies (a).

(b) Suppose that $(hf)G$ and $(hipf)G$ are in $\mathfrak{C}$, then

$$h^{-1}(B_{\mathfrak{C}})/f(G) \in \mathfrak{C} \leftrightarrow \frac{(hi)^{-1}B_{\mathfrak{C}}}{(pf)G} \in \mathfrak{C}.$$

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Since

$$\frac{(hi)^{-1}B_{\mathfrak{C}}}{(pf)G} = \frac{i^{-1}(h^{-1}(B_{\mathfrak{C}}))}{p(f(G))} = \frac{p(h^{-1}(B_{\mathfrak{C}}))}{p(f(G))}$$

and since p is a (mod $\mathfrak{C}$) isomorphism, this last group is isomorphic mod $\mathfrak{C}$ to $h^{-1}(B_{\mathfrak{C}})/f(G)$.

Suppose that $A_1 \approx_{\mathfrak{C}} A_2$. Let $p_j\colon A_j \to \bar{A}_j$ and $i_j\colon \bar{A}_j \to A_j$ be the canonical projections and their right inverses. Let $s\colon \bar{A}_1 \to \bar{A}_2$ be an isomorphism.

COROLLARY 3.1.

$$G \xrightarrow{f} A_1 \xrightarrow{h} B$$

is exact (mod $\mathfrak{C}$) *if and only if*

$$G \xrightarrow{i_2 s p_1 f} A_2 \xrightarrow{h i_1 s^{-1} p_2} B$$

is exact (mod $\mathfrak{C}$).

Thus, using the method of changing homomorphisms described here, one can replace a group in a (mod $\mathfrak{C}$) exact sequence by a group isomorphic (mod $\mathfrak{C}$) to it without destroying the (mod $\mathfrak{C}$) exactness. In future, the same symbols will generally be used for the original and the altered homomorphisms.

This corollary and Lemma 3 seem to justify the definition of (mod $\mathfrak{C}$) exactness given above. The usual definition is

$$G \xrightarrow{f} A \xrightarrow{h} B$$

is exact (mod $\mathfrak{C}$) if and only if $hf = 0$ and $\ker h/\operatorname{Im} f \in \mathfrak{C}$. We present an example to show that these two concepts are different. $G = Z$, $A = Z + Z_2$ with generators a_1 (of infinite order) and a_2 ($2a_2 = 0$), and $B = Z_2$ (generator b). Define

$$f\colon Z \to Z + Z_2$$

by $f(1) = a_1 + a_2$, and

$$h\colon Z + Z_2 \to Z_2$$

by $h(a_1) = b$, $h(a_2) = b$. Then

$$Z \xrightarrow{f} Z + Z_2 \xrightarrow{h} Z_2$$

is exact (mod $\mathfrak{C}(2)$); in fact, it is exact. Let $p\colon Z + Z_2 \to \overline{Z + Z_2} = Z$ be the canonical projection and $i\colon Z \to Z + Z_2$ its right inverse. Consider the sequence

$$Z \xrightarrow{pf} \overline{Z + Z_2} \xrightarrow{hi} Z_2.$$

$hipf(1) = b \neq 0$; thus, in the old definition, this is not exact (mod $\mathfrak{C}(2)$). It is possible that Corollary 3.1 will remain true using the old definition, but the homomorphisms would have to be changed in a more complicated way.

Corollary 3.1 is essential to much of what follows in this section and will generally be used without explicit mention.

LEMMA 4. *If* $G = G_n \supset G_{n-1} \supset \ldots \supset G_1 \supset G_0 = 0$ *and* $F_i = G_i/G_{i-1} \in \mathfrak{C}$

except for $i = r, s$ $(r > s)$, *then there exists a sequence* $0 \to F_s \to G \to F_r \to 0$ *which is exact* (mod $\mathfrak{C}$).

Proof. $0 \to G_r/G_{r-1} \to G_n/G_{r-1} \to G_n/G_r \to 0$ is exact. Lemma 1 implies that $G_n/G_r \in \mathfrak{C}$. Therefore, $F_r = G_r/G_{r-1} \approx_{\mathfrak{C}} G_n/G_r$. Furthermore,

$$0 \to G_s \to G_{r-1} \to G_{r-1}/G_s \to 0$$

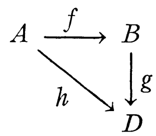
and

$$0 \to G_{s-1} \to G_s \to G_s/G_{s-1} \to 0$$

are exact. Using Lemma 1 again we have that G_{r-1}/G_s and G_{s-1} are in $\mathfrak{C}$. Combining isomorphisms, this implies that $G_{r-1} \approx_{\mathfrak{C}} G_s/G_{s-1} = F_s$. The lemma now follows by substitution in the exact sequence $0 \to G_{r-1} \to G \to G/G_{r-1} \to 0$.

Definition. A homomorphism $f: A \to B$ is in a class of finite abelian groups $\mathfrak{C}$ if and only if f, as an element of $\operatorname{Hom}(A, B)$, is in $\mathfrak{C}$. This is equivalent to the condition that $f(A) \in \mathfrak{C}$.

Definition. The triangle



is commutative (mod $\mathfrak{C}$) if and only if $h - gf \in \mathfrak{C}$.

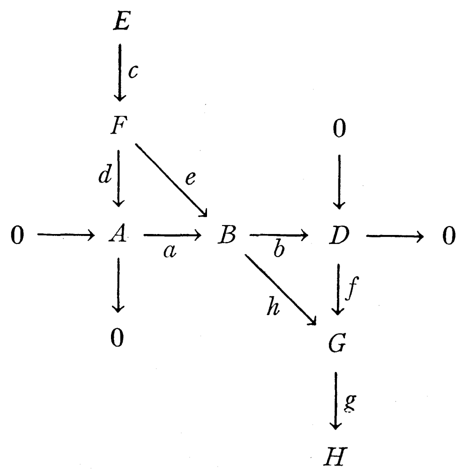
If f and g are two homomorphisms from A to B and $f - g \in \mathfrak{C}$, then $f^{-1}(B_{\mathfrak{C}}) = g^{-1}(B_{\mathfrak{C}})$ and $\overline{f(A)} = \overline{g(A)}$. This follows since

(i) $A \xrightarrow{f-g} B \xrightarrow{p} \bar{B}$ is the zero homomorphism,

(ii) $f^{-1}(B_{\mathfrak{C}}) = \ker pf = \ker pg = g^{-1}(B_{\mathfrak{C}})$, and

(iii) $\overline{f(A)} = pf(A) = pg(A) = \overline{g(A)}$.

LEMMA 5. *If the diagram*



is commutative (mod $\mathfrak{C}$) *and the vertical and horizontal lines are exact* (mod $\mathfrak{C}$), *then the sequence*

$$E \xrightarrow{c} F \xrightarrow{e} B \xrightarrow{h} G \xrightarrow{g} H$$

is exact (mod $\mathfrak{C}$).

Proof. (A) $e - ad$ and dc are in $\mathfrak{C}$. Hence, $ec = (e - ad)c + adc \in \mathfrak{C}$. Since a is a (mod $\mathfrak{C}$) monomorphism, $a^{-1}(B_{\mathfrak{C}}) = A_{\mathfrak{C}}$.

$$\frac{e^{-1}(B_{\mathfrak{C}})}{c(E)} = \frac{(ad)^{-1}B_{\mathfrak{C}}}{c(E)} = \frac{d^{-1}(a^{-1}(B_{\mathfrak{C}}))}{c(E)};$$

hence,

$$\frac{d^{-1}(A_{\mathfrak{C}})}{c(E)} \in \mathfrak{C}.$$

(B) $h - fb$, $e - ad$, and ba are all in $\mathfrak{C}$. Therefore, $he = (h - fb)e + fb(e - ad) + fbad \in \mathfrak{C}$. We are given that $b^{-1}(D_{\mathfrak{C}})/a(A) \in \mathfrak{C}$. Since $d: F \to A$ is onto (mod $\mathfrak{C}$), the inclusion $ad(F) \to a(A)$ is a (mod $\mathfrak{C}$) isomorphism and induces a (mod $\mathfrak{C}$) isomorphism

$$\frac{b^{-1}(D_{\mathfrak{C}})}{ad(F)} \to \frac{b^{-1}(D_{\mathfrak{C}})}{a(A)}.$$

Now

$$\frac{b^{-1}(D_{\mathfrak{C}})}{ad(F)} \approx_{\mathfrak{C}} \frac{\overline{b^{-1}(D_{\mathfrak{C}})}}{\overline{ad(F)}} = \frac{\overline{b^{-1}(D_{\mathfrak{C}})}}{\overline{e(F)}} \approx_{\mathfrak{C}} \frac{b^{-1}(D_{\mathfrak{C}})}{e(F)}.$$

Hence, this last group is in $\mathfrak{C}$. Since f is a (mod $\mathfrak{C}$) monomorphism, $f^{-1}(G_{\mathfrak{C}}) = D_{\mathfrak{C}}$. Therefore,

$$\frac{b^{-1}(D_{\mathfrak{C}})}{e(F)} = \frac{b^{-1}(f^{-1}(G_{\mathfrak{C}}))}{e(F)} = \frac{(fb)^{-1}G_{\mathfrak{C}}}{e(F)} = \frac{h^{-1}(G_{\mathfrak{C}})}{e(F)}.$$

(C) $h - fb$ and gf are in $\mathfrak{C}$. Thus, $gh = g(h - fb) + gfb \in \mathfrak{C}$. Using the same reasoning as in part (B), we have:

$$\frac{g^{-1}(H_{\mathfrak{C}})}{h(B)} \approx_{\mathfrak{C}} \frac{\overline{g^{-1}(H_{\mathfrak{C}})}}{\overline{h(B)}} = \frac{\overline{g^{-1}(H_{\mathfrak{C}})}}{\overline{fb(B)}} \approx_{\mathfrak{C}} \frac{g^{-1}(H_{\mathfrak{C}})}{fb(B)} \approx_{\mathfrak{C}} \frac{g^{-1}(H_{\mathfrak{C}})}{f(D)}$$

(since b is onto mod $\mathfrak{C}$) and this last group is in $\mathfrak{C}$. Therefore

$$\frac{g^{-1}(H_{\mathfrak{C}})}{h(B)} \in \mathfrak{C}.$$

THE FIVE LEMMA (mod $\mathfrak{C}$). *Suppose that*

$$\begin{array}{ccccccccc} A_1 & \to & A_2 & \to & A_3 & \to & A_4 & \to & A_5 \\ {\scriptstyle c_1}\downarrow & & {\scriptstyle c_2}\downarrow & & {\scriptstyle c_3}\downarrow & & {\scriptstyle c_4}\downarrow & & \downarrow{\scriptstyle c_5} \\ B_1 & \to & B_2 & \to & B_3 & \to & B_4 & \to & B_5 \end{array}$$

is commutative (mod $\mathfrak{C}$), *each row is exact* (mod $\mathfrak{C}$) *and* c_1, c_2, c_4, *and* c_5 *are isomorphisms* (mod $\mathfrak{C}$). *Then* c_3 *is an isomorphism* (mod $\mathfrak{C}$).

Proof. Reduce every group (mod $\mathfrak{C}$). mod $\mathfrak{C}$ exactness is preserved, and now each double composition is trivial. mod $\mathfrak{C}$ commutativity becomes regular commutativity. The vertical maps remain (mod $\mathfrak{C}$) isomorphisms. (In fact, they are monomorphisms and (mod $\mathfrak{C}$) epimorphisms.) The reduced diagram now satisfies the hypotheses of (**5**, p. 309). Thus, the reduced map $\bar{c}_3$: $\bar{A}_3 \to \bar{B}_3$ is a (mod $\mathfrak{C}$) isomorphism. This implies that c_3: $A_3 \to B_3$ is a (mod $\mathfrak{C}$) isomorphism.

4. A spectral sequence studied (mod $\mathfrak{C}$)**.** In the following section, I will use the definitions, notation, and some of the results of (**5**, Chapter VIII, § 6).

Definition. A bigraded exact couple is a system

$$C = \langle D, E; i, j, k \rangle,$$

where D and E are bigraded abelian groups, i, j, and k are homogeneous homomorphisms, and

$$\begin{array}{ccc} D & \xrightarrow{\ i\ } & D \\ & {\scriptstyle k}\nwarrow \quad \swarrow{\scriptstyle j} & \\ & E & \end{array}$$

is exact. When the degree of i is $(1, -1)$, of j is $(0, 0)$, and of k is $(-1, 0)$, the couple is called a ∂-couple. A ∂-couple is regular if $D_{p,q} = 0$ when $p < 0$ and $E_{p,q} = 0$ when $q < 0$.

Define

$$H_{p,q} = H_{p,q}(C) = D^{q+2}_{p+q+1,-1}$$

and $H_m = H_{m,0}$. Hu showed (**5**, p. 238) that for a regular ∂-couple we have, for each m > 0,

(A) $$H_m = H_{m,0} \supset H_{m-1,1} \supset \ldots \supset H_{0,m} \supset H_{-1,m+1} = 0$$

and also

(B) $$H_{p,q}/H_{p-1,q+1} = E^{\infty}_{p,q}.$$

Using the algebraic lemmas above, we can now obtain a (mod $\mathfrak{C}$) version of a standard exact sequence.

THEOREM 1. *Let* $\langle D, E; i, j, k \rangle$ *be a regular* ∂*-couple. Suppose that* $E^2_{p,q} \in \mathfrak{C}$ *for* $p + q \leqq r$ *unless* $\langle p, q \rangle$ *is of the form* $(0, a)$ *or* $(b, 0)$. *Then*

$$E^2_{0,r} \to H_r \to E^2_{r,0} \to E^2_{0,r-1} \to \ldots$$

is exact (mod $\mathfrak{C}$).

Proof. Notice that $E^2_{p,q} \in \mathfrak{C}$ implies that $E^n_{p,q} \in \mathfrak{C}$ for all $n \geqq 2$, for $E^{k+1}_{p,q}$ is a quotient of a subgroup of $E^k_{p,q}$.

Fix $m \geqq 2$. Looking at (A) and using (B),

$$H_{m-k,k}/H_{m-k-1,k+1} \approx E^{\infty}_{m-k,k} \in \mathfrak{C}$$

unless $k = 0$ or m, we obtain, by Lemma 4, that

(1) $0 \to E^{\infty}_{0,m} \to H^{\infty}_{m} \to E_{m,0} \to 0$ is exact (mod $\mathfrak{C}$).

For a ∂-couple, d^k has degree $(-k, k-1)$; consequently, $E^k_{n,0}$ contains no boundaries and $E^k_{0,n}$ contains only cycles (for all n, k). For $2 \leqq n \leqq m$,

$$d^n(E^n_{m+1,0}) \subset E^n_{m-n+1,n-1} \in \mathfrak{C}$$

and

$$0 \to \ker d^n_{m+1,0} \to E^n_{m+1,0} \to d^n(E^n_{m+1,0}) \to 0$$
$$\ker d^n_{m+1,0} = E^{n+1}_{m+1,0}$$

is exact. Iterating this argument, we obtain

$$E^2_{m+1,0} \approx_{\mathfrak{C}} E^{m+1}_{m+1,0}.$$

The same reasoning yields

$$E^2_{m,0} \approx_{\mathfrak{C}} E^m_{m,0}.$$

For $2 \leqq n \leqq m$, $E^n_{n,m-n+1} \in \mathfrak{C}$, and therefore $d^n(E^n_{n,m-n+1}) \in \mathfrak{C}$ [here $m < r$] and

$$0 \to d^n(E^n_{n,m-n+1}) \to E^n_{0,m} \to E^{n+1}_{0,m} \to 0$$

is exact. Iteration then yields: $E^2_{0,m} \approx_{\mathfrak{C}} E^{m+1}_{0,m}$. Similarly,

$$E^2_{0,m-1} \approx_{\mathfrak{C}} E^m_{0,m-1} \quad \text{and} \quad E^n_{n,m-n+1} \to E^n_{0,m} \to E^{n+1}_{0,m} \to 0$$

is exact. However, for $n \geqq m+2$, $E^n_{n,m-n+1} = 0$, and consequently $E^{m+2}_{0,m} \approx_{\mathfrak{C}} E^{\infty}_{0,m}$. For $n \geqq m+1$, $d^n_{m,0} = 0$. Thus $E^{m+1}_{m,0} \approx_{\mathfrak{C}} E^{\infty}_{m,0}$.

Since $E^{m+1}_{0,m}$ contains only cycles,

(2) $E^{m+1}_{m+1,0} \to E^{m+1}_{0,m} \to E^{m+2}_{0,m} \to 0$ is exact.

Since $E^m_{m,0}$ has no boundaries,

(3) $0 \to E^{m+1}_{m,0} \to E^m_{m,0} \to E^m_{0,m-1}$ is exact.

Putting together sequences (1), (2), and (3) and the isomorphisms, we obtain

$$\begin{array}{ccccccccc}
 & & E^2_{m+1,0} & & & & & & \\
 & & \downarrow & & & & & & \\
 & & E^2_{0,m} & & & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & E^{\infty}_{0,m} & \longrightarrow & H_m & \longrightarrow & E_{m,0} & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & E^2_{m,0} & & \\
 & & & & & & \downarrow & & \\
 & & & & & & E^2_{0,m-1} & &
\end{array}$$

where the straight lines are exact (mod $\mathfrak{C}$). (When $m = r$, we must write $E_{0,r}^{r+1}$ in place of $E_{0,m}^2$.) Then by Lemma 5,

$$(*) \qquad E_{m+1,0}^2 \to E_{0,m}^2 \to H_m \to E_{m,0}^2 \to E_{0,m-1}^2$$

is exact (mod $\mathfrak{C}$). This is true for all $m < r$ and the homomorphisms $E_{m,0}^2 \to E_{0,m-1}^2$ are the same for different short sequences. (In each case it is $d_{m,0}^m$ preceded and followed by the same (mod $\mathfrak{C}$) isomorphisms.) Thus, combining the short sequences, we have that

$$E_{r,0}^2 \to E_{0,r-1}^2 \to H_{r-1} \to E_{r-1,0}^2 \to \ldots$$

is exact (mod $\mathfrak{C}$).

Since all the exact sequences and all the isomorphisms (except (*)) hold when $m = r$, we can extend this sequence slightly to the left.

It is not true that $E_{0,r}^2 \approx_{\mathfrak{C}} E_{0,r}^{r+1}$; however, since $E_{0,r}^n$ has no boundaries (for any n), $E_{0,r}^2$ is mapped onto $E_{0,r}^{r+1}$ by $d^r d^{r-1} \ldots d^2$. Thus, the five-term exact sequence which results from the diagram above can be replaced by a four-term sequence,

$$E_{0,r}^2 \to H_r \to E_{r,0}^2 \to E_{0,r-1}^2,$$

which is exact (mod $\mathfrak{C}$). This completes the proof.

5. Some standard theorems (mod $\mathfrak{C}$). In this section, as before, $\mathfrak{C}$ will be a class of abelian groups. Coefficient groups for homology, when suppressed, will be understood to be the integers. We wish to prove the following result.

THEOREM 2 (*The Serre exact sequence* (mod $\mathfrak{C}$)). Suppose that

$$F \xrightarrow{i} X \xrightarrow{g} B$$

is a Serre fibring, where B is path-connected and $H_1(B)$ operates simply on $H_*(F)$, and suppose that for some class $\mathfrak{C}$, $H_i(B) \in \mathfrak{C}$ for $0 < i < q$ and $H_i(F) \in \mathfrak{C}$ for $0 < i < p$. Then there exists a (mod $\mathfrak{C}$) *exact sequence*

$$H_{p+q-1}(F) \xrightarrow{i_*} H_{p+q-1}(X) \xrightarrow{g_*} H_{p+q-1}(B) \xrightarrow{\alpha} H_{p+q-2}(F) \to \ldots .$$

Proof. Serre (**7**) has shown that there is a regular ∂-couple D associated with the fibring in which

$$E_{a,b}^2 = H_a(B, H_b(F)) \quad \text{and} \quad H(D) = H(X).$$

Taking $r = p + q - 1$, the exactness of the sequence follows from Theorem 1. The assertion about the homomorphisms i_* and g_* is proved in (**5**, p. 271). (α is just a name.)

The Eilenberg-MacLane computation of $H_*(Z_p, 1)$, which used no (mod $\mathfrak{C}$) theory, shows that $H_i(Z_p, 1; Z)$ is finitely generated (**4**). In fact, since every element is of order p, $H_i(Z_p, 1) \in \mathfrak{C}(p)$. Using the fibring

$$K(Z_p, 1) \to K(Z_{p^r}, 1) \to (Z_{p^{r-1}}, 1),$$

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the Serre exact sequence (mod $\mathfrak{C}(p)$), and induction, we obtain $H_i(Z_{p^r},1) \in \mathfrak{C}(p)$. Using the same technique on the fibring

$$K(Z_{p^r}, k - 1) \to E \to K(Z_{p^r}, k),$$

we obtain $H_i(Z_{p^r}, k) \in \mathfrak{C}(p)$ for all r, k. (Here, E is the space of paths in $K(Z_{p^r}, k)$ starting at a fixed base-point.)

LEMMA 6. *If $G \in \mathfrak{C}$, then $H_i(G, k; Z) \in \mathfrak{C}$.*

Proof. Write G as a direct sum of cyclic groups of prime power order:

$$G = \sum_j Z_{p_j}{}^{n_j}.$$

Then $Z_{p_j} \in \mathfrak{C}$ for all j,

$$K(G, k) = \prod_j K(Z_{p_j}{}^{n_j}, k) \quad \text{and} \quad H_i(K(Z_{p_j}{}^{n_j}, k); Z) \in \mathfrak{C}.$$

Therefore, by the Künneth formulas, $H_i(G, k; Z) \in \mathfrak{C}$.

LEMMA 7. *Suppose that X has a finite number of non-zero homotopy groups, and $\pi_i(X) \in \mathfrak{C}$ for all i. Then $H_i(X) \in \mathfrak{C}$ for all i.*

Proof. The proof is by induction on the number n of non-zero homotopy groups of X. Lemma 6 establishes the result for the case $n = 1$. Suppose that it holds for all simply connected spaces which have no more than j non-zero homotopy groups. Let Y be a simply connected space with non-zero homotopy groups in dimensions $n_1 < n_2 < \ldots < n_j < n_{j+1}$ with $\pi_{n_i}(Y) \in \mathfrak{C}$, $i = 1, \ldots, j + 1$.There is a fibring

$$F \to Y \to K(\pi_{n_1}(Y), n_1)$$

such that

$$p_*: \pi_{n_1}(Y) \to \pi_{n_1}(K(\pi_{n_1}(Y), n_1))$$

is an isomorphism. Using the homotopy exact sequence of this fibring we obtain

$$\pi_{n_i}(F) \approx \pi_{n_i}(Y), \qquad 2 \leqq i \leqq j + 1,$$

and that these are the only non-zero homotopy groups of F. Then, by the induction hypothesis, $H_i(F) \in \mathfrak{C}$ for all i.

Since both F and $K(\pi_{n_1}(Y); n_1)$ are infinitely connected (mod $\mathfrak{C}$), the (mod $\mathfrak{C}$) Serre exact sequence is infinite, and this implies that $H_i(Y) \in \mathfrak{C}$ for all i. This completes the induction.

THEOREM 3 (*The Hurewicz theorem* (mod $\mathfrak{C}$)). *Suppose that $\pi_i(X) \in \mathfrak{C}$ for $i < n$. Then $\pi_n(X) \approx_{\mathfrak{C}} H_n(X)$.*

Proof. Take a Postnikov system for X. Let B^{n-1} be that space in the Postnikov system made up of the homotopy groups of X up to dimension $n - 1$. Then $p: X \to B^{n-1}$ is a fibring which induces isomorphisms in homotopy

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up to dimension $n-1$, and $i\colon F \to X$ (the inclusion of the fibre) induces isomorphisms in homotopy in dimensions n and above. Furthermore, $\pi_i(F) = 0$ for all $i < n$ and $\pi_n(F) \approx H_n(F)$. B^{n-1} is infinitely connected (mod $\mathfrak{C}$); thus, by the Serre exact sequence (mod $\mathfrak{C}$) we obtain

$$H_n(F) \approx_{\mathfrak{C}} H_n(X).$$

Using $\pi_n(F) \approx \pi_n(X)$ and combining the isomorphisms, we obtain

$$\pi_n(X) \approx_{\mathfrak{C}} H_n(X);$$

and the isomorphism is induced by the Hurewicz homomorphism since the following diagram is commutative

$$\begin{array}{ccc} \pi_n(F) & \xrightarrow{i_*} & \pi_n(X) \\ h\downarrow & & \downarrow h \\ H_n(F) & \xrightarrow{i_*} & H_n(X) \end{array}$$

THEOREM 4 (*The Whitehead theorem* (mod $\mathfrak{C}$)). *Suppose that X and Y are simply connected and $f\colon X \to Y$ is a continuous map; then statements* (i) *and* (ii) *are equivalent:*

(i) $f_*\colon H_i(X) \to H_i(Y)$ *is a* (mod $\mathfrak{C}$) *isomorphism for $i < n$ and*

$$H_n(Y)/f_*H_n(X) \in \mathfrak{C};$$

(ii) $f_\#\colon \pi_i(X) \to \pi_i(Y)$ *is a* (mod $\mathfrak{C}$) *isomorphism for $i < n$ and*

$$\pi_n(Y)/f_\#\pi_n(X) \in \mathfrak{C}.$$

Proof. (A) (i) implies (ii). Take f to be a fibre map with fibre F. Suppose that $H_i(F) \in \mathfrak{C}$ for all $i < p \leqq n-1$. By assumption, $H_i(Y) = 0$ for $i < 2$. Using Theorem 2, the sequence

$$H_{p+1}(F) \to H_{p+1}(X) \xrightarrow{f_*^{p+1}} H_{p+1}(Y) \to H_p(F) \to H_p(X) \xrightarrow{f_*^{p}} H_p(Y)$$

is exact (mod $\mathfrak{C}$). Since $p \leqq n-1$, f_*^p is a (mod $\mathfrak{C}$) isomorphism and f_*^{p+1} is certainly a mod $\mathfrak{C}$ epimorphism. Therefore, $H_p(F) \in \mathfrak{C}$. Repeating this argument, we obtain $H_i(F) \in \mathfrak{C}$ for all $i \leqq n-1$. Furthermore, $H_1(F) = \pi_1(F) \in \mathfrak{C}$. Using the (mod $\mathfrak{C}$) Hurewicz theorem, we have $\pi_i(F) \in \mathfrak{C}$ for all $i \leqq n-1$. The result now follows from the homotopy exact sequence of the fibring

$$F \to X \xrightarrow{f} Y.$$

(B) (ii) implies (i). Using the homotopy exact sequence of the fibring, we see that $\pi_i(F) \in \mathfrak{C}$ for all $i \leqq n-1$. Therefore, $H_i(F) \in \mathfrak{C}$ for $i \leqq n-1$. By hypothesis, $H_1(Y) = 0$. The result clearly follows from the (mod $\mathfrak{C}$) Serre exact sequence.

THEOREM 5 (*The suspension theorem* (mod $\mathfrak{C}$)). *Suppose that X is connected and $\pi_i(X) \in \mathfrak{C}$ for all $i < n$. Then $i_\#: \pi_j(X) \to \pi_j(\Omega SX)$ is a* (mod $\mathfrak{C}$) *isomorphism for $j < 2n - 1$ and a* (mod $\mathfrak{C}$) *epimorphism for $j = 2n - 1$ ($i: X \to \Omega SX$ is the natural inclusion).*

Proof. Consider the acyclic fibring $\Omega SX \to E \to SX$. We have $H_i(SX) = H_{i-1}(X) \in \mathfrak{C}$ for $i < n + 1$. Therefore, $\pi_i(SX) \in \mathfrak{C}$ for $i < n + 1$ and $\pi_j(\Omega SX) \in \mathfrak{C}$ for $j < n$. Consequently, $H_j(\Omega SX) \in \mathfrak{C}$ for $j < n$. Applying Theorem 2 we obtain:

$$\begin{array}{ccccccccc} H_{2n}(\Omega SX) & \to & H_{2n}(E) & \to & H_{2n}(SX) & \xrightarrow{\alpha} & H_{2n-1}(\Omega SX) & \to & H_{2n-1}(E) \to \dots \\ & & \| & & & & & & \| \\ & & 0 & & & & & & 0 \end{array}$$

is exact (mod $\mathfrak{C}$). That is, $\alpha: H_{j+1}(SX) \to H_j(\Omega SX)$ is a (mod $\mathfrak{C}$) isomorphism for $j \leqq 2n - 1$.

Let $\Sigma: H_i(X) \to H_{i+1}(SX)$ be the suspension isomorphism. Then

$$\alpha\Sigma = \pm i_*^j: H_j(X) \to H_j(\Omega SX);$$

see (**1**). Therefore, the i_*^j induce (mod $\mathfrak{C}$) homology isomorphisms for $j \leqq 2n - 1$. Then, by the (mod $\mathfrak{C}$) Whitehead theorem, $i_\#: \pi_j(X) \to \pi_j(\Omega SX)$ is a (mod $\mathfrak{C}$) isomorphism when $j < 2n - 1$ and a (mod $\mathfrak{C}$) epimorphism when $j = 2n - 1$.

6. A Whitehead-type theorem. In this section, we prove the following theorems.

THEOREM 6A. *Suppose that X and Y are simply connected spaces and that $f: X \to Y$ is a continuous map. Then the following four statements are equivalent:*

(i) *$f_*: H_i(X) \to H_i(Y)$ is a* (mod $\mathfrak{C}$) *isomorphism for all i;*

(ii) *$f_\#: \pi_i(X) \to \pi_i(Y)$ is a* (mod $\mathfrak{C}$) *isomorphism for all i;*

(iii) *for any space L with $H_*(L)$ finitely generated, the homomorphism*

$$(\Omega^2 f)_*: [L, \Omega^2 X] \to [L, \Omega^2 Y]$$

is a (mod $\mathfrak{C}$) *isomorphism;*

(iv) *for any space P with $\pi_*(P)$ finitely generated, the homomorphism*

$$(S^2 f)^*: [S^2 Y, P] \to [S^2 X, P]$$

is a (mod $\mathfrak{C}$) *isomorphism.*

THEOREM 6B. *Suppose that X and Y are simply connected spaces and that $f: X \to Y$ is a continuous map. Then the following two statements are equivalent:*

(v) *$f_*: H_*(X) \to H_*(Y)$ is a* (mod $\mathfrak{C}$) *isomorphism;*

(vi) *for any space P, the homomorphism*

$$(S^2 f)^*: [S^2 Y, P] \to [S^2 X, P]$$

is a (mod $\mathfrak{C}$) *isomorphism.*

THEOREM 6C. *Suppose that X and Y are simply connected spaces and that $f: X \to Y$ is a continuous map. Then the following two statements are equivalent:*

(vii) $f_\#: \pi_*(X) \to \pi_*(Y)$ *is a* (mod $\mathfrak{C}$) *isomorphism;*

(viii) *for any space L, the homomorphism*

$$(\Omega^2 f)_*: [L, \Omega^2 X] \to [L, \Omega^2 Y]$$

is a (mod $\mathfrak{C}$) *isomorphism.*

The proof of these theorems will proceed from a series of lemmas. First recall some standard homotopy theory.

If $f: X \to Y$ is a fibring with fibre F, then for any space L we have a long exact sequence

$$\ldots [L, \Omega^i F] \to [L, \Omega^i X] \xrightarrow{(\Omega^i f)_*} [L, \Omega^i Y] \to [L, \Omega^{i-1} F] \to \ldots$$
$$\to [L, F] \to [L, X] \xrightarrow{f_*} [L, Y].$$

If $f: X \to Y$ has mapping cone C_f, then, for any space P, there is a long exact sequence

$$\ldots \to [S^i C_f, P] \to [S^i Y, P]^{(S^i f)^*} \to [S^i X, P] \to [S^{i-1} C_f, P] \to \ldots$$
$$\to [C_f, P] \to [Y, P] \to [X, P].$$

Let G be a finitely generated abelian group. Then $M(G, n)$ $(n \geqq 2)$ will be a simply connected space with $H_i(M(G, n)) = 0$ for all $i \neq n$ and $H_n(M(G, n)) = G$. For $n = 1$, we have the condition that

$$\pi_1(M(G, 1)) = H_1(M(G, 1)).$$

Such spaces exist and are well-determined up to homotopy type (**6**).

Let $n: S^k \to S^k$ be a map of degree n. Then C_n has the homotopy type of an $M(Z_n, k)$, and the sequence

$$\pi_k(P) \xrightarrow{n\#} \pi_k(P) \to [M(Z_n, k), P] \to \pi_{k+1}(P) \xrightarrow{n\#} \pi_{k+1}(P)$$

is exact. Therefore, if $Z_n \in \mathfrak{C}$, then $[M(Z_n, k), P] \in \mathfrak{C}$ for $k \geqq 3$ (and $[M(Z_n, 2), P] \in \bar{\mathfrak{C}}$).

LEMMA 8. *If $G \in \mathfrak{C}$, then $[M(G, k), P] \in \mathfrak{C}$ for $k \geqq 3$.*

Proof. Suppose that

$$G = \sum_{i=1}^{s} Z_{n_i}.$$

Then $M(G, k) = \vee_i M(Z_{n_i}, k)$ (where the wedge product is the one-point union), and

$$[M(G, k), P] = \sum_i [M(Z_{n_i}, k), P].$$

Now by the remark above, each summand is in $\mathfrak{C}$ and therefore

$$[M(G, k), P] \in \mathfrak{C}.$$

LEMMA 9. *Suppose that* (a) $H_*(A) \in \mathfrak{C}$ *or that* (b) $H_i(A) \in \mathfrak{C}$ *for all* i *and that* $\pi_*(P)$ *is finitely generated. Using either hypothesis,* $[SA, P] \in \bar{\mathfrak{C}}$.

Proof. The Eckmann-Hilton decomposition of SA is the suspension of that of A, i.e. all the spaces and maps that occur are suspensions:

$$\begin{array}{ccccccccc}
M(H_1(A), 2) & \to & SA_2 & \to \dots \to & SA_r & \xrightarrow{Si_r} & SA_{r+1} & \to \dots \\
\| & & \downarrow & & & & \downarrow Sf_{r+1} & \\
SA_1 & & M(H_2(A), 3) & & & & M(H_{r+1}(A), r+2) &
\end{array}$$

and for all r, $M(H_{r+1}(A), r+2) = C_{Si_r}$ (**3**). The proof is by induction on r.

Assume that $[SA_r, P] \in \bar{\mathfrak{C}}$. Then

$$[M(H_{r+1}(A), r+2), P] \xrightarrow{(Sf_{r+1})^*} [SA_{r+1}, P] \xrightarrow{(Si_r)^*} [SA_r, P]$$

is exact. Using either hypothesis, $H_{r+1}(A) \in \mathfrak{C}$, and hence by Lemma 8, $[M(H_{r+1}(A), r+2), P] \in \mathfrak{C}$. Therefore, $[SA_{r+1}, P] \in \bar{\mathfrak{C}}$. Now, again using either hypothesis, for large enough k, $[M(H_k(A), k+1), P] = 0$. That is, only a finite number of extensions are necessary to build up to $[SA, P]$, and therefore $[SA, P] \in \bar{\mathfrak{C}}$.

LEMMA 10. *Suppose that* $f: X \to Y$ *either* (a) *induces a* (mod $\mathfrak{C}$) *isomorphism* $f_*: H_*(X) \to H_*(Y)$ *or* (b) *induces* (mod $\mathfrak{C}$) *isomorphisms* $f_*: H_i(X) \to H_i(Y)$ *for all* i *and* $\pi_*(P)$ *is finitely generated. Using either hypothesis,*

$$(S^2f)^*: [S^2Y, P] \to [S^2X, P]$$

is an isomorphism (mod $\mathfrak{C}$).

Proof. Let $A = C_f$. Then

$$[S^2A, P] \to [S^2Y, P] \xrightarrow{(S^2f)^*} [S^2X, P] \to [SA, P]$$

is an exact sequence of groups. Now either $H_*(A) \in \mathfrak{C}$ or $H_i(A) \in \mathfrak{C}$ (all i) and $\pi_*(P)$ is finitely generated. Thus, using Lemma 9, $[SA, P] \in \bar{\mathfrak{C}}$ and $[S^2A, P] \in \mathfrak{C}$ (since it is abelian). Now coker $(S^2f)^*$ is abelian and is in $\bar{\mathfrak{C}}$. Therefore, it is in $\mathfrak{C}$ and $(S^2f)^*$ is a (mod $\mathfrak{C}$) isomorphism.

LEMMA 11. *Suppose that* $G \in \mathfrak{C}$ *and* L *is any space, then* $[L, K(G, n)] \in \bar{\mathfrak{C}}$.

Proof. $[L, K(G, n)] = H^n(L; G) = H^n(L) \otimes G \oplus \mathrm{Tor}(H^{n+1}(L), G)$ which is in $\bar{\mathfrak{C}}$.

LEMMA 12. *If* $\pi_i(F) \in \mathfrak{C}$ *and* $H_*(L)$ *is finitely generated, or if* $\pi_*(F) \in \mathfrak{C}$ *and* L *is any space, then* $[L, \Omega F] \in \bar{\mathfrak{C}}$.

Proof. The Postnikov system for ΩF is the "loop" of the Postnikov system for F. That is, the spaces and maps that occur are the "loops" of those that occur for F:

$$
\begin{array}{ccc}
 & & \Omega F \\
 & & \downarrow \\
K(\pi_i(F), i-1) = \Omega(K(\pi_i(F)), i) & \xrightarrow{\Omega j_i} & \Omega X_i \\
 & & \Big\downarrow \Omega p_i \\
 & & \Omega X_{i-1} \\
 & & \downarrow \\
 & & \Omega K(\pi_2(F), 2) = K(\pi_2(F), 1)
\end{array}
$$

and for each i,

$$K(\pi_i(F), i-1) \xrightarrow{\Omega j_i} \Omega X_i \xrightarrow{\Omega p_i} \Omega X_{i-1}$$

is a fibring.

The proof is by induction on i. Suppose that $[L, \Omega X_{i-1}] \in \bar{\mathfrak{C}}$. The sequence

$$[L, K(\pi_i(F), i-1)] \to [L, \Omega X_i] \to [L, \Omega X_{i-1}]$$

is exact. The two outside groups are in $\bar{\mathfrak{C}}$. Therefore $[L, \Omega X_i] \in \bar{\mathfrak{C}}$. Either hypothesis assures us that only a finite number of non-trivial extensions are involved as we build up to $[L, \Omega F]$. Therefore $[L, \Omega F] \in \bar{\mathfrak{C}}$.

LEMMA 13. *Suppose that* $f: X \to Y$ *either*

(a) *induces a* (mod $\mathfrak{C}$) *isomorphism* $f_{\#}: \pi_*(X) \to \pi_*(Y)$ *or*

(b) *induces* (mod $\mathfrak{C}$) *isomorphisms* $f_{\#}: \pi_i(X) \to \pi_i(Y)$ *for all* i *and* $H_*(L)$ *is finitely generated.*

Then

$$(\Omega^2 f)_*: [L, \Omega^2 X] \to [L, \Omega^2 Y]$$

is an isomorphism (mod $\mathfrak{C}$).

Proof. Take f to be a fibre map with fibre F. Then either $\pi_*(F) \in \mathfrak{C}$ or $\pi_i(F) \in \mathfrak{C}$ (all i) and $H_*(L)$ is finitely generated. In either case, Lemma 5 implies that $[L, \Omega F] \in \mathfrak{C}$ and $[L, \Omega^2 F] \in \mathfrak{C}$. Now we have an exact sequence of groups

$$[L, \Omega^2 F] \to [L, \Omega^2 X] \xrightarrow{(\Omega^2 f)_*} [L, \Omega^2 Y] \to [L, \Omega F].$$

coker $(\Omega^2 f)_*$ is abelian and is in $\bar{\mathfrak{C}}$. Therefore, it is in $\mathfrak{C}$ and $(\Omega^2 f)_*$ is a (mod $\mathfrak{C}$) isomorphism.

Proof of Theorem 6A. (i) $\Rightarrow$ (ii). (Whitehead theorem);
(i) $\Rightarrow$ (iv) (Lemma 3);
(ii) $\Rightarrow$ (iii) (Lemma 6);
(iii) $\Rightarrow$ (ii) since we can take $L = S^n$, $n = 0, 1, 2, \ldots$;
(iv) $\Rightarrow$ (i) since we can take $P = K(Z, n)$, $n = 2, 3, \ldots$. This shows that we have (mod $\mathfrak{C}$) isomorphisms in cohomology and implies the result for homology.

Proof of Theorem 6B. (v) $\Rightarrow$ (vi) (Lemma 3).
(vi) $\Rightarrow$ (v), since we can take

$$P = \prod_{i=2}^{\infty} K(Z, i).$$

Proof of Theorem 6C. (vii) $\Rightarrow$ (viii) (Lemma 6).
(viii) $\Rightarrow$ (vii) since we can take

$$L = \bigvee_{i=0}^{\infty} S^i.$$

That the "finiteness" conditions are necessary can be seen from the following example. Let $\mathfrak{C} = \mathfrak{C}(p)$, and let

$$X = Y = \bigvee_{i=2}^{\infty} S^i.$$

Let $f_i \colon S^i \to S^i$ be a map of degree p^i, and let $f \colon X \to Y$ be the map

$$\bigvee_{i=2}^{\infty} f_i \colon \bigvee_{i=2}^{\infty} S^i \to \bigvee_{i=2}^{\infty} S^i.$$

Clearly, $f_* \colon H_i(X) \to H_i(Y)$ $(i \geqq 2)$ is multiplication by p^i and is an isomorphism (mod $\mathfrak{C}(p)$). Let

$$P = \prod_{i=4}^{\infty} K(Z, i)$$

and consider the induced homomorphism

$$(S^2 f)^* \colon [S^2 Y, P] \to [S^2 X, P].$$

This is a monomorphism and the co-kernel is $\prod_{i=2}^{\infty} Z_{p^i}$. However, this group is not in $\mathfrak{C}(p)$. The problem, of course, is that the classes we have discussed are not closed under the operation of taking limits.

7. The general (mod $\mathfrak{C}$) **suspension theorem.** This section completes the proof of the general (mod $\mathfrak{C}$) suspension theorem. We begin with the following result.

LEMMA 14. *Suppose that $H_*(X)$ is finitely generated and that $H^i(X) \in \mathfrak{C}$ for $i > k$; furthermore, suppose that $\pi_i(F) \in \mathfrak{C}$ for $i < n$; then, when $k + r < n$ $(r \geqq 1)$, $[S^r X, F] \in \mathfrak{C}$ (or in $\bar{\mathfrak{C}}$ when $r = 1$).*

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Proof. Take a Postnikov system for F:

$$\begin{array}{ccc} & & F \\ & & \downarrow \\ & & \vdots \\ & & \downarrow \\ K(\pi_s(F), s) & \xrightarrow{j_s} & F^s \\ & & \downarrow f_s \\ & & F^{s-1} \\ & & \downarrow \\ & & K(\pi_1(F), 1) \end{array}$$

(In this proof read $\bar{\mathfrak{C}}$ for $\mathfrak{C}$ when $r = 1$.) The proof is by induction.

Suppose that $[S^rX, F^{s-1}] \in \mathfrak{C}$. If $s < n$, then $\pi_s(F) \in \mathfrak{C}$ and

$$[S^2X, K(\pi_s(F), s)] \in \mathfrak{C}$$

(Lemma 4). If $s \geqq n$, then $s > k + r$ and $H^s(S^rX)$ and $H^{s+1}(S^rX)$ are both in $\mathfrak{C}$. However,

$$\begin{aligned}[S^rX, K(\pi_s(F), s)] &= H^s(S^rX; \pi_s(F)) \\ &= H^s(S^rX) \otimes \pi_s(F) \oplus \mathrm{Tor}(H^{s+1}(S^rX); \pi_s(F)),\end{aligned}$$

and hence is in $\mathfrak{C}$. Therefore, in the exact sequence

$$[S^rX, K(\pi_s(F), s)] \to [S^rX, F^s] \to [S^rX, F^{s-1}]$$

the two extreme groups are in $\mathfrak{C}$. Thus, $[S^rX, F^s] \in \mathfrak{C}$. As before, because of the assumption on $H_*(X)$, only a finite number of extensions are required to build up to $[S^rX, F]$ and this last group is in $\mathfrak{C}$.

THEOREM 7 (*The general* (mod $\mathfrak{C}$) *suspension theorem*). *Suppose that*

(i) $\pi_i(Y) \in \mathfrak{C}$ *for all* $i < n$,

(ii) $H_*(X)$ *is finitely generated,*

(iii) $H^i(X) \in \mathfrak{C}$ *for all* $i > k$.

Then the suspension homomorphism

$$E\colon [S^rX, Y] \to [S^{r+1}X, SY]$$

is a (mod $\mathfrak{C}$) *monomorphism for* $2 \leqq r \leqq 2n - k - 2$ (*when* $r = 1$, $\ker E \in \bar{\mathfrak{C}}$); *it is a* (mod $\mathfrak{C}$) *epimorphism for* $2 \leqq r \leqq 2n - k - 1$.

Proof. Let $j\colon Y \to \Omega SY$ be the natural inclusion and make it a fibre map with fibre F. The homotopy exact sequence of the fibring, together with Theorem 4, implies that $\pi_i(F) \in \mathfrak{C}$ for $i < 2n - 1$. The sequence

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$$\begin{array}{ccc} [S^{r-1}X, \Omega F] & \rightarrow & [S^{r-1}X, \Omega Y] \\ & & \downarrow (\Omega j)_* \\ & & [S^{r-1}X, \Omega^2 SY] \\ & & \downarrow \\ & & [S^{r-1}X, F] \end{array}$$

is exact. $\pi_i(\Omega F) \in \mathfrak{C}$ for $i < 2n - 2$. Using Lemma 1 we have:
$[S^{r-1}X, \Omega F] \in \bar{\mathfrak{C}}$ when $r \geqq 1$ and $k + r - 1 < 2n - 2$, and
$[S^{r-1}X, F] \in \bar{\mathfrak{C}}$ for $r \geqq 2$ and $k + r - 1 < 2n - 1$.

The homomorphism E may be defined by the following diagram:

$$\begin{array}{ccc} [S^{r-1}X, \Omega Y] & \xrightarrow{(\Omega j)_*} & [S^{r-1}X, \Omega^2 SY] \\ \| & & \| \\ [S^r X, Y] & \xrightarrow{E} & [S^{r+1}X, SY] \end{array}$$

where the equalities indicate the natural equivalence.

Rewriting the inequalities above, we have that $(\Omega j)_*$ is a (mod $\bar{\mathfrak{C}}$) monomorphism for $1 \leqq r \leqq 2n - k - 2$ and is a (mod $\bar{\mathfrak{C}}$) epimorphism for $2 \leqq r \leqq 2n - k - 1$. However, all the kernels and co-kernels involved are clearly abelian (except for ker E: $[SX, Y] \rightarrow [S^2X, SY]$), and hence the statements hold (mod $\mathfrak{C}$).

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